Power Laws and Self-Similarity in Tornadogenesis

Misha Shvartsman, University of St. Thomas, joint work with
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SIAM GS19, CP2 Climate I
Houston, TX
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Outline

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Tornado Pic

Tornado in Campo, CO on 05/31/2010

Definition and Properties

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- **Self - Similarity**

$$\frac{f(x)}{x^b} = \frac{f^*(x,z)}{x^b} = \phi\left(\frac{z}{x^{b^*}}\right) = A$$

Weatherquake

M. Walter [2010], Earthquakes and Weatherquakes: Mathematics and Climate Change. *Notices of the AMS*, **57**, v. 10, 1278–1284

$$\Delta F = b \ln\left(\frac{C}{C_0}\right), \quad e^{\Delta F} = \left(\frac{C}{C_0}\right)^b, \quad e^{-\Delta F} = \left(\frac{C_0}{C}\right)^b$$

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Weatherquake Hypothesis: (X is the magnitude of the event)

$$N(x) = \alpha p^x,\ 0 \le p \le 1,\ 0 \le x < \infty,\ \alpha = -\ln p$$

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$$P(X > a) = \frac{1}{2},\ a = -\frac{\ln 2}{\ln p}$$

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Fractal Dimension

$$f(\varepsilon) = \varepsilon^{-D}$$

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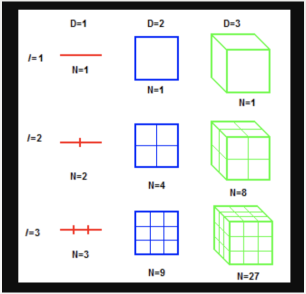

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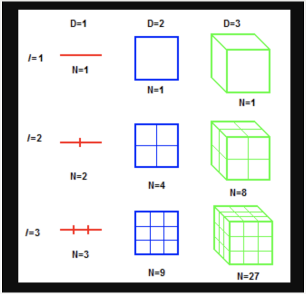

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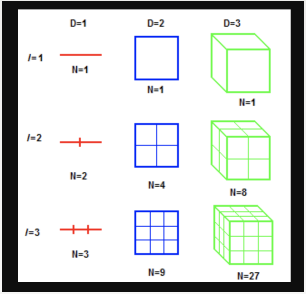

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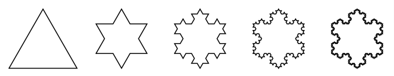
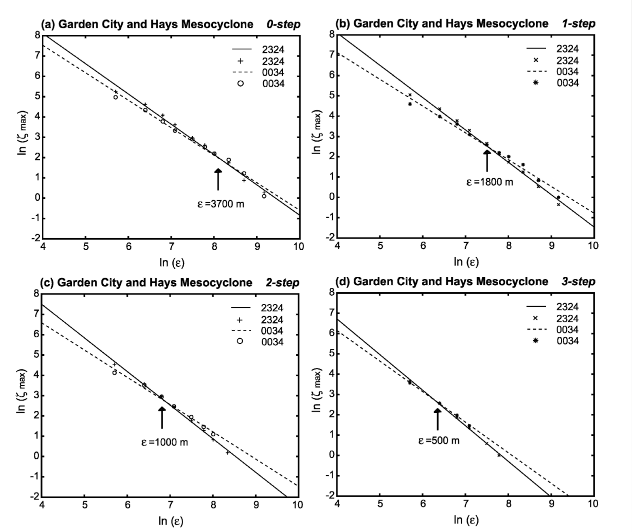


Outline

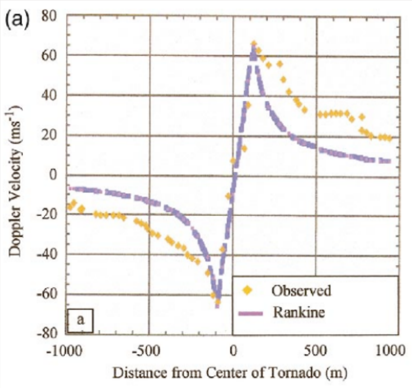
H. Cai [2005], Comparison between tornadic and nontornadic mesocyclones using the vorticity (pseudovorticity) line technique, *Mon. Wea. Rev.*, **133**, 2535–2551

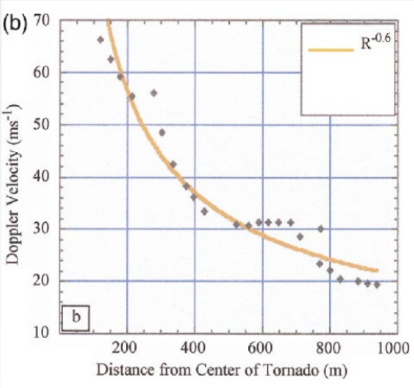
$$\zeta = A\varepsilon^{-b}$$



J. Wurman and S. Gill [2000], Finescale Radar Observations of the Dimmitt, Texas (2 June 1995), Tornado, *Mon. Wea. Rev.*, **128**, 2135–2164

Dimmit, TX, 1995, tornado: Velocity drop-off $\propto r^{-0.6}$





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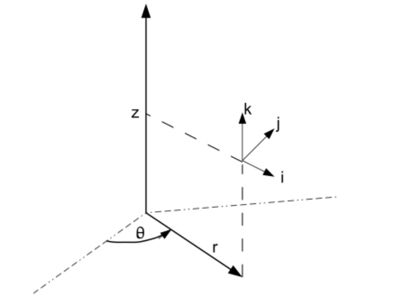
J. Serrin [1972], The swirling vortex. *Phil. Trans. Roy. Soc. London, Series A, Math & Phys. Sci.*, **271**, 325–360

$$\text{velocity} \sim \frac{1}{r}$$



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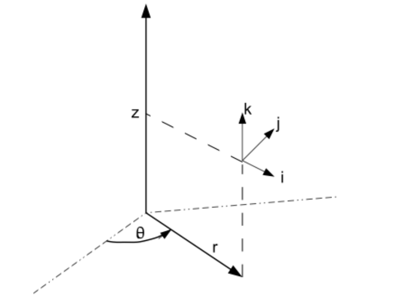

Spherical coordinates: (R, α, θ)

$$v_R = \frac{G(x)}{r} \qquad v_\alpha = \frac{F(x)}{r} \qquad v_\theta = \frac{\Omega(x)}{r}$$

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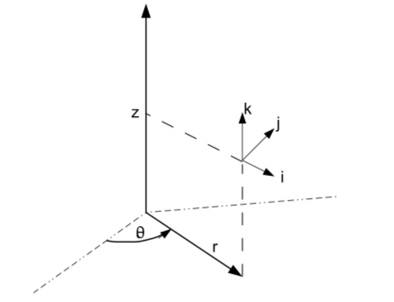

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Case $\nu > 0$, $b = 1$

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- Downdraft core with radial outflow
- Downdraft core with a compensating radial inflow
- Updraft core with radial inflow

Case $\nu > 0$, $b \neq 1$ and Case $\nu = 0$

Bělík et al. [2014], Fractal powers in Serrin's vortex solutions, *Asymptotic Analysis*, **90**, No. 1, p. 53–82.

$\nu > 0$ and $b \neq 1$, no solutions of the form $\mathbf{v} = \dfrac{\mathbf{K}(x)}{r^b}$ exist.

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$$\Omega \equiv C_\omega, \qquad F = c\sqrt{x(1-x)}, \qquad G = c\,\frac{(1-2x)\sqrt{1+x}}{2\sqrt{x}}.$$

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$\nu = 0$ and $0 < b < 1$, numerical simulations indicate the existence of solutions that are stable with respect to axisymmetric perturbations.

Outline

Singular set is not too "large"

M. Cannone [2004], Handbook of MFD, **3**, Chapter 3, p. 164

One may imagine that blow-up of initially regular solutions never happens, or that there is blow-up, but only on a very "thin" set. Clay Mathematical Institute is offering a prize for the answer. Fefferman remarks that finite blow-up in the Euler equation of an "ideal" fluid is an open and challenging mathematical problem as it is for the Navier-Stokes equations. Constantin suggests that it is finite time blow-up in the Euler equations that is the physically more important problem, since blow-up requires large gradients in the limit of zero viscosity. The best result in this direction concerning the possible loss of smoothness for the Navier-Stokes equations was obtained by Caffarelli, Kohn and Nirenberg, who proved that the one-dimensional Hausdorff measure of the singular set is zero.

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Y. G. Sinai [2014], *Private Communication*, "Tornado is a (possibly point) singularity in 3D"

Kolmogorov Theory [1941]

Velocity: $\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}'$, Kinetic energy: $\int_0^\infty \langle E \rangle(k)\, dk$

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Kolmogorov: Midrange Scales: $\langle E\rangle(k)$ is a function of **only** L and ε.

$$\langle E\rangle(k) = C\varepsilon^{\alpha}k^{\beta}, \quad \alpha = \frac{2}{3}, \quad \beta = -\frac{5}{3}, \quad k = 2\pi/L$$

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A. Chorin [1994], Vorticity and Turbulence, Springer Verlag.

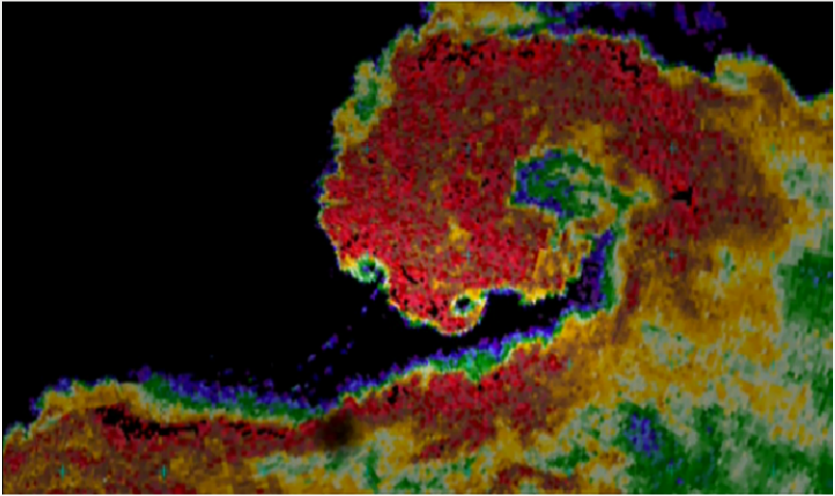
$$-\beta \approx \text{fractal dimension}, \quad \langle E \rangle(k) = Ck^{-D_\Sigma - 1}$$

D_Σ is the dimension of the vortex cross section.

Direct and inverse cascades

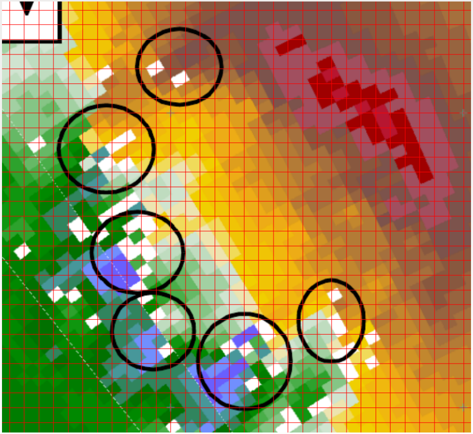
Reflectivity Image

© **J. Wurman**, PBS NOVA, *Hunt for Supertwister*, 03/30/2004



Box Counting Dimension

$$D_{\mathrm{B}} = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln \frac{1}{\varepsilon}}$$



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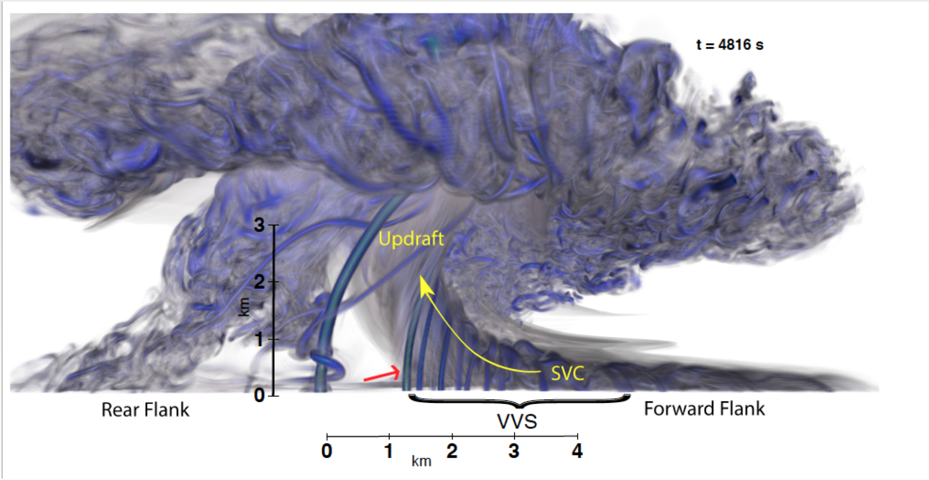
L. Orf et al. [2017], Evolution of a long-track violent tornado within a simulated supercell, *B. Am. Meterol. Soc.*, **98**, 55–68



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THANK YOU!