Parallel simulation of channel network

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Motivation

Goal:

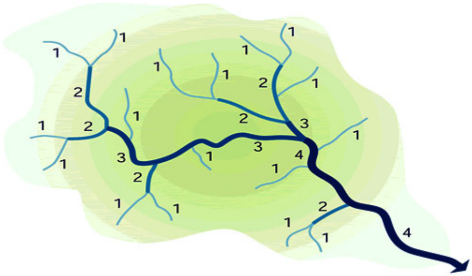
- parallel simulation of network
- couple rainfall with the current tools of storm surge simulation, ADCIRC

Current models:

- HEC-RAS, implicit solver
- GSSHA, Manning's equation

Computational challenges:

- establish an explicit method to solve the Shallow water Equations
- include physical characteristics (e.g., infiltration rates, friction coefficients, etc.) and the geometrical characteristics (e.g., bathymetry and topography)
- algorithmic performance, parallel simulation



River network

Outline

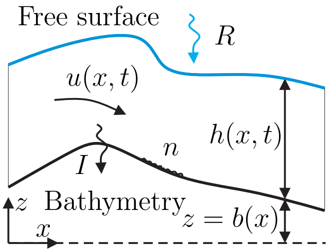
1. Mathematical framework: flow in a single reach
   - 1D Shallow Water Equations
   - Source term
2. Verification
   - Flow over a bump
3. River network simulation
   - Junction simulation
   - Flow in a synthetic channels
4. Conclusions and remarks

Outline

Governing equations

One-dimensional shallow water equation in the conservative form

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial (uh)}{\partial x} & = R - I, \\ \dfrac{\partial (uh)}{\partial t} + \dfrac{\partial (u^2 h + 0.5 g h^2)}{\partial x} & = gh(S_0 - S_f) \end{cases}$$



1D Shallow Water Equation

x :	distance
t :	time
$h = h(x,t)$:	height
$u = u(x,t)$:	velocity
R :	Rainfall
I :	Infiltration
g :	ground acceleration
$z = b(x)$:	bottom surface elevation
$S_0 = -\dfrac{\partial z}{\partial x}$:	slope of the bottom
$S_f = \dfrac{n^2 u\|u\|}{h^{\frac{4}{3}}}$	friction slope
n :	Manning's number

Governing equations

In the compact form, (unknown: $\mathbf{w}(x,t)$)

$$\frac{\partial \mathbf{w}(x,t)}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{w};x,t)}{\partial x} = \mathbf{s}(\mathbf{w};x,t)$$

where, $\quad \mathbf{w}\ \&\ \mathbf{f}\ \&\ \mathbf{s} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{m=2}$

$$\mathbf{w}(x,t) = \begin{bmatrix} h \\ uh \end{bmatrix}, \qquad \mathbf{f}(\mathbf{w},x,t) = \begin{bmatrix} uh \\ u^2h + 0.5gh^2 \end{bmatrix}, \qquad \mathbf{s}(\mathbf{w},x,t) = \begin{bmatrix} R - I \\ gh(S_0 - S_f) \end{bmatrix}$$

Using the chain rule, to linearize the system:

$$\frac{\partial \mathbf{f}(\mathbf{w},x,t)}{\partial x} = \frac{\partial \mathbf{f}(\mathbf{w},x,t)}{\partial \mathbf{w}} \frac{\partial \mathbf{w}}{\partial x} = \mathbf{A}\frac{\partial \mathbf{w}}{\partial x}$$

where,

$$\mathbf{A}(x,t) = \begin{bmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{bmatrix}, \qquad c = \sqrt{gh}$$

Finally:

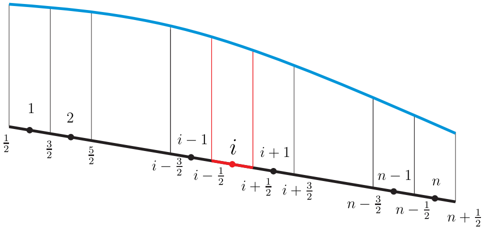
$$\frac{\partial \mathbf{w}(x,t)}{\partial t} = -\mathbf{A}\frac{\partial \mathbf{w}}{\partial x} + \mathbf{s}(\mathbf{w},x,t)$$

Finite Volume Method

- Uniform discretization of the $x-t$ domain: $\Delta x = x_{i+1} - x_i$, $\Delta t = t_{n+1} - t_n$
- A mesh cell $\mathcal{C}_i$ denoted by (x_i, t_n), bounded by $x_{i-1/2}, x_{i+1/2}$ $(x_{i+1/2} = x_i + \frac{\Delta x}{2})$
- Discretize the eq. by integrating it over space-time rectangle $[x_{i-1/2}, x_{i+1/2}]$, $[t_n, t_{n+1}]$:

$$\int_t \int_x \left(\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = \mathbf{s}\right) dx\, dt$$

$$\int_x \mathbf{w}(x, t_{n+1}) dx = \int_x \mathbf{w}(x, t_n) dx + \int_t (\mathbf{f}(x_{i+1/2}, t_n) - \mathbf{f}(x_{i-1/2}, t_n)) dt + \int_t \int_x \mathbf{s}\, dx\, dt$$



Finite volume discretization a single reach

Finite Volume Method

Now, we express the variables in terms of spatial and temporal mean value of $\mathbf{w}$ and $\mathbf{f}$:

$$\mathbf{U}_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{w}(x, t_n)dx,$$

$$\mathbf{F}(\mathbf{U}_i^n; i+1/2) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{f}(x_{i+1/2}, t_n)dt.$$

$$\frac{\partial \mathbf{U}}{\partial t} = -\frac{\partial \mathbf{F}}{\partial x} + \mathbf{S} = -\mathbf{A}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{S}$$

Final equation based on fluxes:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\big(\mathbf{F}(\mathbf{U}, i+1/2) - \mathbf{F}(\mathbf{U}, i-1/2)\big) + \mathbf{S}$$

How to choose $\mathbf{F}$?

First-order method

Upwind method (Low resolution)
Flow to the right:

$$\mathbf{F}^{L}_{i+1/2} = \mathbf{F}_i$$

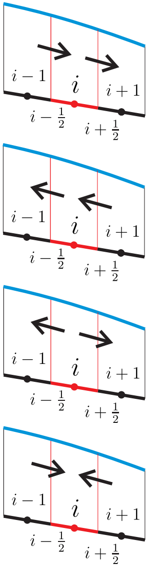
$$\mathbf{U}^{n+1}_i = \mathbf{U}^n_i - \frac{\Delta t}{\Delta x}\left(\mathbf{AU}^n_i - \mathbf{AU}^n_{i-1}\right)$$

Flow to the left:

$$\mathbf{F}^{L}_{i+1/2} = \mathbf{F}_{i+1}$$

$$\mathbf{U}^{n+1}_i = \mathbf{U}^n_i - \frac{\Delta t}{\Delta x}\left(\mathbf{AU}^n_{i+1} - \mathbf{AU}^n_i\right)$$

- Pros: no oscillation near a discontinuity, convergence
- Cons: only a first-order method, highly diffusive, less accurate



various flow conditions

Second-order method

- Pros: High-resolution method,
- Cons: Solution is oscillatory at the discontinuities.

Taylor expansion of $\mathbf{U}$ for each cell $\mathcal{C}_i = (x_{i-1/2}, x_{i+1/2})$:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \Delta t (\frac{\partial \mathbf{U}}{\partial t})_i^n + \frac{\Delta t^2}{2} (\frac{\partial^2 \mathbf{U}}{\partial t^2})_i^n + \dots$$

We have from the SWE:

$$\frac{\partial \mathbf{U}}{\partial t} = -\mathbf{A} \frac{\partial \mathbf{U}}{\partial x} + \mathbf{S}$$

$$\begin{aligned}
\frac{\partial^2 \mathbf{U}}{\partial t^2} &= \frac{\partial}{\partial t}(\frac{\partial \mathbf{U}}{\partial t}) = \frac{\partial}{\partial t}(-\frac{\partial \mathbf{F}}{\partial x} + \mathbf{S}) = -\frac{\partial^2 \mathbf{F}}{\partial x \partial t} + \frac{\partial \mathbf{S}}{\partial t} \\
&= -\frac{\partial}{\partial x}(\frac{\partial \mathbf{F}}{\partial \mathbf{U}} \frac{\partial \mathbf{U}}{\partial t}) + \frac{\partial \mathbf{S}}{\partial t} \\
&= \frac{\partial}{\partial x}(\mathbf{A}^2 \frac{\partial \mathbf{U}}{\partial x}) - \frac{\partial (\mathbf{A}\mathbf{S})}{\partial x} + \frac{\partial \mathbf{S}}{\partial t}
\end{aligned}$$

Second-order method

Substituting in back in the Taylor expansion, and dropping the third- and higher-order terms:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \Delta t(-\mathbf{A}\frac{\partial \mathbf{U}}{\partial x})_i^n + \frac{\Delta t^2}{2}(\frac{\partial}{\partial x}(\mathbf{A}^2\frac{\partial \mathbf{U}}{\partial x}))_i^n$$
$$+\Delta t \mathbf{S}_i^n - \frac{\Delta t^2}{2}\frac{\partial}{\partial x}(\mathbf{A}\mathbf{S})_i^n + \frac{\Delta t^2}{2}\frac{\partial \mathbf{S}_i^n}{\partial t}$$

Let's drop the terms corresponding to $\mathbf{S}$ for a moment, substitute:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{2\Delta x}\mathbf{A}(\mathbf{U}_{i+1}^n - \mathbf{U}_{i-1}^n) + \frac{\Delta t^2}{2\Delta x^2}\mathbf{A}^2(\mathbf{U}_{i+1}^n - 2\mathbf{U}_i^n + \mathbf{U}_{i-1}^n)$$

Rearrange to find the fluxes-Lax-Wendroff method:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\{[\frac{1}{2}\mathbf{A}(\mathbf{U}_{i+1}^n + \mathbf{U}_i^n) - \frac{\Delta t}{2\Delta x}\mathbf{A}^2(\mathbf{U}_{i+1}^n - \mathbf{U}_i^n)]$$
$$-[\frac{1}{2}\mathbf{A}(\mathbf{U}_i^n + \mathbf{U}_{i-1}^n) - \frac{\Delta t}{2\Delta x}\mathbf{A}^2(\mathbf{U}_i^n - \mathbf{U}_{i-1}^n)]\}$$

Flux limiter

To combine the advantages of both first- and second-order methods:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\left\{\mathbf{F}_{i+1/2}^n - \mathbf{F}_{i-1/2}^n\right\}$$

Rewrite the high-resolution flux:

$$\mathbf{F}_{i+1/2}^H = \mathbf{F}_{i+1/2}^L + \left(\mathbf{F}_{i+1/2}^H - \mathbf{F}_{i+1/2}^L\right) \qquad \mathbf{F}_{i-1/2}^H = \mathbf{F}_{i-1/2}^L + \left(\mathbf{F}_{i-1/2}^H - \mathbf{F}_{i-1/2}^L\right)$$

$$\mathbf{F}_{i+1/2}^H = \mathbf{F}_{i+1/2}^L + \phi_{i+1/2}\left(\mathbf{F}_{i+1/2}^H - \mathbf{F}_{i+1/2}^L\right) \qquad \mathbf{F}_{i-1/2}^H = \mathbf{F}_{i-1/2}^L + \phi_{i-1/2}\left(\mathbf{F}_{i-1/2}^H - \mathbf{F}_{i-1/2}^L\right)$$

The flux limiter term:

$$\phi(\mathbf{U}) = \begin{cases} 1 & \text{high-order method (Lax-Wendroff)} \\ 0 & \text{low-order method (upwind)} \end{cases}$$

Outline

Source term

Now let's focus on the second part of $\mathbf{U}_i^{n+1}$ (source term)

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \Delta t(-\mathbf{A}\frac{\partial \mathbf{U}}{\partial x})_i^n + \frac{\Delta t^2}{2}(\frac{\partial}{\partial x}(\mathbf{A}^2\frac{\partial \mathbf{U}}{\partial x}))_i^n$$
$$+\Delta t\mathbf{S}_i^n - \frac{\Delta t^2}{2}\frac{\partial}{\partial x}(\mathbf{AS})_i^n + \frac{\Delta t^2}{2}\frac{\partial \mathbf{S}_i^n}{\partial t}$$

The second part of the equation:

$$\Delta t\mathbf{S}_i^n - \frac{\Delta t^2}{2}\frac{\partial}{\partial x}(\mathbf{AS})_i^n + \frac{\Delta t^2}{2}\frac{\partial \mathbf{S}_i^n}{\partial t} = \Delta t(\mathbf{S}_i^n + \frac{\Delta t}{2}\frac{\partial \mathbf{S}_i^n}{\partial t}) - \frac{\Delta t^2}{2}\frac{\partial}{\partial x}(\mathbf{AS})_i^n$$

Taylor expansion of S_i^n:

$$\mathbf{S}_i^{n+1} = \mathbf{S}_i^n + \Delta t(\frac{\partial \mathbf{S}}{\partial t})_i^n + \ldots \Rightarrow \mathbf{S}_i^{n+1} + \mathbf{S}_i^n = 2(\mathbf{S}_i^n + \frac{\Delta t}{2}(\frac{\partial \mathbf{S}}{\partial t})_i^n)$$

The second part of the equation:

$$\frac{\Delta t}{2}(\mathbf{S}_i^n + \mathbf{S}_i^{n+1}) - \frac{\Delta t^2}{2}\frac{(\mathbf{AS})_{i+1/2}^n - (\mathbf{AS})_{i-1/2}^n}{\Delta x}$$

The issue here is $\mathbf{S}_i^{n+1}$

Final equations

Final equation:

$$\mathbf{U}_i^{n+1} = (\mathbf{I} - \frac{\Delta t^2}{2}\mathbf{B}_i^n)^{-1}(\mathbf{U}_i^n - \frac{\Delta t}{\Delta x}(\mathbf{A}^-\Delta\mathbf{U}_{i+1/2} + \mathbf{A}^+\mathbf{U}_{i-1/2}) - \frac{\Delta t}{\Delta x}(\tilde{\mathcal{F}}_{i+1/2} - \tilde{\mathcal{F}}_{i-1/2})$$
$$+ \Delta t(\mathbf{S}_i^n - \frac{1}{2}\mathbf{B}_i^n\mathbf{U}_i^n) - \frac{\Delta t^2}{2}\frac{(\mathbf{AS})_{i+1/2}^n - (\mathbf{AS})_{i-1/2}^n}{\Delta x})$$

where,

$$\tilde{\mathcal{F}}_{i-1/2} = \frac{1}{2}\sum_{p=1}^{m=2}|s_{i-1/2}^p|(1 - \frac{\Delta t}{\Delta x}|s_{i-1/2}^p|)\tilde{\mathcal{W}}_{i-1/2}^p$$

$$\mathbf{A}^-\Delta\mathbf{U}_{i+1/2}^n = \sum_{p=1}^{m=2}(s_{i+1/2}^p)^-\alpha_{i-1/2}^p\mathbf{r}^p$$

$$\mathbf{B}_i^n = \begin{bmatrix} 0 & 0 \\ g(S_0 + \frac{7}{3}S_f) & -\frac{2gS_f}{u} \end{bmatrix}_i^n$$

Boundary conditions

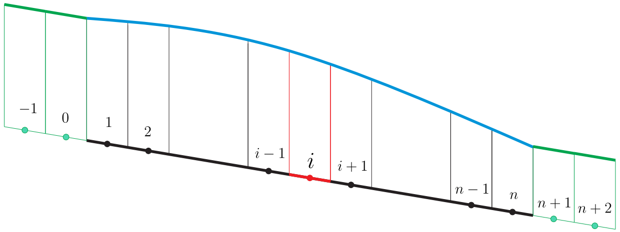
Periodic boundary condition:

$$\begin{cases} U_{-1}^n = U_{N-1}^n, & U_0^n = U_N^n \\ U_{N+1}^n = U_1^n, & U_{N+2}^n = U_2^n \end{cases}$$

Zero-order extrapolating from the interior solution:

$$\begin{cases} U_{-1}^n = U_1^n, & U_0^n = U_1^n \\ U_{n+1}^n = U_N^n, & U_{N+2}^n = U_N^n \end{cases}$$

First-order extrapolating from the interior solution.



Ghost cells

Outline

Flow over a bump

Example 1: Smooth subcritical flow
Domain: $0\mathrm{m} < x < 25\mathrm{m}$
Bathymetry:

$$z(x,y) = \begin{cases} -0.2 + 0.05(x-10)^2 & 8\mathrm{m} < x < 12\mathrm{m} \\ 0 & \text{else} \end{cases}$$

Manning's n: 0
Initial condition:
Surface elevation: $h = 2\mathrm{m}$
Flux: $Q = 0\mathrm{m}^3/\mathrm{s}$
Boundary conditions:
at $x = 0\mathrm{m}$ $\quad Q = 4.42\mathrm{m}^3/\mathrm{s}$
at $x = 25\mathrm{m}$ $\quad h = 2\mathrm{m}$

Example 1: Smooth subcritical flow
Upwind method ($\phi = 0$)

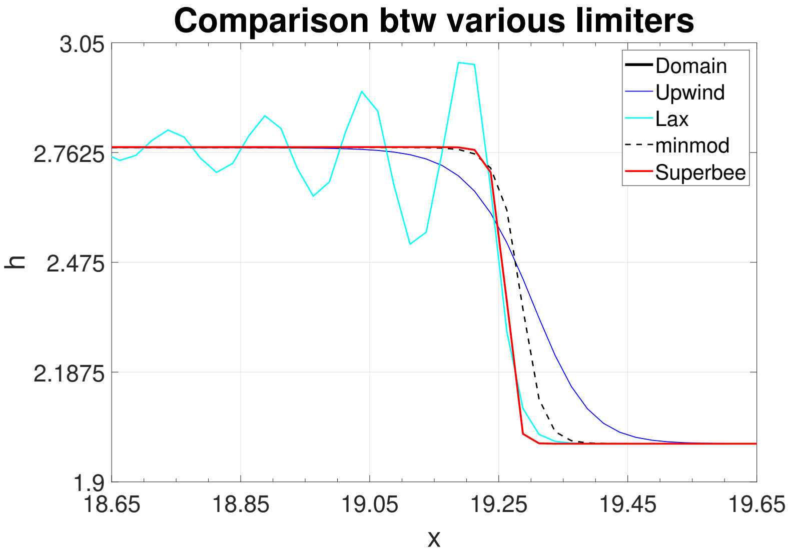
Flow over a bump

Example 1: Smooth subcritical flow
Lax-Wendroff method ($\phi = 1$)

Flow over a bump

Example 1: Smooth subcritical flow
minmod limiter, most diffusive ($\phi = ...$)

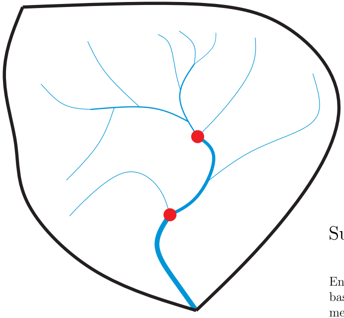
Flow over a bump



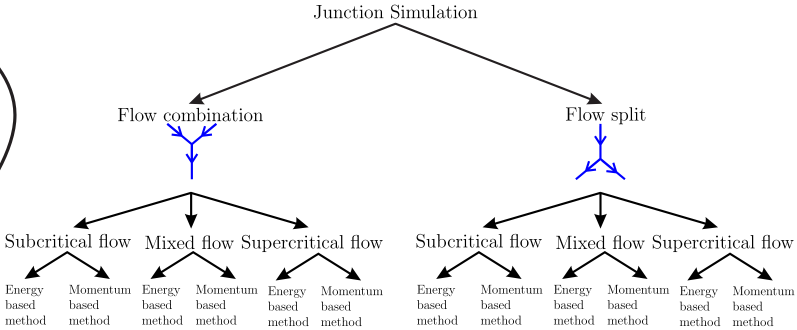
Outline

Network/Junction simulation

Network simulation

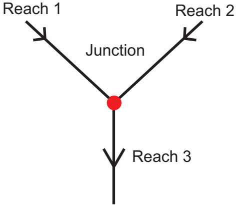


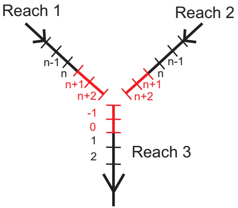
Junction simulation in HEC-RAS



Junction simulation

Energy-based method for flow combination in HEC-RAS





- Reach 1 & 2:
  h from the energy balance,
  uh from cell n.
- Reach 3:
  h from cell 1,
  uh from conservation of mass.

$$h_u + \frac{\alpha^2 V_u^2}{2g} = h_d + \frac{\alpha^2 V_d^2}{2g}$$

$$Q_3 = Q_1 + Q_2$$

Parallel implementation

- Partitioner:
  - based on METIS 4.0,
  - creates input file for each rank separately,
  - creates geometry files for visualization.
- Parallel engine (simulator):
  - hybrid parallelization: MPI and OpenMP,
  - reach and junction simulation.
- Visualization
  - based on XDMF and PHDF5,
  - partitioner code: geometry data,
  - simulator creates the results file for each rank.

Outline

Synthetic channel

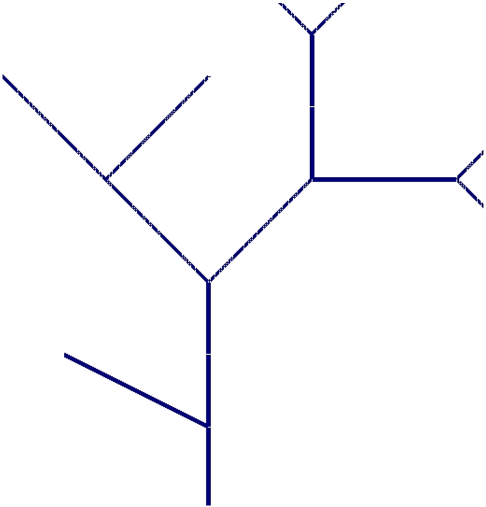
Network:

- no. of reaches: 16
- no. of junctions: 15
- no. of cells: 2395
- duration: 3600 sec
- time step: 0.1 sec
- total steps: 36000
- total length: 9650 m
- longest reach: 4350 m
- initial height: 4 m
- initial velocity: 0 m/s
- flow: 8 m/s

Scalability of the model for HPC:

No. rank	Simulation time
2	200 sec
4	134 sec
8	95 sec

Synthetic network



Synthetic channel

Animation water height/velocity

Conclusions

The developed model is:

- Explicit,
- Scalable,
- Flow in in a single reach or river network,
- Coupled with storm surge models (ADCIRC)

The code is:

- Fortran,
- OOP,
- Parallal (OMP/MPI)

References

- Finite Volume Methods for Hyperbolic Problems, by Randall J. Leveque
- Advances toward a multi-dimensional discontinuous Galerkin method for modeling Hurricane storm surge induced flooding in coastal watersheds, PhD Dissertation, Prapti Neupane, UT Austin

The End

Thanks for your attention