Efficient Solution of Coupled Flow and Porous Media Problems by Monolithic Multigrid Methods

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Outline

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- Coupled Darcy-Stokes problem
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  - Staggered grids
  - Discretization of the interface
- Numerical Method
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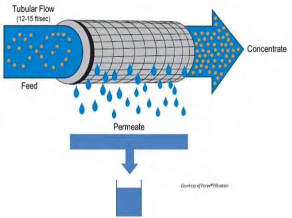
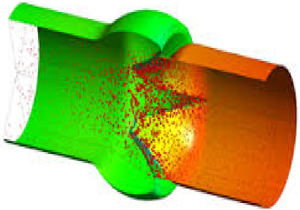
  - Uzawa smoother
  - Local Fourier Analysis (LFA)
  - Multiblock multigrid algorithm
- Numerical Experiments
- Coupled Stokes Flow and Deformable Porous Medium System.
- Conclusions

Application



(a) filtration process



(b) blood flow simulation

(c) flooding simulation

(d) waste water treatment

Introduction

COUPLED PROBLEM:

Free flow $\overset{interface}{\Longleftrightarrow}$ Flow in the porous medium

DIFFERENT APPROACHES to solve the coupled problem:

- Domain Decomposition Methods:
  Decoupling the global problem so that mainly independent subproblems are to be solved.
- Monolithic Methods:
  Simultaneous solution of the coupled multi-physics system. Preconditioners and Multigrid methods.

Basic equations

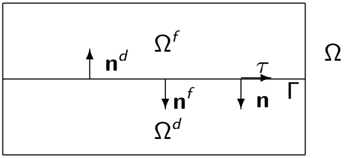


Figure: Geometry of the coupled Darcy/Stokes problem.

Porous medium description

$$\mathbb{K}^{-1}\mathbf{u}^d + \nabla p^d = \mathbf{0} \quad \text{in } \Omega^d \,,$$
$$\nabla \cdot \mathbf{u}^d = f^d \quad \text{in } \Omega^d \,.$$

- $\mathbf{u}^d = (u^d, v^d)$ and p^d.
- The hydraulic conductivity tensor $\mathbb{K} = K\mathbb{I}$, $K > 0$.

Free flow description:

$$-\nabla \cdot \boldsymbol{\sigma}^f = \mathbf{f}^f \quad \text{in } \Omega^f \,,$$
$$\nabla \cdot \mathbf{u}^f = 0 \quad \text{in } \Omega^f \,.$$

- $\mathbf{u}^f = (u^f, v^f)$ and p^f.
- $\boldsymbol{\sigma}^f = -p^f \mathbf{I} + 2\nu \mathbf{D}(\mathbf{u}^f)$, $\mathbf{D}(\mathbf{u}^f) = (\nabla \mathbf{u}^f + (\nabla \mathbf{u}^f)^T)/2$.

Interface conditions

We fix the normal vector to the interface to be $\mathbf{n} = \mathbf{n}^f = -\mathbf{n}^d$ and we denote $\boldsymbol{\tau}$ as the tangential unit vector at the interface Γ.

- Mass conservation:
$$\mathbf{u}^f \cdot \mathbf{n} = \mathbf{u}^d \cdot \mathbf{n} \quad \text{on } \Gamma \ .$$
- Balance of normal stresses:
$$-\mathbf{n} \cdot \boldsymbol{\sigma}^f \cdot \mathbf{n} = p^d \quad \text{on } \Gamma.$$
- Beavers-Joseph-Saffman condition: (α is a parameter)
$$\alpha \mathbf{u}^f \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \boldsymbol{\sigma}^f \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \ ,$$

  No-slip condition:
$$\mathbf{u}^f \cdot \boldsymbol{\tau} = 0 \quad \text{on } \Gamma \ .$$

Staggered grids

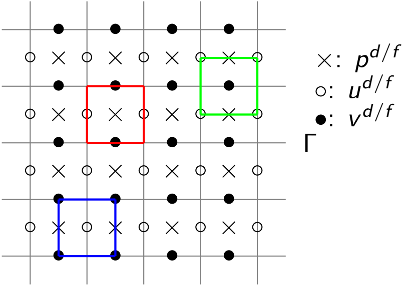


Figure: Staggered grid location of unknowns for the coupled model, and corresponding control volumes.

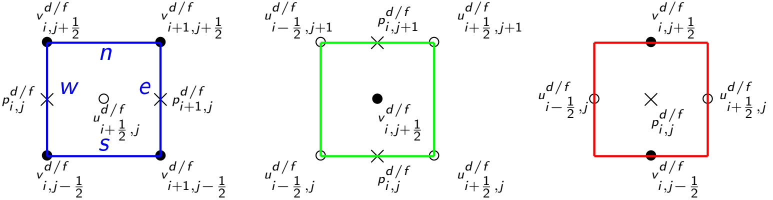


Figure: Control volumes for $u^{d/f}$ (left), $v^{d/f}$ (middle), $p^{d/f}$ (right).

Discretization at the interface

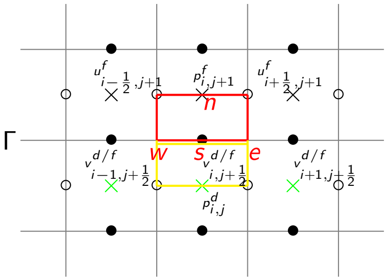


Figure: Staggered grid location of the unknowns for the interface conditions.

$$-\frac{(\sigma_{xy})_e-(\sigma_{xy})_w}{h}-\frac{(\sigma_{yy})_n-(\sigma_{yy})_s}{h/2}=(f_2^f)_{i,j+\frac{1}{2}}$$

- $(\sigma_{yy})_n$
- $(\sigma_{yy})_s=-p_s^d$
- $(\sigma_{xy})_e$ and $(\sigma_{xy})_w$

Beavers-Joseph-Saffman condition:

$$\alpha u_e^f-\nu\left(\frac{u^f_{i+\frac{1}{2},j+1}-u^f_e}{h/2}+\frac{v^f_{i+1,j+\frac{1}{2}}-v^f_{i,j+\frac{1}{2}}}{h}\right)=0$$

- Peiyao Luo, Carmen Rodrigo, Francisco J. Gaspar, Cornelis W. Oosterlee, *Uzawa smoother in multigrid for the coupled Porous Medium and Stokes Flow System*, *SIAM Journal on Scientific Computing*, 2017.

Saddle point system

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ f \end{pmatrix}$$

- B^T: discrete gradient. B: minus discrete divergence.
- A: discrete $-\nu\Delta$ for the Stokes equation.
  discrete $K^{-1}I$ for the Darcy equation.

Coupled system

$$\begin{pmatrix} A^d & 0 & (B^d)^T & 0 \\ 0 & A^f & R & (B^f)^T \\ B^d & R & 0 & 0 \\ 0 & B^f & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}^d \\ \mathbf{u}^f \\ p^d \\ p^f \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{f}^f \\ f^d \\ 0 \end{pmatrix},$$

$$A = \begin{pmatrix} A^d & 0 \\ 0 & A^f \end{pmatrix}, \quad B = \begin{pmatrix} B^d & R \\ 0 & B^f \end{pmatrix}, \quad B^T = \begin{pmatrix} (B^d)^T & 0 \\ R & (B^f)^T \end{pmatrix},$$

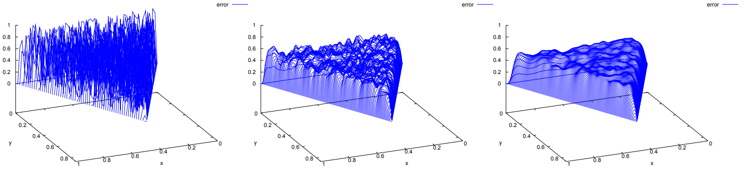
where R contains the relations given by the interface discretization

Introduction to Multigrid

Multigrid methods are among the fastest iterative methods for solving PDEs

Two principles:

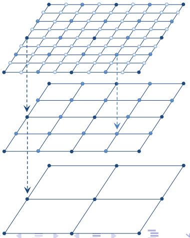
- Smoothing property:



- Coarse-grid correction principle

Multigrid components

- Choice of coarse grids and operators
- Inter-grid transfer operators
- Type of cycle
- Smoother
- Number of iterations of pre- and post-smoothing



Uzawa smoother

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} = \begin{pmatrix} M_A & \\ B & -\omega^{-1} I \end{pmatrix} - \begin{pmatrix} M_A - A & -B^T \\ & -\omega^{-1} I \end{pmatrix},$$

- ω: some positive parameter.
- M_A: Symmetric Gauss-Seidel for velocities

$$M_A = (D_A + L_A) D_A^{-1} (D_A + U_A)$$

The decoupled iteration can be described as:

$$\begin{pmatrix} M_A & \\ B & -\omega^{-1} I \end{pmatrix} \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} M_A - A & -B^T \\ & -\omega^{-1} I \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} + \begin{pmatrix} \mathbf{g} \\ f \end{pmatrix}$$

- apply smoother M_A to relax the system $A\mathbf{u} = \mathbf{g} - B^T p$;
  i.e., $\hat{\mathbf{u}} = \mathbf{u} + M_A^{-1}(\mathbf{g} - A\mathbf{u} - B^T p)$;
- update the pressure: $\hat{p} = p + \omega(B\hat{\mathbf{u}} - f)$.

- Optimal Parameter ω_{opt}?

Comparison between LFA and asymptotic results

- Darcy: $\omega_{opt} = \dfrac{2}{\dfrac{8K}{h^2} + \dfrac{2K}{h^2}} = \dfrac{h^2}{5K}$
- Stokes: $\omega_{opt} = \dfrac{2}{\dfrac{1}{\nu} + \dfrac{1}{\nu}} = \nu$

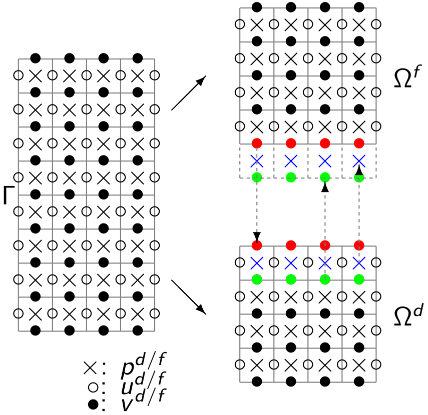
	Darcy		Stokes	
$\nu_1 + \nu_2$	$K = 1$	$K = 10^{-6}$	$\nu = 1$	$\nu = 10^{-6}$
2	0.600	0.600	0.304	0.304
3	0.360	0.360	0.143	0.143
4	0.216	0.216	0.081	0.081

Table: Two-grid convergence factors, ρ predicted by LFA.

	K	1		10^{-6}	
	ν	1	10^{-6}	1	10^{-6}
$\nu_1 + \nu_2$	2	0.59	0.59	0.59	0.59
	3	0.36	0.36	0.36	0.36
	4	0.21	0.21	0.21	0.21

Table: Asymptotic convergence factors, ρ_h, for the coupled problem.

Multiblock multigrid algorithm



Multiblock two-grid algorithm: (with only pre-smoothing)

1. Relax velocity unknowns.
2. *Stokes to Darcy*: $v^f \to v^d$ ($\bullet$).
3. Update pressure unknowns.
4. *Darcy to Stokes*: $p^d \to p^f$ ($\times$).
5. Compute the residual.
6. *Darcy to Stokes*: $r^d \to r^f$ ($\bullet$).
7. Restrict the residual.
8. Solve exactly the defect equation on the coarsest grid.
9. *Stokes to Darcy*: $e^f \to e^d$.
10. Interpolation and correction.

Beavers-Joseph-Saffman interface condition

Analytical solution

$$\mathbf{u}^d(x,y) = \begin{pmatrix} u^d(x,y) \\ v^d(x,y) \end{pmatrix} = \begin{pmatrix} -Ke^y \cos x \\ -Ke^y \sin x \end{pmatrix},$$

$$p^d(x,y) = e^y \sin x,$$

$$\mathbf{u}^f(x,y) = \begin{pmatrix} u^f(x,y) \\ v^f(x,y) \end{pmatrix} = \begin{pmatrix} \lambda'(y) \cos x \\ \lambda(y) \sin x \end{pmatrix},$$

$$p^f(x,y) = 0,$$

where $\lambda(y) = -K - \frac{g\,y}{2\nu} + (-\frac{\alpha}{4\nu^2} + \frac{K}{2})y^2$.

- $\Omega = (0,1) \times (-1,1)$, $\Omega^d = (0,1) \times (-1,0)$, $\Omega^f = (0,1) \times (0,1)$.
- Interface $\Gamma = (0,1) \times \{0\}$.
- Free flow: Dirichlet conditions for u^f and v^f at the outer boundaries.
- Porous medium: fixed p^d at the bottom, Dirichlet conditions for u^d and v^d at the lateral walls.

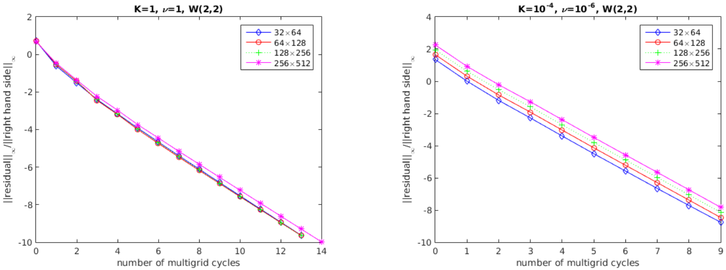
Beavers-Joseph-Saffman interface condition

	64×128	128×256	256×512
u^d	1.42×10^{-5}	3.63×10^{-6}	9.19×10^{-7}
v^d	4.09×10^{-5}	1.19×10^{-5}	3.38×10^{-6}
p^d	9.11×10^{-6}	2.32×10^{-6}	5.84×10^{-7}
u^f	1.21×10^{-5}	3.06×10^{-6}	7.71×10^{-7}
v^f	2.97×10^{-5}	7.66×10^{-6}	1.95×10^{-6}
p^f	4.74×10^{-3}	2.38×10^{-3}	1.19×10^{-3}

Table: Maximum norm errors of variables $u^{d/f}$, $v^{d/f}$, $p^{d/f}$ for different grid-sizes, by considering fixed values $\nu = 1$ and $K = 1$, and prescribing the Beavers-Joseph-Saffman condition at the interface with $\alpha = 1$.

Beavers-Joseph-Saffman interface condition

$K = 1,\ \nu = 1$ $\qquad$ $K = 10^{-4},\ \nu = 10^{-6}$



(a) (b)

Figure: History of the convergence of the $W(2,2)-$multigrid method for different values of the physical parameters.

Realistic problem: cross-flow membrane filtration model

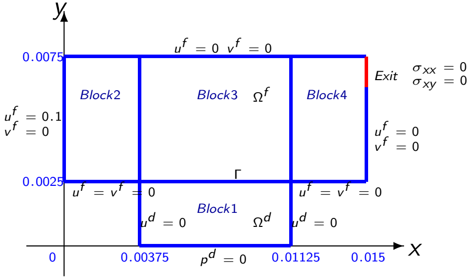
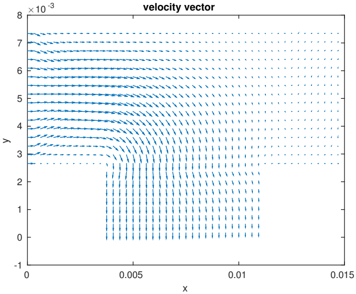


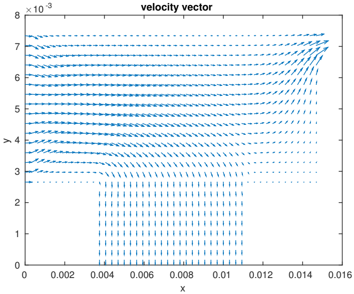
Figure: Geometry of the coupled problem.

- 4 blocks, $K = 0.1$ or $K = 10^{-6}$, $\nu = 10^{-6}$.
- Beavers-Joseph-Saffman interface condition.
- Communications on each level.
- Excellent multigrid convergence factor 0.2 for $W(2,2)$-cycle for the coupled system.

Realistic problem: cross-flow membrane filtration model



(a) $K = 0.1$



(b) $K = 10^{-6}$

Figure: Velocity vectors over the cross-flow filtration domain with different values of permeability.

Heterogeneity test

To simulate heterogeneity in the porous medium, a Gaussian model characterized by parameters λ_g and σ_g^2 is considered, i.e.,

$$C(d_g) = \sigma_g^2 \exp\left(-\frac{d_g^2}{\lambda_g}\right) ,$$

where d_g is the distance between two points, λ_g defines the correlation length and σ_g^2 represents the variance.

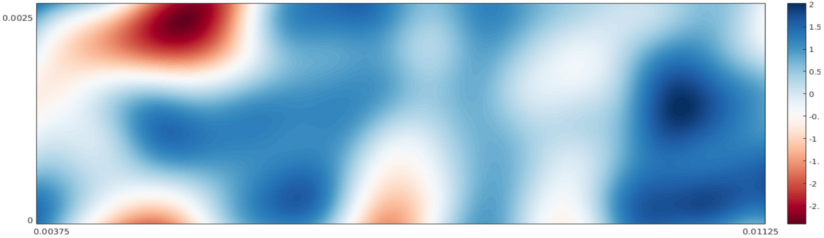


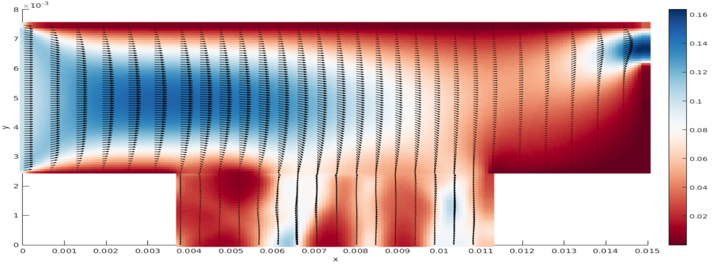
Figure: Example of random field of hydraulic conductivity K in log-scale, with parameters $\lambda_g = 0.3$ and $\sigma_g^2 = 1$.

Heterogeneity test

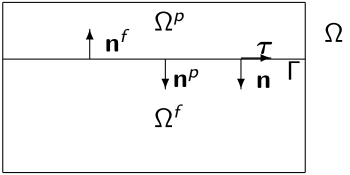
- Two different values for parameter λ_g: $\lambda_g = 0.1$ denotes a more heterogeneous porous medium than $\lambda_g = 0.3$.
- 50 realizations of the random field are generated and we record the multigrid convergence factors of the $W(2,2)$-cycle.

h^{-1}	$\lambda_g = 0.3$	$\lambda_g = 0.1$
25600	0.19	0.20
12800	0.19	0.21
6400	0.20	0.29

Table: Mean value of the multigrid convergence factors after 50 realizations.



Coupled Stokes and Deformable Porous Medium System



Deformable Porous Media

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{\sigma}^p &= \mathbf{f}^p \quad \text{in } \Omega^p \\
\frac{\partial}{\partial t}(\nabla \cdot \mathbf{u}^p) + \nabla \cdot \mathbf{q}^p &= f^p \quad \text{in } \Omega^p \\
\mathbf{q}^p &= -K \nabla p^p \quad \text{in } \Omega^p
\end{aligned}
$$

- $\mathbf{u}^p = (u^p, v^p)$ and p^p
- $\boldsymbol{\sigma}^p = \boldsymbol{\sigma}^E - p^p \mathbf{I}$
- $\boldsymbol{\sigma}^E(\mathbf{u}^p) = 2\mu \mathbf{D}(\mathbf{u}^p) + \lambda tr(\mathbf{D}(\mathbf{u}^p))\mathbf{I}$

Stokes Flow

$$
\begin{aligned}
\rho \frac{\partial \mathbf{u}^f}{\partial t} - \nabla \cdot \boldsymbol{\sigma}^f &= \mathbf{f}^f \quad \text{in } \Omega^f \\
\nabla \cdot \mathbf{u}^f &= 0 \quad \text{in } \Omega^f
\end{aligned}
$$

- $\mathbf{u}^f = (u^f, v^f)$ and p^f
- $\boldsymbol{\sigma}^f = -p^f \mathbf{I} + 2\nu \mathbf{D}(\mathbf{u}^f)$
- $\mathbf{D}(\mathbf{u}^f) = (\nabla \mathbf{u}^f + (\nabla \mathbf{u}^f)^T)/2$

Interface conditions

- Mass conservation:

$$(\mathbf{u}^f - \frac{\partial \mathbf{u}^p}{\partial t}) \cdot \mathbf{n} = \mathbf{q}^p \cdot \mathbf{n} \ ,$$

- Balance of normal stresses in the fluid phase:

$$\mathbf{n} \cdot \boldsymbol{\sigma}^f \mathbf{n} = -p^p$$

- Conservation of momentum:

$$\mathbf{n} \cdot \boldsymbol{\sigma}^f \mathbf{n} - \mathbf{n} \cdot \boldsymbol{\sigma}^p \mathbf{n} = 0$$

  and

$$\boldsymbol{\tau} \cdot \boldsymbol{\sigma}^f \mathbf{n} - \boldsymbol{\tau} \cdot \boldsymbol{\sigma}^p \mathbf{n} = 0$$

- Beavers-Joseph-Saffman interface condition:

$$-\boldsymbol{\tau} \cdot \boldsymbol{\sigma}^f \mathbf{n} = \beta(\mathbf{u}^f - \frac{\partial \mathbf{u}^p}{\partial t}) \cdot \boldsymbol{\tau}$$

- No-slip condition:

$$\mathbf{u}^f \cdot \boldsymbol{\tau} = \frac{\partial \mathbf{u}^p}{\partial t} \cdot \boldsymbol{\tau}$$

Saddle point structure

At each time step: $\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ f \end{pmatrix}$

- B^T and $B \equiv$ discrete gradient and the negative discrete divergence
- For the poroelastic system:
  - A is $-\mu\Delta - \nabla(\lambda+\mu)\nabla\cdot$ and C corresponds to $-\tau\nabla\cdot(K\nabla p)$
- For the Stokes system:
  - A represents $\frac{\rho}{\tau}I - \nu\Delta$ and C is a zero block

$$\begin{pmatrix} A^f & R^T & (B^f)^T & (R')^T \\ R & A^p & 0 & (B^p)^T \\ B^f & 0 & 0 & 0 \\ R' & B^p & 0 & -C^p \end{pmatrix} \begin{pmatrix} \mathbf{u}^f \\ \mathbf{u}^p \\ p^f \\ p^p \end{pmatrix} = \begin{pmatrix} \mathbf{f}^f \\ \mathbf{f}^p \\ 0 \\ f^p \end{pmatrix}$$

$$A = \begin{pmatrix} A^f & R^T \\ R & A^p \end{pmatrix}, B = \begin{pmatrix} B^f & 0 \\ R' & B^p \end{pmatrix}, -C = \begin{pmatrix} 0 & 0 \\ 0 & -C^p \end{pmatrix},$$

wheere R and R' contain the coupling at and near the interface.

Monolithic multigrid

- Uzawa smoother
- Optimal relaxation parameter
  - Poroelasticity system:

$$\omega^p = \frac{h^2(\lambda + 2\mu)}{5K\tau(\lambda + 2\mu) + h^2}$$

  - Stokes system:

$$\omega^f = \nu + \frac{\rho h^2}{8\tau}$$

Relaxation parameters do not only depend on the model coefficients but also on the grid size and on time step τ, thus ω^p and ω^f are different on each grid of the hierarchy in the multigrid method

Analytical test. No-slip condition

Analytical solution

$$u^f = u^p = (y^2 - y)e^t$$
$$v^f = v^p = 0$$
$$p^f = p^p = xe^t$$

- $\Omega = (0,1) \times (0,2)$, $\Omega^f = (0,1) \times (0,1)$, $\Omega^p = (0,1) \times (1,2)$
- Interface $\Gamma = (0,1) \times \{1\}$
- Dirichlet boundary conditions for displacements and pressure at the lateral boundaries of Ω^p.
- Stress conditions at the top of Ω^p, where the fluid pressure is fixed
- In Ω^f, stress conditions at both inlet and outlet, while a symmetric boundary condition is imposed at the bottom.
- Interface conditions with the simplified no-slip interface condition

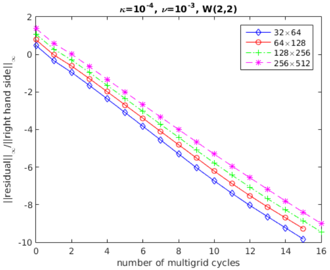
Analytical test. No-slip condition

	$64 \times 128 \times 4$	$128 \times 256 \times 8$	$256 \times 512 \times 16$
u^f	2.01×10^{-4}	9.73×10^{-5}	4.76×10^{-5}
v^f	1.20×10^{-4}	4.47×10^{-5}	2.31×10^{-5}
p^f	3.16×10^{-3}	1.63×10^{-3}	7.95×10^{-4}
u^p	6.77×10^{-3}	3.46×10^{-3}	1.75×10^{-3}
v^p	6.38×10^{-4}	3.26×10^{-4}	1.65×10^{-4}
p^p	3.87×10^{-3}	1.68×10^{-3}	7.75×10^{-4}

Table: Maximum norm errors of variables $u^{f/p}$, $v^{f/p}$ and $p^{f/p}$ for different grid sizes with parameters $K = 1$, $\lambda = 1$, $\mu = 1$, $\nu = 1$ and $\rho = 1$.

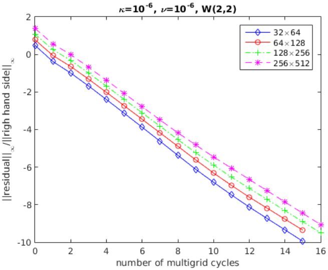
Analytical test. No-slip condition

$K = 10^{-4}$, $\nu = 10^{-3}$



(a)

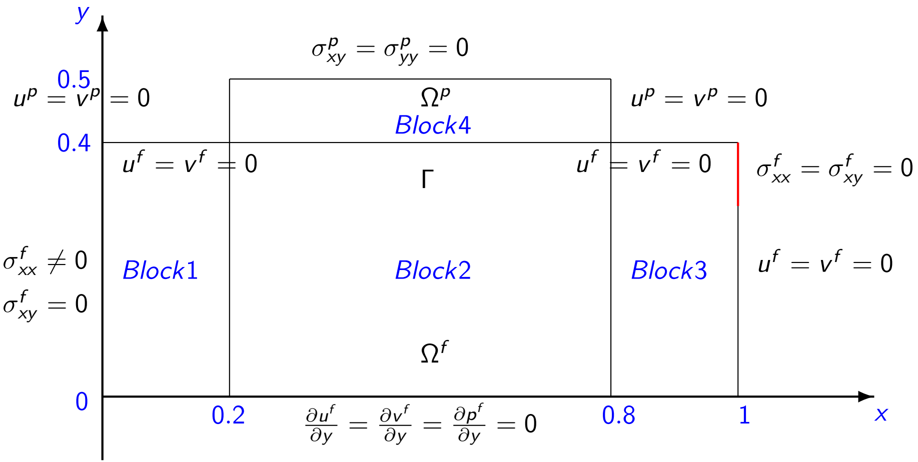
$K = 10^{-6}$, $\nu = 10^{-6}$



(b)

Figure: History of the convergence of the $W(2,2)-$multigrid method for different values of the physical parameters

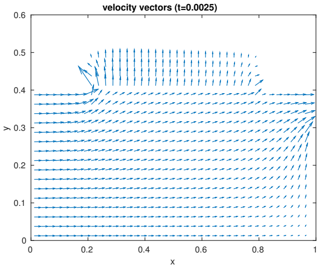
Multi-block realistic test

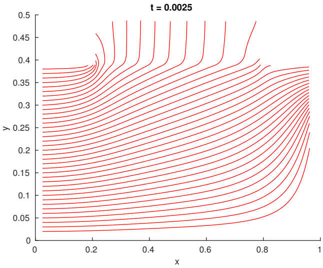


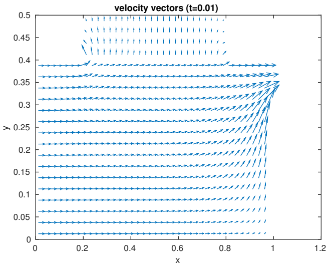
- Fluid inflow in Ω^f: $\sigma^f_{xx} = -20000$
- Small exit at the right vertical boundary (stress-free boundary)
- $K = 10^{-4}$, $\lambda = 10^6$, $\mu = 2.5 \times 10^5$, $\nu = 0.0035$ and $\rho = 1$.

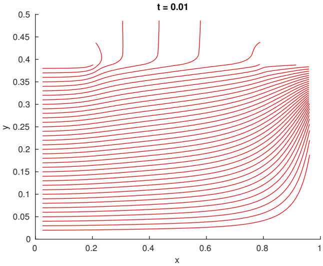
Drained conditions on the exterior of Ω^p

- Drained conditions ($p^p = 0$) for pressure on the exterior of Ω^p





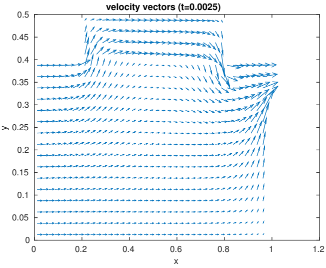




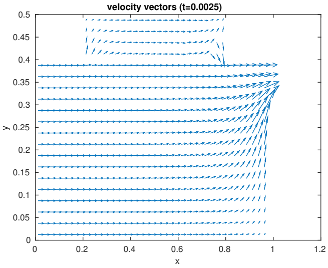
Impermeable conditions on the exterior of Ω^p

- Impermeable conditions on the exterior of Ω^p

$K = 0.01$



$K = 10^{-4}$



Conclusions

- A coupled model based on the Darcy equation and the Stokes equations with appropriate internal interface conditions is formulated.
- An efficient monolithic multigrid solution technique with a decoupled Uzawa smoother is employed for the coupled system.
- LFA smoothing analysis is applied to determine the optimal parameters in the smoother.
- The proposed method is independent from the physical parameters, which is more robust than other existing strategies.
- Same idea applied for a more complicated coupled model based on Deformable porous media and Stokes equations.

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THANK YOU FOR YOUR ATTENTION!!