

Model Reduction - Trouble with Scales?

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SIAM CSE15, March 14, 2015

Outline

- 1 Scales and Spatial Dimensionality
 - Upscaling and Transport
 - Transpiration Cooling

- 2 How to Best Sample the Solution Manifold?
 - Model Reduction - Reduced Basis Method
 - Transport Dominated Problems

What is this Talk about?...

...some obstructions...

- range of relevant scales
(geometric micro structures, turbulence, porous media...)
- high spatial dimensionality
(parameter dependence, high dimensional phase space...)

... prediction beyond experiment... any mathematical “shortcuts”?

~> **Certified Reduced Bases** for transport dominated problems...

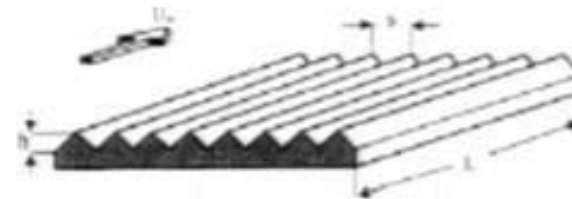
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Turbulent Drag Reduction - FOR 1779 - DFG

(Joint work with G. Deolmi, S. Müller)



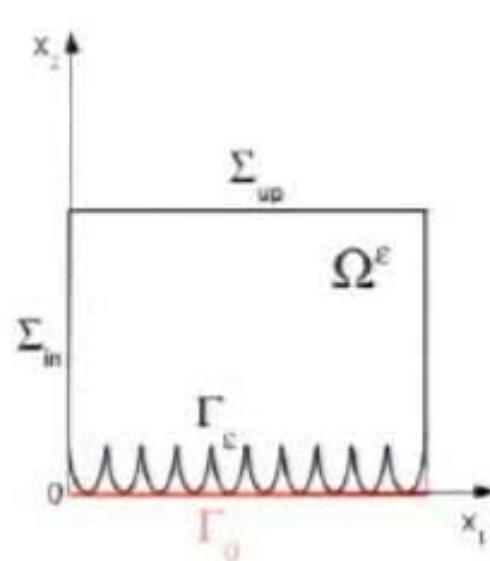
P.R.Viswanath "Aircraft viscous drag reduction using riblets",
Progress in Aerospace Sciences 38 (2002) 571-600

Challenges:

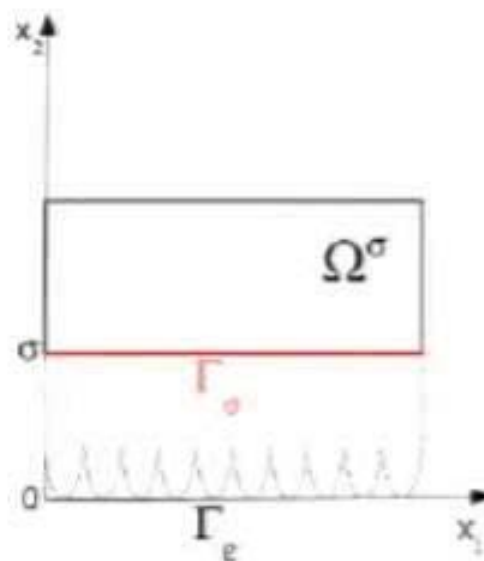
- limited experimental accessibility...
- One cannot resolve turbulent flow...
- One cannot discretize flow boundary...
- How to find the best parameters y : frequency, amplitude... ?
Feedback control?...

Upscaling... (Dahmen/Deolmi/Müller)

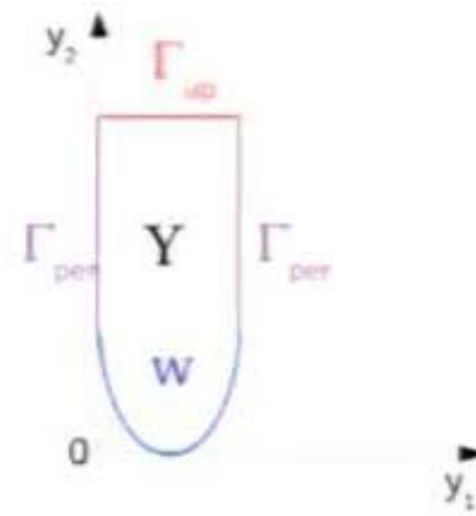
- turbulence modeling $\leftrightarrow$ geometric upscaling
- regularized compressible NS equations on **smooth** domain Ω^σ
 - find effective** boundary conditions depending on $y \in \{\text{frequency, amplitude, wavelength, ...}\}$
 - cell problem $\rightsquigarrow$ parameter dependent **convection-diffusion equation**



$$F(u^\epsilon, \rho^\epsilon, p^\epsilon) = 0,$$

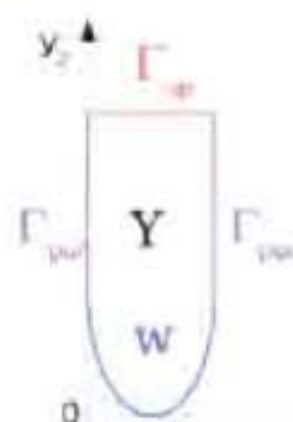


$$F(u^{\text{eff}}, \rho^{\text{eff}}, p^{\text{eff}}) = 0,$$



$$F(u^0, \rho^0, p^0) = 0$$

The Cell Problem



$$\mathbf{y} := \frac{\mathbf{x}}{\epsilon} \quad \eta_\epsilon := \frac{\eta}{\epsilon Re}, \quad k_\epsilon := \frac{k}{\epsilon Re}$$

Beavers-Joseph cond:

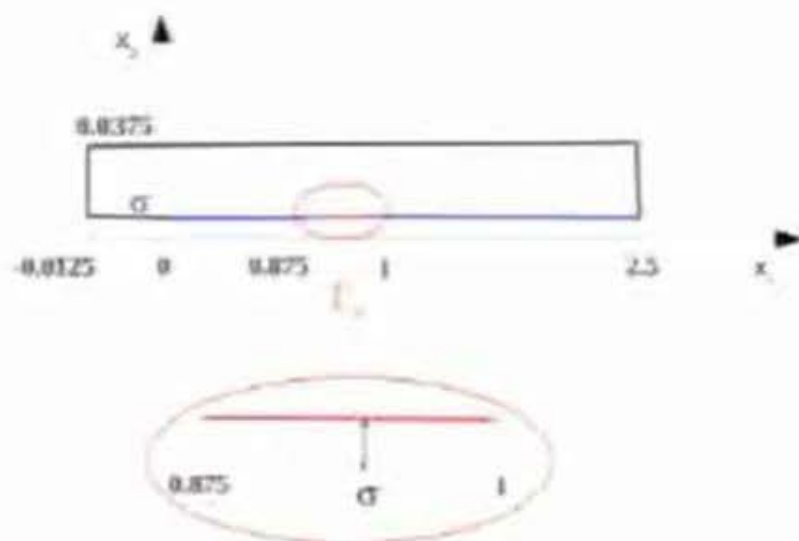
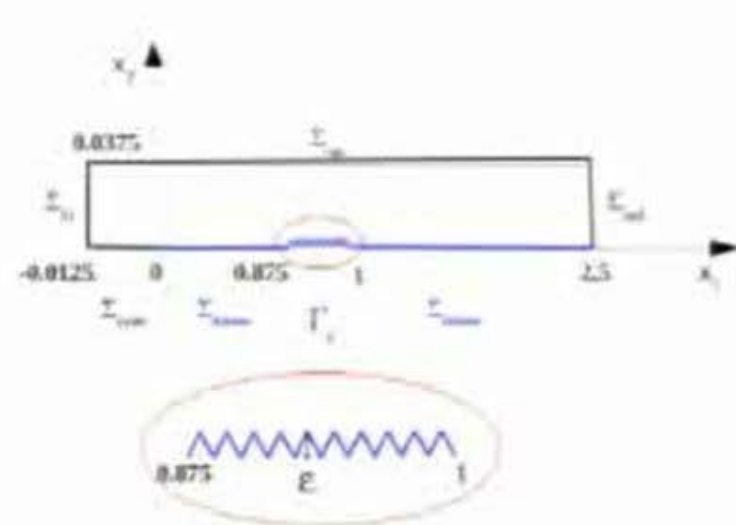
$$\frac{\partial \rho^{eff}}{\partial n} = 0, \quad \frac{\partial p^{eff}}{\partial n} = 0, \quad \mathbf{u}^{eff} = \frac{\partial u_1^{eff}}{\partial x_2} (\sigma \mathbf{e}_1 + \epsilon \langle \chi \rangle)$$

Asymptotic expansions $\leadsto \dots$

$$\begin{aligned} (\mathbf{u}^0 \cdot \nabla_y) \phi + \rho^0 \nabla_y \cdot \chi &= 0, \\ \rho^0 (\mathbf{u}^0 \cdot \nabla_y) \chi + \nabla_y \pi &= \eta_\epsilon (\Delta_y \chi + \frac{1}{3} \nabla_y (\nabla_y \cdot \chi)), \quad \text{in } Y \\ (\mathbf{u}^0 \cdot \nabla_y) \pi + \gamma \rho^0 \nabla_y \cdot \chi &= \frac{\gamma k_y}{Pr \rho^0} \left(\Delta_y \pi - \frac{\epsilon^2}{\rho^0} \Delta_y \phi \right) \end{aligned}$$

$\dots$ + BC's $\dots \leadsto$ **Parameter dependent convection-diffusion equation**

Laminar Flow over Rough Plate: $Re = 5 \cdot 10^5$, $Ma = 0.3$, $\epsilon = 6.25 \cdot 10^{-4}$, $\frac{\epsilon}{\delta} \approx 9\%$ (Dahmen/Doolmi/Müller)



$\epsilon = 6.25 \cdot 10^{-4}$	DNS	Effective
$s = 10\epsilon$	$1.15799 \cdot 10^{-3}$	$1.159905 \cdot 10^{-3}$
$s = 20\epsilon$	$1.199297 \cdot 10^{-3}$	$1.159265 \cdot 10^{-3}$

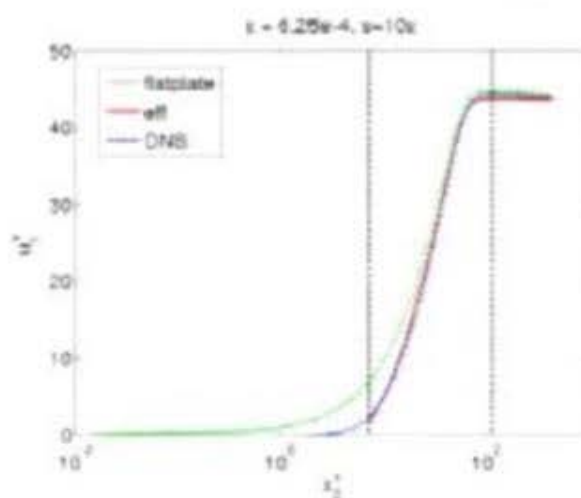
$\epsilon = 6.25 \cdot 10^{-4}$	DNS	Effective
$s = 10\epsilon$	59738	10924
$s = 20\epsilon$	53469	10978

Dynamic drag coefficient: Flat plate: $c_D = 1.224175 \cdot 10^{-3}$

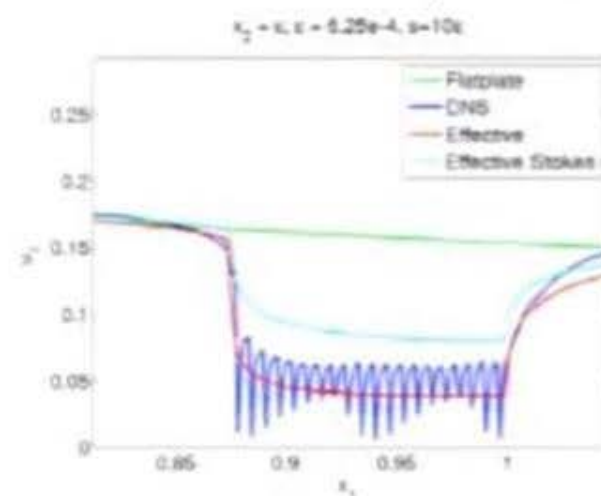
Effective Problem: $\epsilon = 6.25 \cdot 10^{-4}$

$S = 10\epsilon$

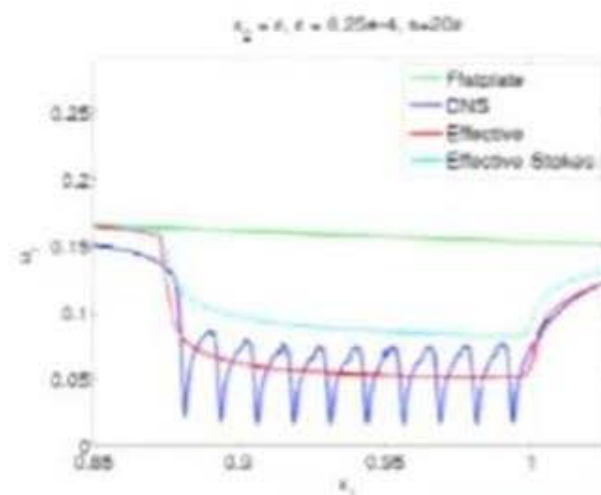
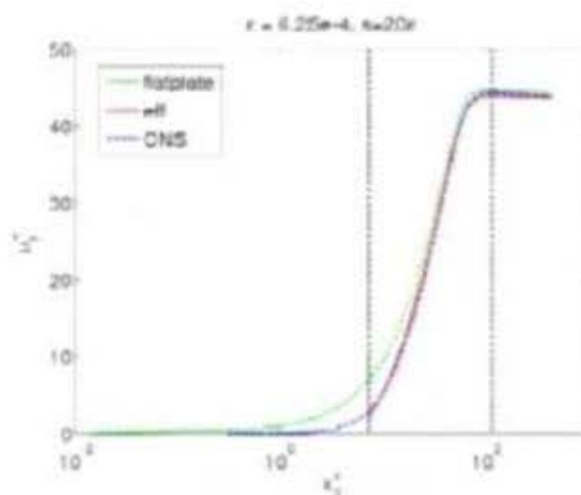
wall-normal direction



streamwise profile



$S = 20\epsilon$



Ongoing...

W. Dahmen, G. Deolmi, S. Müller, Effective boundary conditions for compressible flows over rough surfaces, to appear in M³AS Mathematical Models & Methods in Applied Sciences

So far: proof of principle for small structured area - single cell problem

- incorporate turbulence
- large structured surface: frequent solution of cell problems depending on macro-flow
- feed back loop: reduce macro-flow model... estimate ϵ, λ ... from model and data...

Key demand: model reduction for **transport dominated** problems

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1 Scales and Spatial Dimensionality

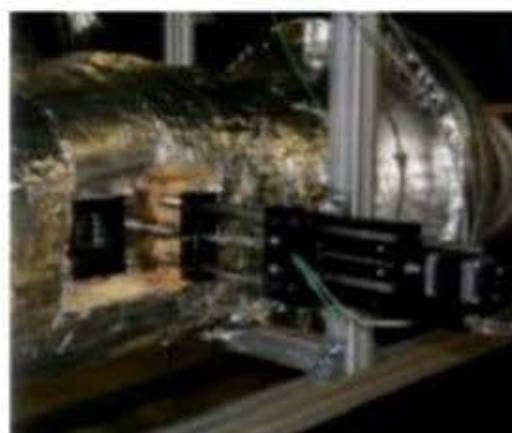
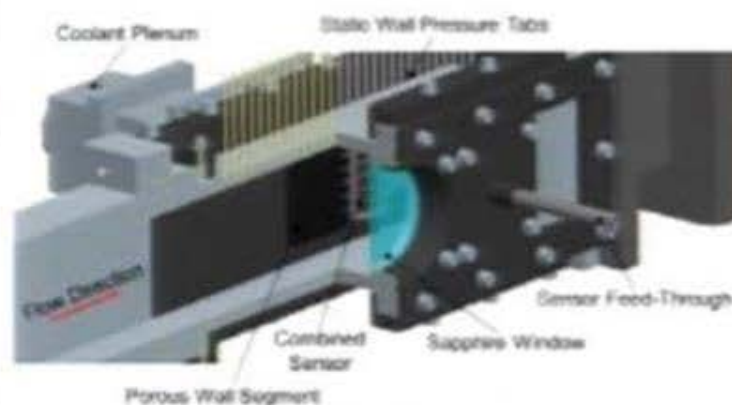
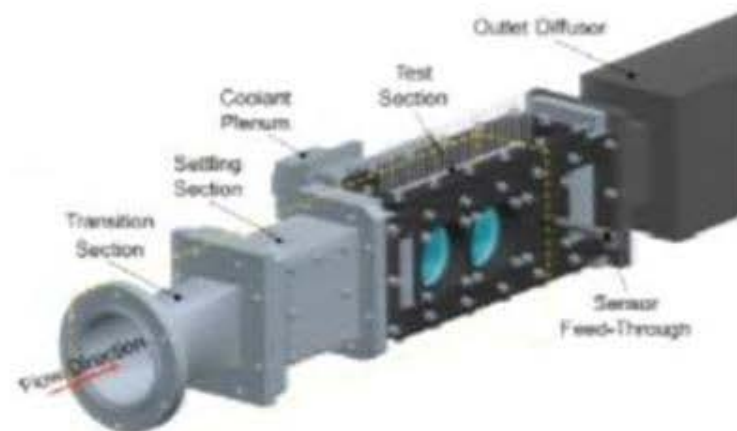
- Upscaling and Transport
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2 How to Best Sample the Solution Manifold?

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Experimental Setup: DFG-TR40

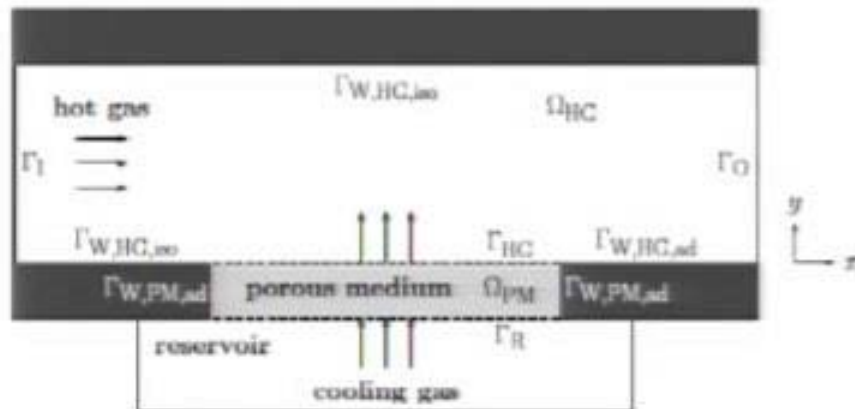
RWTH Aachen, TU Braunschweig, TU Munich, Un. Stuttgart ... Eckert/Livingood



Main Issues

- subsonic turbulent hot gas flow
(Reynolds averages Navier Stokes equation, Wilcox k, ω -model)
- compressible porous medium
(Darcy-Forchheimer momentum balance with quadratic drag)
- capture non-equilibrium temperature effects
- no pre-modeled boundary conditions at hot gas interface
~> more detailed information of skin-friction and heat flux reduction
- goal: predictability beyond experiment

Computational Model Dahmen/Gotzen/Müller/Rom



step 5 $L_{NS}(U_{NS}) = 0$ HG

- $\frac{\partial U_{NS}}{\partial t} + L_{NS}(U_{NS}) = 0$
- $(\Delta t)^{-1} U_{NS}^{n+1} + L_{NS}(U_{NS}^{n+1}) = (\Delta t)^{-1} U_{NS}^n \hookrightarrow$

step 2 P_{HG}
 T_{HG}
 ∇T_{HG} Γ_{HG} v_{PM}
 T_{LPM} step 4

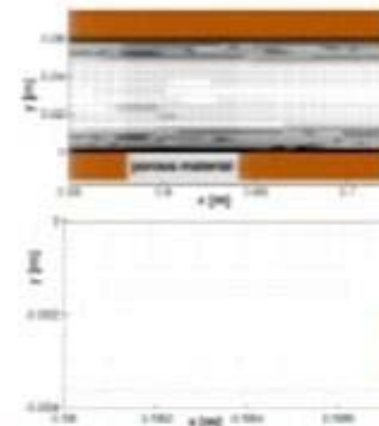
step 3 $a_{PM}(U_{PM}, \Theta_{PM}) = F(\Theta_{PM}, T_{HG}, \nabla T_{HG})$ PM

- $U_{PM} = (U, T)$ with $U = (\rho_f, v)$, $T = (T_s, T_f)$
- $a_T(U^n, T^{n+1}, \Theta_{PM}) = F(\Theta_{PM}, T_{HG}, \nabla T_{HG}) + a_U(U_{PM}^n, \Theta_{PM})$
- $a_U(U^{n+1}, T^n, \Theta_{PM}) = F(\Theta_{PM}, T_{HG}, \nabla T_{HG}) + a_T(U_{PM}^n, \Theta_{PM}) \hookrightarrow$

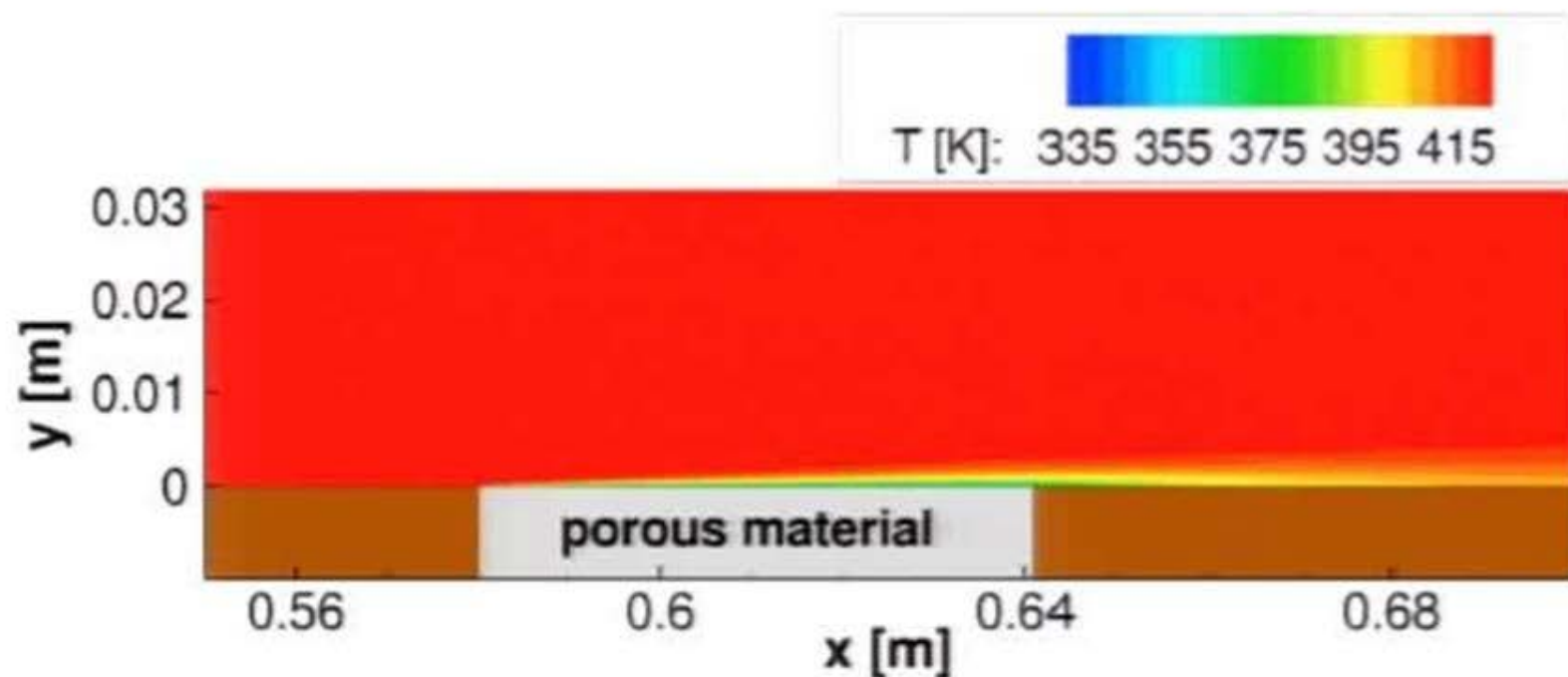
Hot gas: compr. Navier Stokes

porous medium: Darcy-Forchheimer

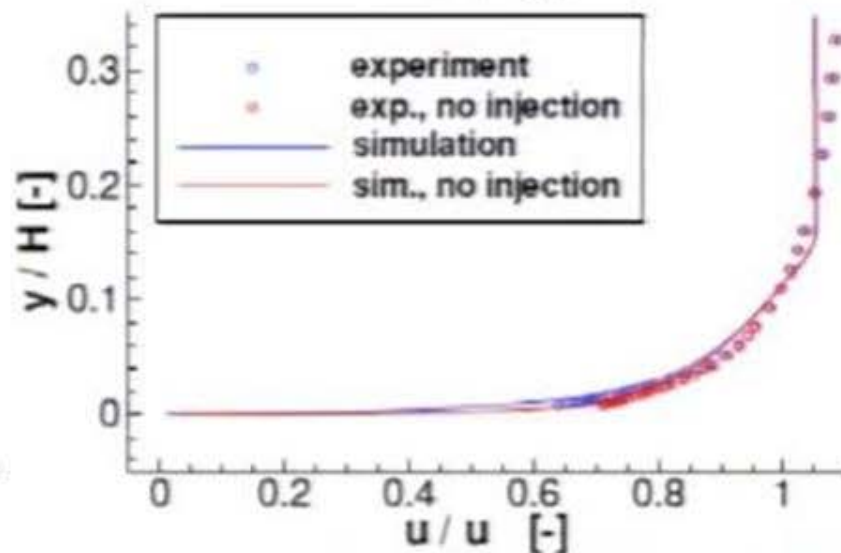
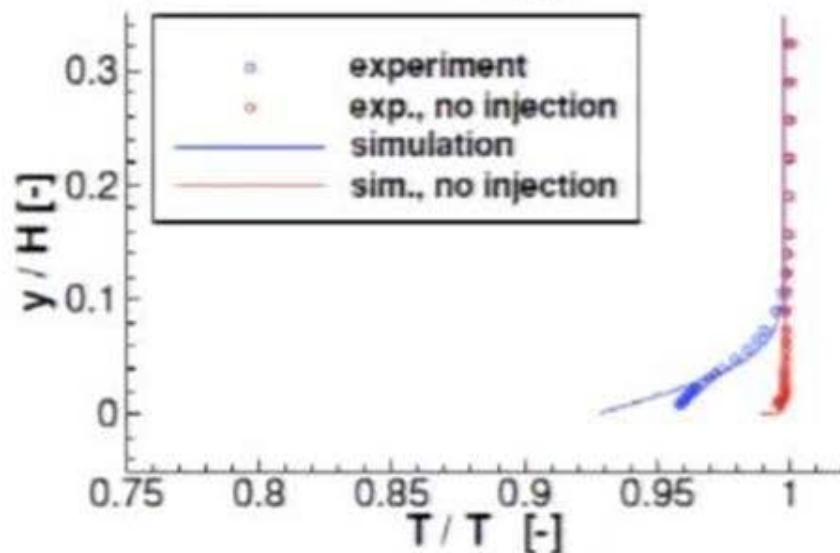
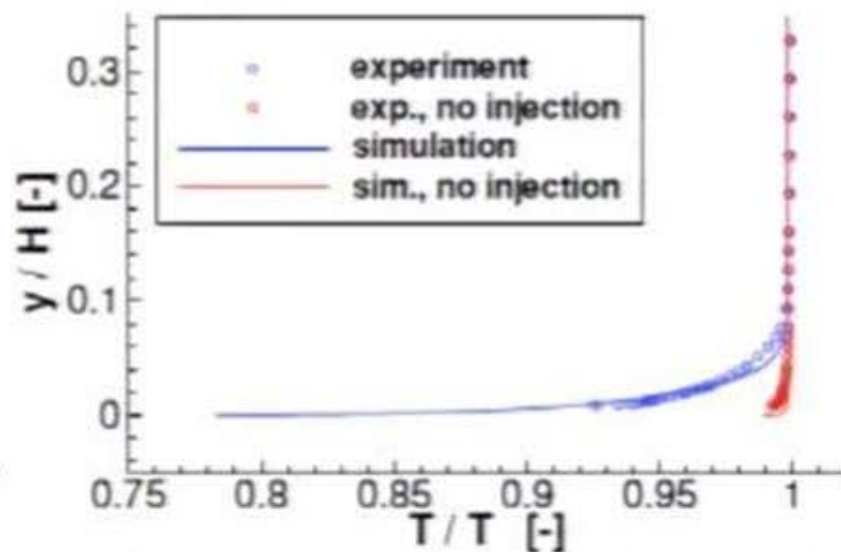
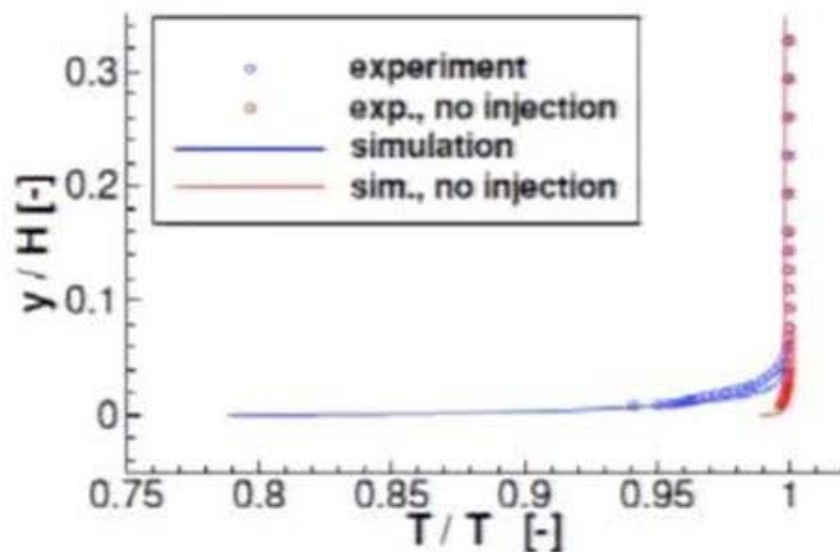
interface: coupling conditions



Some Numerical Results



Some Numerical Results



Some Numerical Results

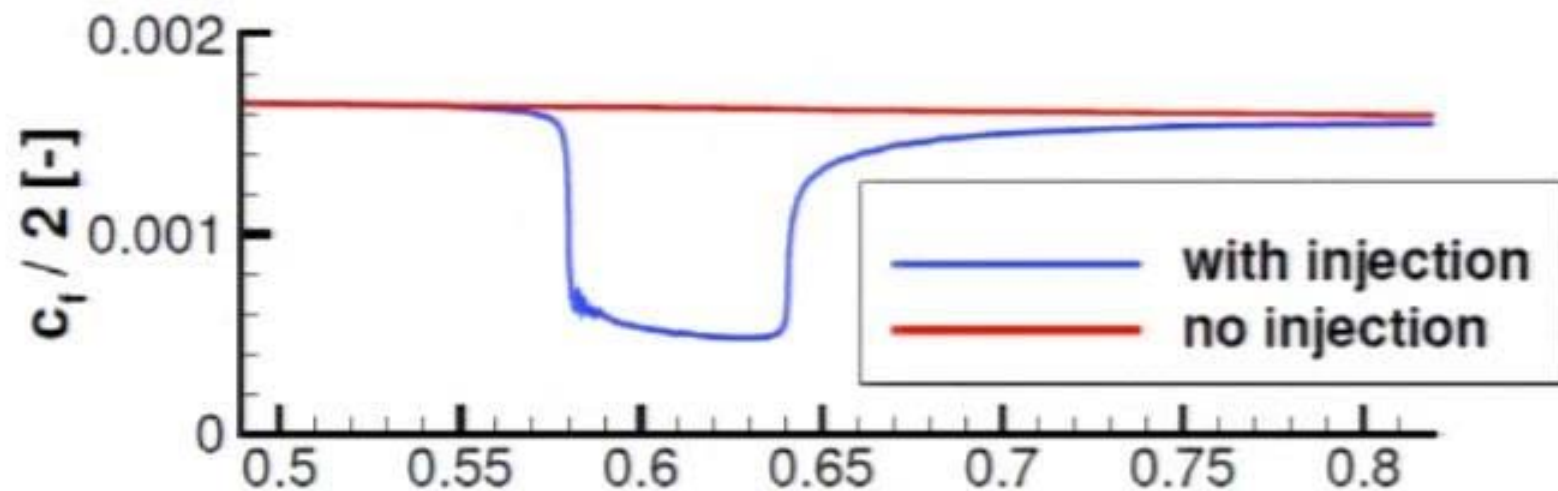


Figure: streamwise variation of skin-friction coefficient

Ongoing...

- parameter (blowing ratio) dependent transport equations in porous medium
- parameter dependent coupling conditions, hot gas flow
- take roughness of porous medium surface into account
- $\rightsquigarrow$ further **model reduction** for porous medium flow

References:

W. Dahmen, T. Gotzen, S. Müller, M. Rom, Numerical simulation of transpiration cooling through porous material, IGPM Preprint, Juli 2013, RWTH Aachen, to appear in International Journal for Numerical Methods in Fluids.

S. Müller, W. Dahmen, M. Rom, S. Schweikert, M. Seizer, J. von Wolfersdorf, Numerical boundary layer investigations of transpiration-cooled turbulent channel flow, to appear in International Journal of Heat and Mass Transfer

What are we up to?...

Scenario: parameter dependent family of transport dominated PDEs:

$$F(u, y) = 0, \quad y \in \mathcal{Y}, \quad \rightsquigarrow u(x, y)$$

- High-dimensionality
sparsity promoting dictionaries
- online parameter optimization: $QI(y) = \ell(u(y)) \rightarrow \text{opt}$
is practically infeasible
frequent query problems, instantaneous solutions
- Data assimilation: state (parameter) estimation based on few measurements
mitigates undersampling - regularization

... calls for Certified Model Reduction ...

Problem: dominating transport

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Abstract Setting

- Parameter dependent family of PDEs

$$\mathcal{B}_y u = f, \quad y \in \mathcal{Y}, \quad \Leftrightarrow \quad b_y(u, v) = f(v), \quad v \in Y', \quad \Rightarrow \quad u = u(y)$$

- Solution manifold:

$$\mathcal{M} = \{u(y) : y \in \mathcal{Y}\} \subset X$$

- Goal: approximate $\mathcal{M}$ uniformly over $\mathcal{Y}$ by possibly small dimensional reduced space X_n , i.e.,

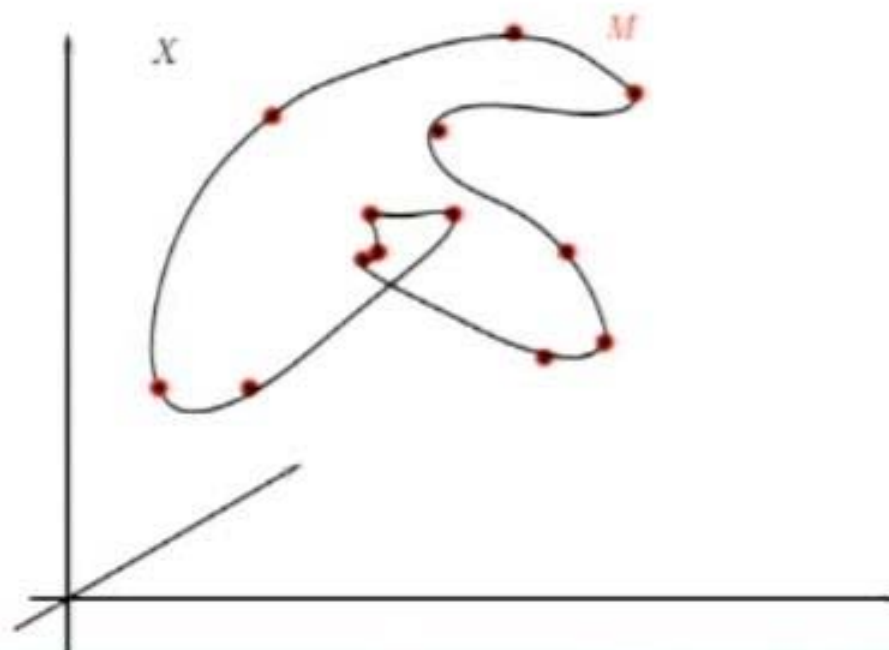
$$\sup_{v \in \mathcal{M}} \inf_{w \in X_n} \|v - w\|_X =: \text{dist}(\mathcal{M}, X_n)_X \stackrel{!}{\leq} \varepsilon, \quad n = n(\varepsilon) \text{ small !}$$

- Benchmark: Kolmogorov n -widths:

$$d_n(\mathcal{M})_X := \inf_{\dim V_n = n} \text{dist}(\mathcal{M}, V_n)_X$$

Philosophy of the Reduced Basis Method (RBM)

- Offline mode: precompute a “reduced basis” consisting of “suitable” solution samples $\rightsquigarrow$ heavy computational cost in the “truth space”
- Online mode: for each parameter query solve only a small reduced system - with certified accuracy



$$\sum_{j=1}^n c_j(y) u(x, y_j) \approx u(x, y)$$

How to get good snapshots?

A Greedy Paradigm... Maday, Patera, Rozza, Prud'homme, Turinici...

- Recall $B_y : X \rightarrow Y' : B_y u(y) = f, \quad \mathcal{M} := \{u(y) := B_y^{-1} f : y \in \mathcal{Y}\}$
- Goal: find $X_n \subset X$ s.t. $\text{dist}(\mathcal{M}, X_n)_X \leq \varepsilon$ for n small
- Suppose that one has a rapidly computable Surrogate:

$$\inf_{v \in X_n} \|u(y) - v\|_X \leq R(y, X_n), \quad y \in \mathcal{Y}, \quad \rightsquigarrow$$



Greedy Algorithm:

- $X_0 := \{0\};$
- for $n = 1, 2, \dots$, given X_{n-1}

$$y_n := \operatorname{argmax}_{y \in \mathcal{P}} R(y, X_{n-1}), \quad X_n := \operatorname{span} \{X_{n-1}, \{u(y_n)\}\}$$

Rate-Optimality

THEOREM: [Binev/Cohen/Dahmen/DeVore/Petrova/Wojtaszczyk]

For $\inf_{w \in X_n} \|u(y) - w\|_X = R(y, X_n)$ one has

$$\text{dist}(\mathcal{M}, X_n)_X \leq \frac{2^{n+1}}{\sqrt{3}} d_n(\mathcal{M})_X \quad n \in \mathbb{N}$$

THEOREM: [Binev/Cohen/Dahmen/DeVore/Petrova/Wojtaszczyk]

If surrogate is **tight**, i.e., $c_S R(y, X_n) \leq \inf_{w \in X_n} \|u(y) - w\|_X \leq R(y, X_n)$

$$d_n(\mathcal{M})_X = \begin{cases} O(n^{-\alpha}), & n \in \mathbb{N}, \\ O(e^{-cn^\alpha}), & n \in \mathbb{N}, \end{cases} \Rightarrow \text{dist}(\mathcal{M}, X_n)_X = \begin{cases} O(n^{-\alpha}), & n \in \mathbb{N}, \\ O(e^{-\tilde{c}n^\alpha}), & n \in \mathbb{N}, \end{cases}$$

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What are good surrogates? ...how to best see the manifold?

- One should have

$$\text{surrogate} \approx \text{best approximation error}$$

- Surrogate should be based on **residuals**: $f - \mathcal{B}_\mu u_n^*(\mu)$
- One should have for $\mathcal{B}_\mu : X \rightarrow Y'$

$$\begin{aligned} \|f - \mathcal{B}_\mu u_n^*(\mu)\|_{Y'} &= \|\mathcal{B}_\mu(u(\mu) - u_n^*(\mu))\|_{Y'} \stackrel{!}{\sim} \|u(\mu) - u_n^*(\mu)\|_X \\ &\stackrel{!}{\sim} \|u(\mu) - P_{X_n} u(\mu)\|_X \end{aligned}$$

- Works well for elliptic problems with Galerkin projection:
 $X = Y = H_0^1(\Omega)$, $Y' = H^{-1}(\Omega)$
- For **transport dominated**, indefinite, singularly perturbed problems
need $X \neq Y$

RBMs for Transport Dominated Problems [Dahmen/Plesken/Welper]

Problem: “elliptic surrogates” don’t work

Remedy: instead of $\|f - B_Y \Pi_{X_n}^Y u(y)\|_{X'}$ use

- $R(y, X_n) := \|f - B_Y \Pi_{X_n}^Y u(y)\|_{Y'}$, where Y is chosen properly
- stable variational formulations $b_Y(\cdot, \cdot) : X \times Y \rightarrow \mathbb{R}$

[D./Huang/Schwab/Welper, Cohen/D./Welper, Barret/Morton,

Cai/Manteuffl..., Demkowicz/Jopalakrishnan... Rozza/Veroy, Patera/Urban (SP problems)...

- grow pairs of spaces $X_n, Y_n, n = 1, 2, 3, \dots$ such that corresponding Petrov-Galerkin projections realize near best approximation in X
- construction of the Y_n through inner stabilizing greedy loop based on equivalent saddle-point formulation

THEOREM: [Dahmen/Plesken/Welper]

The “double greedy” RBM is rate optimal.

An Example

(Pure) Transport Equation: Dahmen/Huang/Schwab/Welper

$$\Gamma_-(\mu) := \{x \in \partial\Omega : n(x) \cdot \mu < 0\}, \quad \mu \in S^{d-1}$$

$$\mu \cdot \nabla u + cu = f^\circ, \quad \text{in } \Omega = (0, 1)^d, \quad u = u_b, \quad \text{on } \Gamma_-(\mu),$$

$$b_\mu(u, v) := \langle u, \underbrace{\mu \cdot \nabla v + cv}_{= \mathcal{B}_\mu^* v} \rangle = \langle f^\circ, v \rangle - \int_{\Gamma_-(\mu)} n \cdot \mu u_b v =: \langle f, v \rangle$$

$$\|v\|_{Y_\mu} := \|\mathcal{B}_\mu^* v\|_{L_2}, \quad \|u\|_{X_\mu} := \|u\|_{L_2}$$

Transport Equation

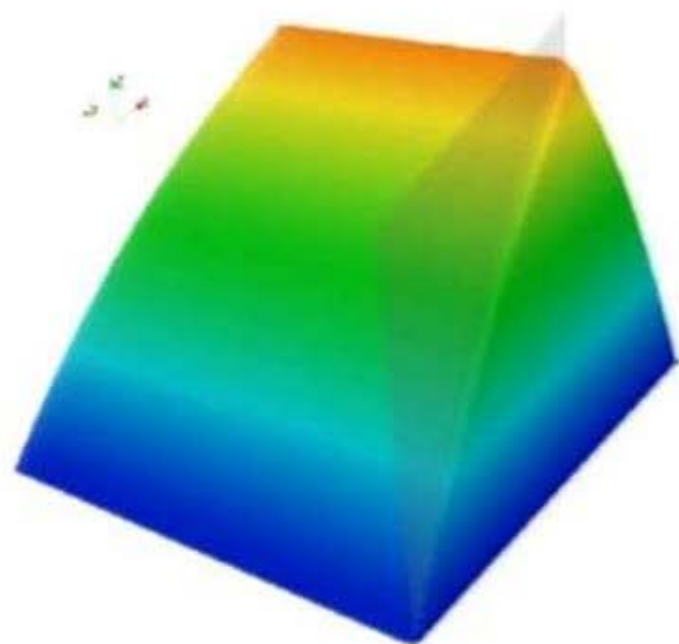


Figure: (1): reduced basis of dimension $n = 24$, $m(n) = 91$, angle $\mu = 0.244579$

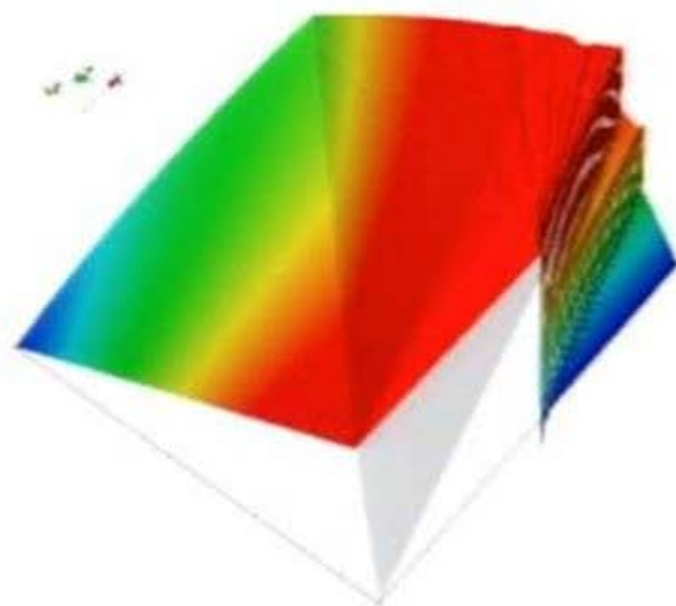
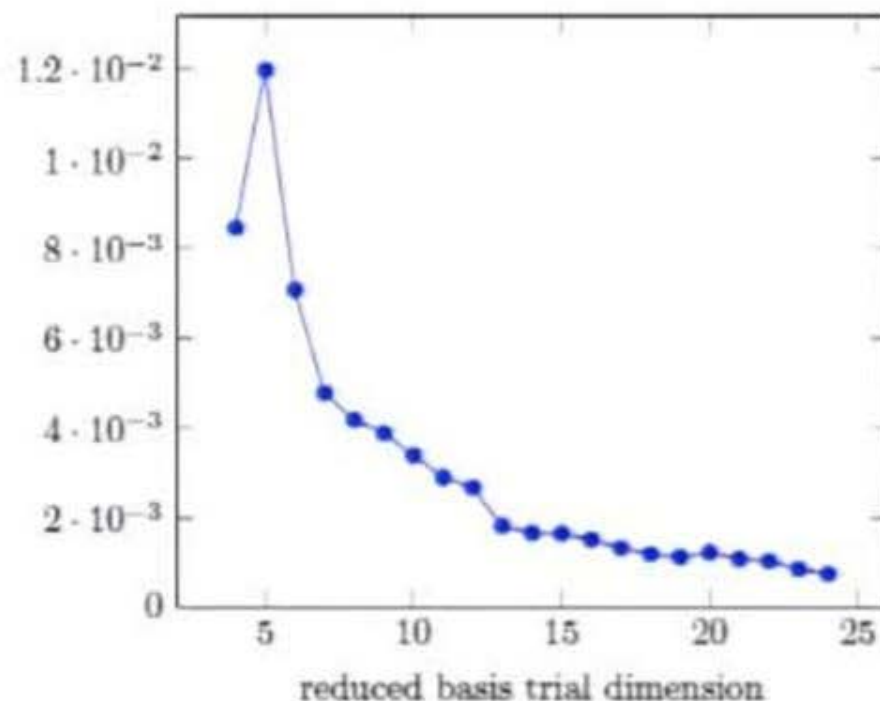


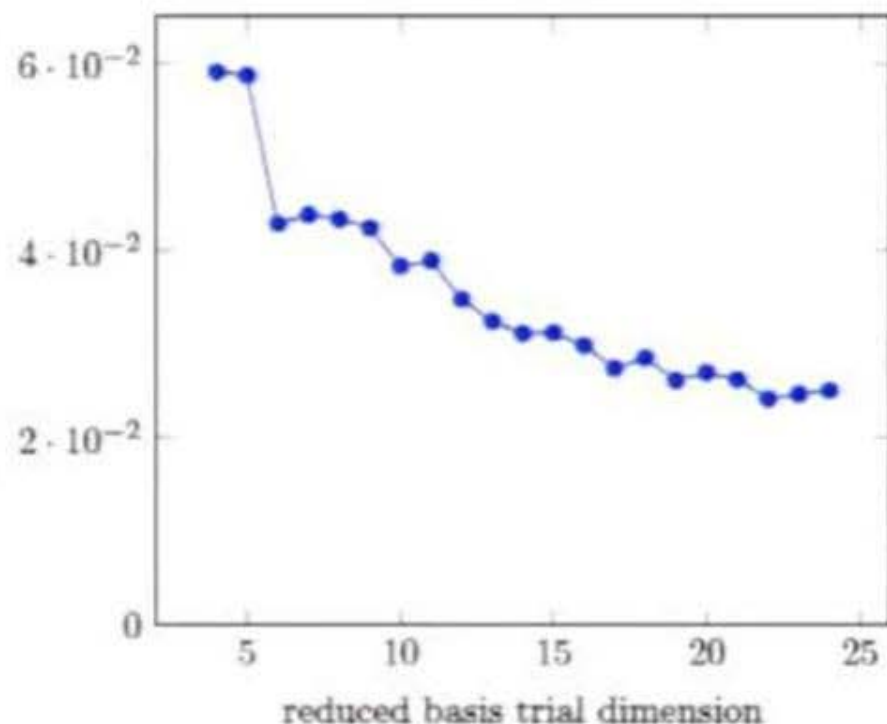
Figure: (2): reduced basis of dimension $n = 24$, $m(n) = 96$, angle $\mu = 0.256311$

Transport Equation: Convergence

zero boundary



jump boundary



(Pure) Transport Equation (2): truth L_2 -error 0.0154

trial	test	δ	surr	rb truth	rb L2	surr/err
4	14	4.97e-01	5.91e-02	1.29e-01	1.30e-01	4.54e-01
8	31	4.29e-01	4.34e-02	7.78e-02	7.95e-02	5.46e-01
12	49	3.71e-01	3.48e-02	7.40e-02	7.53e-02	4.63e-01
16	64	3.74e-01	2.99e-02	6.20e-02	6.41e-02	4.67e-01
20	81	3.73e-01	2.71e-02	5.46e-02	5.62e-02	4.82e-01
24	96	3.91e-01	2.51e-02	4.51e-02	4.79e-02	5.25e-01

Convection-Diffusion Equations

$$-\epsilon \Delta u + \begin{pmatrix} \cos \mu \\ \sin \mu \end{pmatrix} \cdot \nabla u + u = 1, \text{ in } \Omega = (0, 1)^2, \quad u = 0, \text{ on } \partial\Omega$$

Choice of the test norm:

$$s_\mu(u, v) := \frac{1}{2} (\langle B_\mu u, v \rangle + \langle B_\mu v, u \rangle),$$

$$\|v\|_{Y_\mu}^2 := s_\mu(v, v) = \epsilon |v|_{H^1(\Omega)}^2 + \left\| \left(c - \frac{1}{2} \operatorname{div} b(\mu) \right)^{1/2} v \right\|_{L_2(\Omega)}^2$$

+ boundary penalization at outflow boundary: (see [Cohen/Dahmen/Welper, M2AN 2012](#))

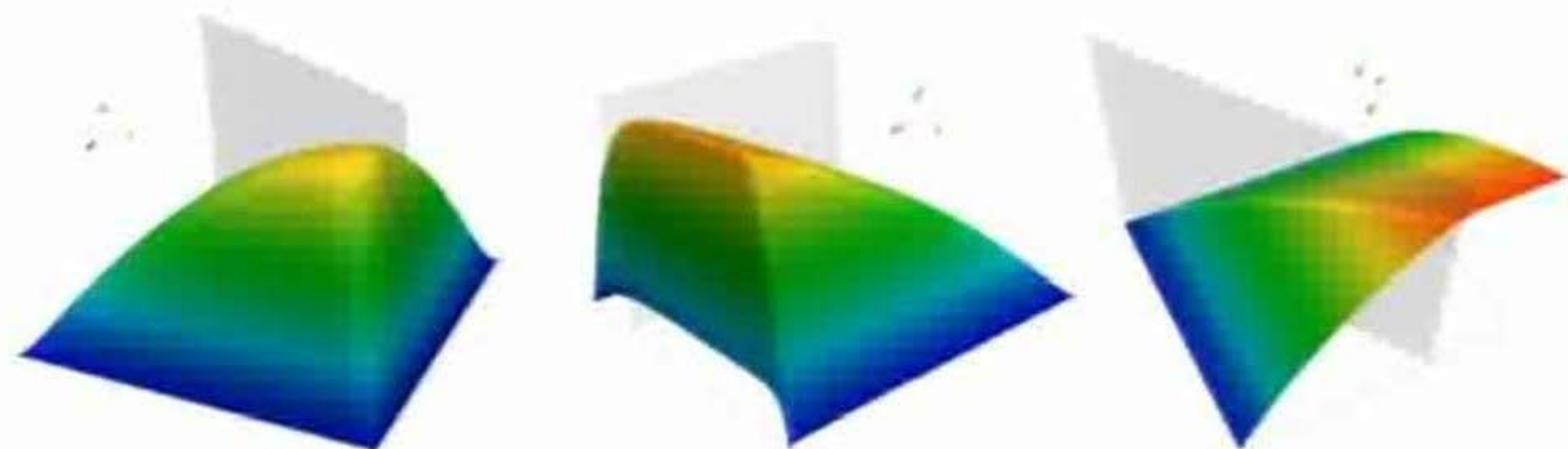
$$\|u\|_{\bar{X}_\mu}^2 := \|\bar{B}_\mu u\|_{\bar{Y}_\mu}^2 = \|\bar{B}_\mu u\|_{Y_\mu}^2 + \lambda \|u\|_{H_b(\mu)}^2$$

$$H_b(\mu) = H_{00}^{1/2}(\Gamma_+(\mu)), \quad \bar{Y}_\mu := Y_\mu \times H_b(\mu)'$$

$$u(\mu) = \operatorname{argmin}_{w \in X_-} \{ \|\bar{B}_\mu w - f\|_{\bar{Y}_\mu}^2 + \lambda \|\gamma w\|_{H_b(\mu)}^2 \}$$

Convection-Diffusion Equations

Robustness with respect to small diffusion

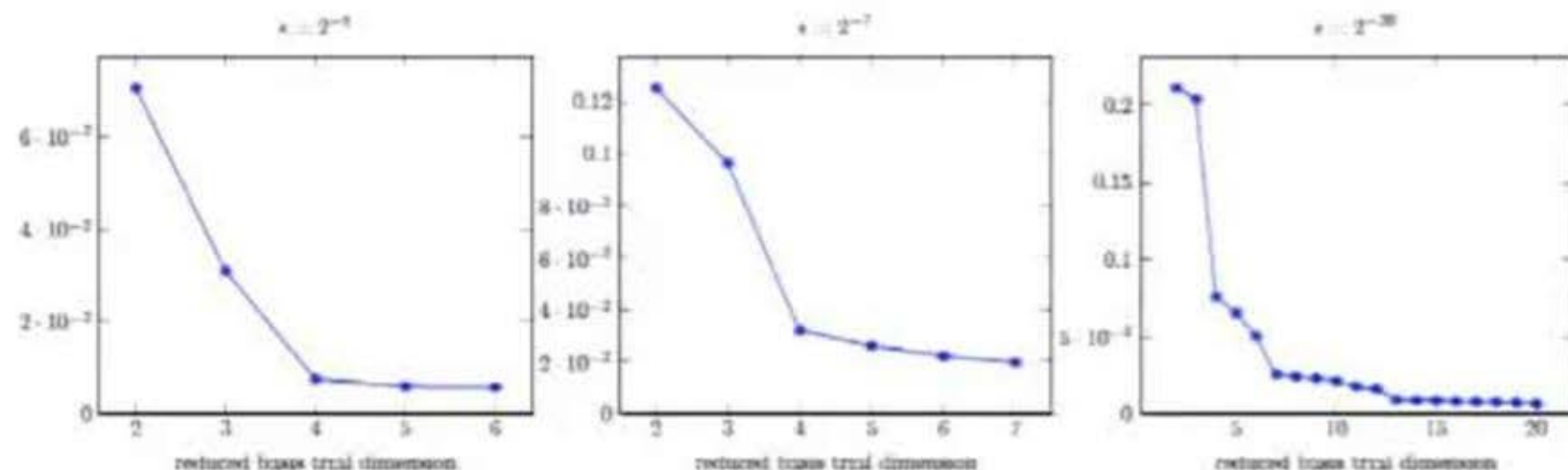


(a) $\epsilon = 2^{-5}$ RB dim $n = 6$, $m(n) = 13$, angle $\mu = 0.885115$,

(b) $\epsilon = 2^{-7}$ RB dim $n = 7$, $m(n) = 20$, angle $\mu = 0.257484$,

(c) $\epsilon = 2^{-26}$ RB dim $n = 20$, $m(n) = 57$, angle $\mu = 0.587137$

Convection-Diffusion Equations



Convection-Diffusion Equations: $\epsilon = 2^{-7}$

a-post. error 0.0197

dim trial	dim test	δ	max surr	surr/a-post
2	5	8.92e-03	1.25e-01	6.37e+00
3	8	1.22e-01	9.65e-02	4.90e+00
4	11	1.13e-02	3.21e-02	1.63e+00
5	14	1.27e-02	2.61e-02	1.32e+00
6	17	5.55e-03	2.21e-02	1.12e+00
7	20	4.82e-03	1.97e-02	1.00e+00

Convection-Diffusion Equations: $\epsilon = 2^{-26}$

a-post. error 0.0011

trial	test	δ	surrogate	a-post	trial	test	δ	surrogate	surri/a-post
2	5	1.35e-03	2.11e-01	2.00e+02	12	33	3.47e-04	1.60e-02	1.52e+01
4	9	1.09e-02	7.58e-02	7.19e+01	14	39	1.10e-04	8.46e-03	8.02e+00
6	15	1.61e-03	5.02e-02	4.76e+01	16	45	9.39e-05	7.87e-03	7.46e+00
8	21	7.99e-04	2.39e-02	2.26e+01	18	51	6.11e-05	7.69e-03	7.29e+00
10	27	3.55e-04	2.10e-02	2.00e+01	20	57	5.28e-05	6.35e-03	6.02e+00

Reference:

W. Dahmen, C. Plesken, G. Welper, Double Greedy Algorithms: Reduced Basis Methods for Transport Dominated Problems, ESAIM: Mathematical Modelling and Numerical Analysis, 48(3) (2014), 623–663. DOI 10.1051/m2an/2013103, <http://arxiv.org/abs/1302.5072>

Outlook...

- reduced modeling and data assimilation
- inversion: assisting rapid conditioned sampling
- construction of **low-rank** approximation to kinetic models

$$b(y) \cdot \nabla u(y) + cu(y) = f + \int_{\mathcal{P}} K(y, y') u(y') d\mu' \quad \text{in } \Omega, \quad u|_{\Gamma_-} = 0$$

view the function $u(x, y)$ as collection of fibers

$$\mathcal{M} = \{u(\cdot, y) =: y \in \mathcal{Y}\}$$