Dual-porosity-Stokes model and finite element methods for coupling dual-porosity flow and free flow

Xiaoming He
Department of Mathematics & Statistics
Missouri University of Science & Technology, Rolla, MO

Joint work with
Baojun Bai, Chaohua Guo, Mingzhen Wei, Jiangyong Hou, Meilan Qiu; Missouri S & T
Md. Abdullah Al Mahbub and Haibiao Zheng
East China Normal University

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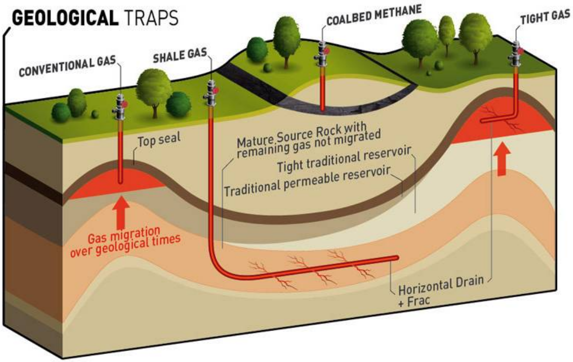
Outline

- Shale/tight reservoirs with dual porosity media and multi-stage fracturing horizontal well:



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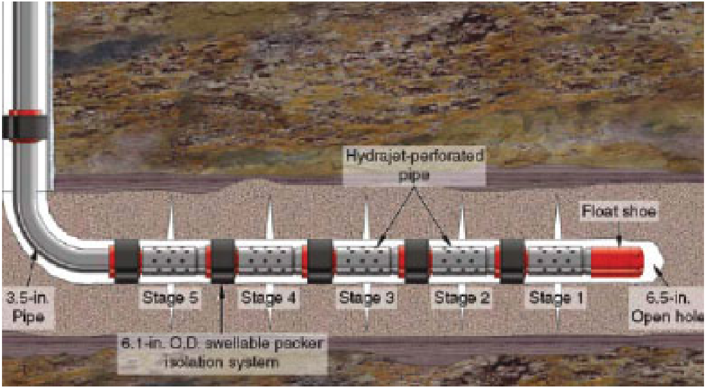
Introduction: applications

- Multistage fracturing horizontal well:



http://www.halliburton.com/public/solutions/contents/Tight_Gas/Case_Histories/images/TG_SwellablePackers_graphic.jpg

Introduction: Basic idea

Modeling the coupled dual porosity flow and free flow:

- Use a dual porosity model to govern the flow in the dual porosity media.
- Use the Stokes or Navier-Stokes equation to govern the flow in the conduits/macro-fractures.
- Propose appropriate interface conditions on the interface between the dual porosity media and the conduits/macro-fractures in order to couple the two flows together.
- Impose appropriate boundary conditions for different applications.

Introduction: dual-porosity-Stokes model

We consider a coupled dual-porosity-Stokes system on a bounded domain $\Omega = \Omega_c \bigcup \Omega_d \subset \mathbb{R}^{\mathrm{N}}, \ (N = 2, 3)$.

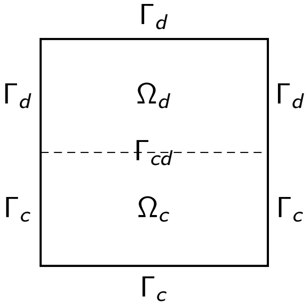


Figure: The interface (dashed line) separates the problem domain into two sub-domains: the dual porous media region and the free flow region.

Introduction: dual-porosity-Stokes model

A traditional dual-porosity model (Warren and Root):

$$\phi_m \gamma_m \frac{\partial p_m}{\partial t} - \nabla \cdot (\frac{k_m}{\mu} \nabla p_m) = -Q,$$
$$\phi_f \gamma_f \frac{\partial p_f}{\partial t} - \nabla \cdot (\frac{k_f}{\mu} \nabla p_f) = Q,$$

where

- $Q = \frac{\sigma k_m}{\mu}(p_m - p_f)$ is a mass exchange term between matrix and micro-fractures
- σ is the shape factor characterizing the morphology and dimension of the microfractures
- p_m and p_f: pressures in matrix and micro-fractures
- k_m and k_f: intrinsic permeability
- ϕ_m and ϕ_f: porosity
- γ_m and γ_f: rock compressibility
- μ: dynamic viscosity

Introduction: dual-porosity-Stokes model

In Ω_c, the fluid flow is assumed to satisfy the traditional Stokes equation

$$\begin{aligned} \frac{\partial \mathbf{u}_c}{\partial t} - \nabla \cdot \mathbb{T}(\mathbf{u}_c, p) &= \mathbf{f}, \ \ t \in [0, T], \\ \nabla \cdot \mathbf{u}_c &= 0. \end{aligned}$$

where

- $\mathbb{T}(\mathbf{u}_c, p) = 2\nu \mathbb{D}(\mathbf{u}_c) - p\mathbb{I}$: stress tensor
- $\mathbb{D}(\mathbf{u}_c) = 1/2(\nabla \mathbf{u}_c + \nabla^T \mathbf{u}_c)$: rate of deformation tensor
- $\mathbf{u}_c$: fluid velocity
- p: kinematic pressure
- $\mathbf{f}$: external body force
- ν: kinematic viscosity of the fluid

Based on a fundamental property of dual-porosity model that the flow in matrix only goes into the micro fractures but does not interact with conduits on the interface, The first interface condition is proposed for the flow in the matrix:

- No exchange condition:

$$-\frac{k_m}{\nu}\nabla p_m \cdot \mathbf{n} \;\; = \;\; 0,$$

  where $\mathbf{n}$ is the unit normal vector on Γ_{cd} pointing from Ω_c to Ω_d.

Introduction: dual-porosity-Stokes model

Two interface conditions in the normal direction for the flow in micro-fractures and the free flow in conduits/macro-fractures:

- Continuity of the normal velocity across the interface (conservation of mass):

$$\mathbf{u}_c \cdot \mathbf{n} \;\; = \;\; -\frac{k_f}{\nu} \nabla p_f \cdot \mathbf{n}$$

- Balance of the force normal to the interface:

$$-\mathbf{n}^T \mathbb{T}(\mathbf{u}_c, p)\mathbf{n} \;\; = \;\; \frac{p_f}{\rho}$$

where ρ is the fluid density.

Three choices for the forth interface condition in tangential direction:

- Beavers-Joseph (BJ):

$$-\boldsymbol{\tau}_j \cdot (\mathbb{T}(\mathbf{u}_c, p) \cdot \mathbf{n}) \quad = \quad \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} \boldsymbol{\tau}_j \cdot (\mathbf{u}_c + \frac{k_f}{\mu} \nabla p_f),$$

- Beavers-Joseph-Saffman-Jones (BJSJ):

$$-\boldsymbol{\tau}_j \cdot (\mathbb{T}(\mathbf{u}_c, p) \cdot \mathbf{n}) \quad = \quad \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} \boldsymbol{\tau}_j \cdot \mathbf{u}_c,$$

- Simplified BJSJ:

$$\boldsymbol{\tau}_j \cdot \mathbf{u}_c \quad = \quad 0,$$

where $\boldsymbol{\tau}_j$ $(j = 1, \cdots, d-1)$ denote mutually orthogonal unit tangential vectors to the interface Γ, and $\prod = k_f \mathbb{I}$.

Introduction: dual-porosity-Stokes model

- Y. Cao, M. Gunzburger, F. Hua and X. Wang, *Coupled Stokes-Darcy model with Beavers-Joseph interface boundary condition*, Comm. Math. Sci., 8(1):1-25, 2010.

- N. Chen, M. Gunzburger and X. Wang. Asymptotic analysis of the differences between the dual-porosity-Stokes system with different interface conditions and the Stokes-Brinkman system, J. Math. Anal. Appl., 368(2): 658-676, 2010.

Introduction: dual-porosity-Stokes model

- dual-porosity-Stokes model with Beavers-Joseph interface condition:

$$
\begin{aligned}
&\phi_m \gamma_m \frac{\partial p_m}{\partial t} - \nabla \cdot (\frac{k_m}{\mu} \nabla p_m) = -Q, \text{ in } \Omega_d, \ t \in [0, T] \\
&\phi_f \gamma_f \frac{\partial p_f}{\partial t} - \nabla \cdot (\frac{k_f}{\mu} \nabla p_f) = Q, \text{ in } \Omega_d, \ t \in [0, T]. \\
&\frac{\partial \mathbf{u}_c}{\partial t} - \nabla \cdot \mathbb{T}(\mathbf{u}_c, p) = \mathbf{f}, \text{ in } \Omega_c, \ t \in [0, T], \\
&\nabla \cdot \mathbf{u}_c = 0, \text{ in } \Omega_c, \\
&-\frac{k_m}{\nu} \nabla p_m \cdot \mathbf{n} = 0, \text{ on } \Gamma_{cd}, \\
&\mathbf{u}_c \cdot \mathbf{n} = -\frac{k_f}{\nu} \nabla p_f \cdot \mathbf{n}, \text{ on } \Gamma_{cd}, \\
&-\mathbf{n} \cdot (\mathbb{T}(\mathbf{u}_c, p) \cdot \mathbf{n}) = \frac{p_f}{\rho}, \text{ on } \Gamma_{cd}, \\
&-\boldsymbol{\tau}_j \cdot (\mathbb{T}(\mathbf{u}_c, p) \cdot \mathbf{n}) = \frac{\alpha \nu \sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}} \boldsymbol{\tau}_j \cdot (\mathbf{u}_c + \frac{k_f}{\mu} \nabla p_f), \text{ on } \Gamma_{cd}.
\end{aligned}
$$

Weak formulation: notations

- Spaces:

$$
\begin{aligned}
&H^1_{\Gamma_d}(\Omega_d) := \{\phi \in H^1(\Omega_d) : \ \phi = 0 \text{ on } \Gamma_d\},\\
&\mathbf{H}^1_{\Gamma_c}(\Omega_c) := \{\mathbf{v} \in H^1(\Omega_c)^N : \ \mathbf{v} = 0 \text{ on } \Gamma_c\},\\
&M := L^2(\Omega_c),\\
&\mathbf{X}_0 := \{\overrightarrow{\mathbf{v}} \in H^1_{\Gamma_d}(\Omega_d) \times H^1_{\Gamma_d}(\Omega_d) \times \mathbf{H}^1_{\Gamma_c}(\Omega_c)\},\\
&\mathbf{L}^2 := L^2(\Omega_d) \times L^2(\Omega_d) \times L^2(\Omega_c)^N.
\end{aligned}
$$

Weak formulation: notations

- Bilinear form $a(\cdot,\cdot) : \mathbf{X}_0 \times \mathbf{X}_0 \to \mathbb{R}$:

$$\begin{aligned} a(\vec{\mathbf{u}}, \vec{\mathbf{v}}) = &\int_{\Omega_d} \left(\frac{k_m}{\nu} \nabla p_m \cdot \nabla \psi_m + \frac{\sigma k_m}{\nu} (p_m - p_f) \psi_m \right) d\Omega \\ &+ \int_{\Omega_d} \left(\frac{k_f}{\nu} \nabla p_f \cdot \nabla \psi_f + \frac{\sigma k_m}{\nu} (p_f - p_m) \psi_f \right) d\Omega \\ &+ \int_{\Gamma_{cd}} \left(\mathbf{u}_c \cdot \mathbf{n} \psi_f + \frac{1}{\rho} p_f \mathbf{v} \cdot \mathbf{n} \right) d\Gamma \\ &+ \int_{\Omega_c} 2\nu \mathbb{D}(\mathbf{u}_c) : \mathbb{D}(\mathbf{v}_c) d\Omega \\ &+ \int_{\Gamma_{cd}} \left(\frac{\alpha \nu \sqrt{N}}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}} P_\tau \left(\mathbf{u}_c + \frac{k_f}{\nu} \nabla p_f \right) \cdot \mathbf{v} \right) d\Gamma, \end{aligned}$$

where P_τ denotes the projection onto the tangent space on Γ_{cd}, i.e.,

$$P_\tau \mathbf{u}_c = \sum_{j=1}^{d-1} (\mathbf{u}_c \cdot \boldsymbol{\tau}_j) \boldsymbol{\tau}_j.$$

Weak formulation: notations

- Bilinear form $b(\cdot,\cdot) : M \times \mathbf{X}_0 \to \mathbb{R}$

$$b(p, \overrightarrow{\mathbf{v}}) = -\int_{\Omega_c} p \nabla \cdot \mathbf{v} \, d\Omega.$$

- linear form $\ell(\cdot) : \mathbf{X}_0 \to \mathbb{R}$

$$\ell(\overrightarrow{\mathbf{v}}) = \begin{pmatrix} \mathbf{0} \\ < \mathbf{f}, \mathbf{v} >_{\Omega_c} \end{pmatrix}, \tag{2.1}$$

where $\mathbf{f}$ is functional on $\mathbf{H}^1_{\Gamma_c}(\Omega_c)$ and $< \mathbf{f}, \mathbf{v} >_{\Omega_c} := \int_{\Omega_c} \mathbf{f} \cdot \mathbf{v} d\Omega$ is the dualities induced by the L^2 inner product on Ω_c;

Weak formulation

- The weak formulation of the coupled dual-porosity-Stokes problem is given as follows: find $(\overrightarrow{\mathbf{u}}, p) \in L^2(0,T;\mathbf{X}_0) \cap H^1(0,T;\mathbf{X}_0') \times L^2(0,T;M)$, a.e. for $t \in (0,T)$, such that

$$
\begin{aligned}
< \partial_t \overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}} > + a(\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}) + b(p, \overrightarrow{\mathbf{v}}) &= \ell(\overrightarrow{\mathbf{v}}), \quad && \forall\, \overrightarrow{\mathbf{v}} \in L^2(0,T;\mathbf{X}_0),\\
b(q, \overrightarrow{\mathbf{u}}) &= 0, && \forall\, q \in L^2(0,T;M),\\
\overrightarrow{\mathbf{u}}(0) &= \overrightarrow{\mathbf{u}}_0,
\end{aligned}
$$

  where $\mathbf{X}_0'$ denotes the dual space of $\mathbf{X}_0$, and $\overrightarrow{\mathbf{u}}_0 = (p_m^0, p_f^0, \mathbf{u}_0)$.

Weak formulation: well-posedness

- In order to show the well-posedness of the proposed model, we should multiply the Stokes equation by a scaling factor δ, which does not change the original equation.
- The scaled inner product $< \partial_t \overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}} >_\delta$

$$< \partial_t \overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}} >_\delta = \left\langle \begin{pmatrix} \partial_t p_m \\ \partial_t p_f \\ \delta \partial_t \mathbf{u} \end{pmatrix}, \overrightarrow{\mathbf{v}} \right\rangle$$

Weak formulation: well-posedness

- The scaled bilnear form $a_\delta(\cdot,\cdot)$ is

$$
\begin{aligned}
a_\delta(\vec{\mathbf{u}},\vec{\mathbf{v}}) = & \int_{\Omega_d}\left(\frac{k_m}{\nu}\nabla p_m\cdot\nabla\psi_m + \frac{\sigma k_m}{\nu}(p_m - p_f)\psi_m\right)d\Omega \\
& + \int_{\Omega_d}\left(\frac{k_f}{\nu}\nabla p_f\cdot\nabla\psi_f + \frac{\sigma k_m}{\nu}(p_f - p_m)\psi_f\right)d\Omega \\
& + \int_{\Gamma_{cd}}\left(\mathbf{u}_c\cdot\mathbf{n}\psi_f + \frac{\delta}{\rho}p_f\mathbf{v}\cdot\mathbf{n}\right)d\Gamma \\
& + \int_{\Omega_c} 2\delta\nu\mathbb{D}(\mathbf{u}_c):\mathbb{D}(\mathbf{v}_c)d\Omega \\
& + \int_{\Gamma_{cd}}\delta\left(\frac{\alpha\nu\sqrt{N}}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}}P_\tau(\mathbf{u}_c + \frac{k_f}{\nu}\nabla p_f)P_\tau(\mathbf{v})\right)d\Gamma.
\end{aligned}
$$

Weak formulation: well-posedness

- The scaled bilinear form $b_\delta(\cdot,\cdot)$ is

$$b_\delta(p, \overrightarrow{\mathbf{v}}) = \delta \int_{\Omega_c} p \nabla \cdot \mathbf{v} \, d\Omega.$$

- The scaled linear form $\ell_\delta(\cdot)$ is

$$\ell_\delta(\overrightarrow{\mathbf{v}}) = \begin{pmatrix} \mathbf{0} \\ \delta < \mathbf{f}, \mathbf{v} >_{\Omega_c} \end{pmatrix}.$$

Weak formulation: well-posedness

- The Garding type inequality holds for small enough δ [1]:

$$a_\delta(\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{v}}) \geq C_\delta^{-1} \| \overrightarrow{\mathbf{v}} \|_{\mathbf{X}_0}^2 - C_\delta \| \overrightarrow{\mathbf{v}} \|_{\mathbf{L}^2}^2, \quad \forall\, \overrightarrow{\mathbf{v}} \in \mathbf{X}_0.$$

- Continuity of $a_\delta(\cdot, \cdot)$ on $\mathbf{X}_0$, i.e., there exists a constant C'_δ such that, for all $\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}} \in \mathbf{X}_0$

$$a_\delta(\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}) \leq C'_\delta \| \overrightarrow{\mathbf{u}} \|_{\mathbf{X}_0} \| \overrightarrow{\mathbf{v}} \|_{\mathbf{X}_0}.$$

- The inf-sup condition, i.e. there exists a constant $\beta > 0$ such that

$$\inf_{0 \neq q \in M} \sup_{\mathbf{0} \neq \overrightarrow{\mathbf{v}} \in \mathbf{X}_0} \frac{b_\delta(q, \overrightarrow{\mathbf{v}})}{\|q\|_0 \| \overrightarrow{\mathbf{v}} \|_{\mathbf{X}_0}} \geq \beta. \qquad (2.2)$$

Weak formulation: well-posedness

Theorem

Suppose that $(p_m^{dir}, p_f^{dir}, \mathbf{u}^{dir})$ *is the boundary value of some function* $\overrightarrow{\mathbf{u}}_\Gamma = (\widetilde{p}_m^{dir}, \widetilde{p}_f^{dir}, \widetilde{\mathbf{u}}^{dir}) \in L^2(0, T; \mathbf{X}) \cap H^1(0, T; \mathbf{X}_0')$ *with* $\nabla \cdot \widetilde{\mathbf{u}}^{dir} = 0$, *and* $(p_m^0, p_f^0, \mathbf{u}_c^0) \in \mathbf{L}^2$. *Then there exists a unique weak solution for the coupled dual-porosity-Stokes model.*

Coupled finite element method

- $\mathbf{X}_h^0$ and M_h are proper finite element spaces of $\mathbf{X}_0$ and M respectively, and satisfy the discrete inf-sup condition. There exists a constant $\beta > 0$ such that

$$\inf_{0\neq q\in M_h} \sup_{\mathbf{0}\neq \overrightarrow{\mathbf{v}}\in \mathbf{X}_h^0} \frac{b(q, \overrightarrow{\mathbf{v}})}{\|q\|_0 \|\overrightarrow{\mathbf{v}}\|_{\mathbf{X}_0}} \geq \beta.$$

- The standard finite element method for the proposed problem seeks $(\overrightarrow{\mathbf{u}}_h, p_h) \in H^1(0, T; \mathbf{X}_h^0) \times L^2(0, T; M_h)$ satisfying

$$< \partial_t \overrightarrow{\mathbf{u}}_h, \overrightarrow{\mathbf{v}}_h > + a(\overrightarrow{\mathbf{u}}_h, \overrightarrow{\mathbf{v}}_h) - b(p_h, \overrightarrow{\mathbf{v}}_h) = \ell(\overrightarrow{\mathbf{v}}_h), \forall \overrightarrow{\mathbf{v}}_h \in \mathbf{X}_h^0,$$
$$b(q_h, \overrightarrow{\mathbf{u}}_h) = 0, \forall q_h \in M_h,$$

Coupled finite element method: analysis

Define $\mathbb{P} = (\mathbb{P}_s \overrightarrow{\mathbf{u}}, \mathbb{P}_p p) : \mathbf{X}_0 \times M \to \mathbf{X}_0^h \times M^h$ such that, for $(\overrightarrow{\mathbf{u}}, p) \in \mathbf{X}_0 \times M$ the projection $(\mathbb{P}_s \overrightarrow{\mathbf{u}}, \mathbb{P}_p p) = (\mathbb{P}_{s1} p_m, \mathbb{P}_{s2} p_f, \mathbb{P}_{s3} \mathbf{u}_c, \mathbb{P}_p p)$ satisfies

$$
\begin{aligned}
& a_\eta(\overrightarrow{\mathbf{u}} - \mathbb{P}_s \overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}_h) + C_{0,\eta} < \overrightarrow{\mathbf{u}} - \mathbb{P}_s \overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}_h >_\eta \\
& \quad + b_\eta(p - \mathbb{P}_p p, \mathbf{v}_h) = 0, \ \forall \overrightarrow{\mathbf{v}}_h = (\psi_{mh}, \psi_{fh}, \mathbf{v}) \in \mathbf{X}_0^h, \\
& b_\eta(q_h, \mathbf{u}_c - \mathbb{P}_{s3} \mathbf{u}_c) = 0, \quad \forall q_h \in M^h.
\end{aligned}
$$

Coupled finite element method: analysis

Lemma

Consider $0 < r \le k$. *Assume that* $(\overrightarrow{\mathbf{u}}, p) = (p_m, p_f, \mathbf{u}_c, p) \in L^q(0,T; H^{r+1}_{\Gamma_d} \times H^{r+1}_{\Gamma_d} \times \mathbf{H}^{r+1}_{\Gamma_c}) \times L^q(0,T;\mathbf{H}^r(\Omega_c))$ *for some* $q \in [1,\infty)$. *Let* $(\mathbb{P}_s\overrightarrow{\mathbf{u}}, \mathbb{P}_p p)$ *be the projection solution of* (3.3)-(3.3), *then we have*

$$\begin{aligned} & \|\overrightarrow{\mathbf{u}} - \mathbb{P}_s\overrightarrow{\mathbf{u}}\|_{L^q(0,T;H^1_{\Gamma_d}\times H^1_{\Gamma_d}\times \mathbf{H}^1_{\Gamma_c})} + \|p - \mathbb{P}_p p\|_{L^q(0,T;M)} \\ \le \quad & Ch^r\left(\|\overrightarrow{\mathbf{u}}\|_{L^q(0,T;H^{r+1}_{\Gamma_d}\times H^{r+1}_{\Gamma_d}\times \mathbf{H}^{r+1}_{\Gamma_c})} + \|p\|_{L^q(0,T;\mathbf{H}^r(\Omega_c))}\right). \end{aligned}$$

Furthermore, assume that $(\overrightarrow{\mathbf{u}}, p) = (p_m, p_f, \mathbf{u}_c, p) \in H^1(0,T; H^{r+1}_{\Gamma_d} \times H^{r+1}_{\Gamma_d} \times \mathbf{H}^{r+1}_{\Gamma_c}) \times H^1(0,T;\mathbf{H}^r(\Omega_c))$, *then we have*

$$\begin{aligned} & \|\partial_t\overrightarrow{\mathbf{u}} - \partial_t\mathbb{P}_s\overrightarrow{\mathbf{u}}\|_{L^2(0,T;H^1_{\Gamma_d}\times H^1_{\Gamma_d}\times \mathbf{H}^1_{\Gamma_c})} + \|\partial_t p - \partial_t\mathbb{P}_p p\|_{L^2(0,T;M)} \\ \le \quad & Ch^r\left(\|\overrightarrow{\mathbf{u}}\|_{H^1(0,T;H^{r+1}_{\Gamma_d}\times H^{r+1}_{\Gamma_d}\times \mathbf{H}^{r+1}_{\Gamma_c})} + \|p\|_{H^1(0,T;\mathbf{H}^r(\Omega_c))}\right). \end{aligned}$$

Coupled finite element method: analysis

Lemma

Under the assumptions of Lemma 2, we have the estimates

$$
\begin{aligned}
& \|\overrightarrow{\mathbf{u}} - \mathbb{P}_s \overrightarrow{\mathbf{u}}\|_{L^q(0,T;\mathbf{L}^2)} + h\|p - \mathbb{P}_p p\|_{L^q(0,T;M)} \\
\leq \quad & Ch^{r+1}\left(\|\overrightarrow{\mathbf{u}}\|_{L^q(0,T;H^{r+1}_{\Gamma_d}\times H^{r+1}_{\Gamma_d}\times \mathbf{H}^{r+1}_{\Gamma_c})} + \|p\|_{L^q(0,T;\mathbf{H}^r(\Omega_c))}\right), \\
& \|\partial_t \overrightarrow{\mathbf{u}} - \partial_t \mathbb{P}_s \overrightarrow{\mathbf{u}}\|_{L^2(0,T;\mathbf{L}^2)} + h\|\partial_t p - \partial_t \mathbb{P}_p p\|_{L^2(0,T;M)} \\
\leq \quad & Ch^{r+1}\left(\|\overrightarrow{\mathbf{u}}\|_{H^1(0,T;H^{r+1}_{\Gamma_d}\times H^{r+1}_{\Gamma_d}\times \mathbf{H}^{r+1}_{\Gamma_c})} + \|p\|_{H^1(0,T;\mathbf{H}^r(\Omega_c))}\right).
\end{aligned}
$$

Coupled finite element method: analysis

Theorem

Assume that $0 < r \le k$, *the solution* $(\overrightarrow{\mathbf{u}}, p)$ *of the system satisfies* $(\overrightarrow{\mathbf{u}}, p) \in H^1(0,T; H^{r+1}_{\Gamma_d} \times H^{r+1}_{\Gamma_d} \times \mathbf{H}^{r+1}_{\Gamma_c}) \times H^1(0,T;\mathbf{H}^r(\Omega_c))$, $(\overrightarrow{\mathbf{u}}, p) \in L^\infty(0,T; H^{r+1}_{\Gamma_d} \times H^{r+1}_{\Gamma_d} \times \mathbf{H}^{r+1}_{\Gamma_c}) \times L^\infty(0,T;\mathbf{H}^r(\Omega_c))$, $\partial_{tt}\overrightarrow{\mathbf{u}} \in L^2(0,T;\mathbf{L}^2)$. *Assume that the initial approximation* $\overrightarrow{\mathbf{u}}^0_h$ *of* $\overrightarrow{\mathbf{u}}(0)$ *satisfies* $\|\overrightarrow{\mathbf{u}}^0_h - \mathbb{P}_s\overrightarrow{\mathbf{u}}(0)\|_{\mathbf{L}^2} \le C^\star h^{r+1}$. *Then we have the following estimate:*

$$
\begin{aligned}
& \|\overrightarrow{\mathbf{u}}(t_n) - \overrightarrow{\mathbf{u}}^n_h\|_{0,\eta} \\
\le \;& C\Big[\triangle t\|\partial_{tt}\overrightarrow{\mathbf{u}}\|_{L^2(0,T;\mathbf{L}^2)} + h^{r+1}\big(\|\overrightarrow{\mathbf{u}}\|_{H^1(0,T;H^{r+1}_{\Gamma_d}\times H^{r+1}_{\Gamma_d}\times\mathbf{H}^{r+1}_{\Gamma_c})} \\
& \quad +\|p\|_{H^1(0,T;\mathbf{H}^r(\Omega_c))} + \|\overrightarrow{\mathbf{u}}\|_{L^\infty(0,T;H^{r+1}_{\Gamma_d}\times H^{r+1}_{\Gamma_d}\times\mathbf{H}^{r+1}_{\Gamma_c})} \\
& \quad +\|p\|_{L^\infty(0,T;\mathbf{H}^r(\Omega_c))}\big)\Big].
\end{aligned}
$$

Decoupled stabilized finite element method

- Consider the mixed formulation for the dual porosity flow:

$$\phi_m \gamma_m \frac{\partial p_m}{\partial t} - \nabla \cdot \mathbf{u}_m = -Q,\ \mathbf{u}_m = \frac{k_m}{\mu} \nabla p_m,\ \text{in } \Omega_d,\ t \in [0, T]$$

$$\phi_f \gamma_f \frac{\partial p_f}{\partial t} - \nabla \cdot \mathbf{u}_f = Q,\ \mathbf{u}_f = \frac{k_f}{\mu} \nabla p_f,\ \text{in } \Omega_d,\ t \in [0, T],$$

$$\frac{\partial \mathbf{u}_c}{\partial t} - \nabla \cdot \mathbb{T}(\mathbf{u}_c, p) = \mathbf{f},\ \text{in } \Omega_c,\ t \in [0, T],$$

$$\nabla \cdot \mathbf{u}_c = 0,\ \text{in } \Omega_c,$$

$$-\frac{k_m}{\nu} \nabla p_m \cdot \mathbf{n} = 0,\ \text{on } \Gamma_{cd},$$

$$\mathbf{u}_c \cdot \mathbf{n} = -\frac{k_f}{\nu} \nabla p_f \cdot \mathbf{n},\ \text{on } \Gamma_{cd},$$

$$-\mathbf{n} \cdot (\mathbb{T}(\mathbf{u}_c, p) \cdot \mathbf{n}) = \frac{p_f}{\rho},\ \text{on } \Gamma_{cd},$$

$$-\boldsymbol{\tau}_j \cdot (\mathbb{T}(\mathbf{u}_c, p) \cdot \mathbf{n}) = \frac{\alpha \nu \sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}} \boldsymbol{\tau}_j \cdot (\mathbf{u}_c + \frac{k_f}{\mu} \nabla p_f),\ \text{on } \Gamma_{cd}.$$

Decoupled stabilized finite element method

- Step 1:

$$
\begin{aligned}
&(\frac{\mathbf{u}_{ch}^{n+1}-\mathbf{u}_{ch}^{n}}{\Delta t},\mathbf{v}_{ch})_{\Omega_c}+a_{\Omega_c}(\mathbf{u}_{ch}^{n+1},\mathbf{v}_{ch})-b(p_h^{n+1},\mathbf{v}_{ch})\\
&-\rho g\int_{\mathbb{I}}\varphi_{fh}^{n}(\mathbf{v}_{ch}\cdot\mathbf{n})\ ds+\frac{\rho g\gamma}{h}\int_{\mathbb{I}}(\mathbf{u}_{ch}^{n+1}-\mathbf{u}_{fh}^{n})\cdot\mathbf{n}(\mathbf{v}_{ch}\cdot\mathbf{n})\ ds\\
&=(\mathbf{f}(t_{n+1}),\mathbf{v}_{ch})_{\Omega},\\
&b(q_h,\mathbf{u}_{ch}^{n+1})=0,
\end{aligned}
$$

where $a_{\Omega_c}(\mathbf{u}_c,\mathbf{v})=\int_{\Omega_c}2\nu\mathbb{D}(\mathbf{u}_c):\mathbb{D}(\mathbf{v})\ d\Omega$.

- Step 2:

$$
\begin{aligned}
&\rho g\phi_f\gamma_f(\frac{p_{fh}^{n+1}-p_{fh}^{n}}{\Delta t},\psi_{fh})_{\Omega_d}+\rho g(\nabla\cdot\mathbf{u}_{fh}^{n+1},\psi_{fh})_{\Omega_d}\\
&+\rho g(\mu k_f^{-1}\mathbf{u}_{fh}^{n+1},\mathbf{v}_{fh})_{\Omega_d}-\rho g(p_{fh}^{n+1},\nabla\cdot\mathbf{v}_{fh})_{\Omega_d}\\
&+\frac{\rho g\sigma k_m}{\mu}(p_{fh}^{n+1}-p_{mh}^{n},\psi_{fh})_{\Omega_d}+\rho g\int_{\mathbb{I}}p_{fh}^{n}(\mathbf{v}_{fh}\cdot\mathbf{n})\ ds\\
&-\frac{\rho g\gamma}{h}\int_{\mathbb{I}}(\mathbf{u}_{ch}^{n+1}-\mathbf{u}_{fh}^{n+1})\cdot\mathbf{n}(\mathbf{v}_{fh}\cdot\mathbf{n})\ ds=0,
\end{aligned}
$$

Decoupled stabilized finite element method

- Step 3:

$$
\begin{aligned}
&\rho g \phi_m \gamma_m \left(\frac{p_{mh}^{n+1} - p_{mh}^{n}}{\Delta t}, \psi_{mh}\right)_{\Omega_d} + \rho g (\nabla \cdot \mathbf{u}_{mh}^{n+1}, \psi_{mh})_{\Omega_d} \\
&+ \rho g (\mu k_m^{-1} \mathbf{u}_{mh}^{n+1}, \mathbf{v}_{mh})_{\Omega_d} - \rho g (p_{mh}^{n+1}, \nabla \cdot \mathbf{v}_{mh})_{\Omega_d} \\
&+ \frac{\rho g \sigma k_m}{\mu} (p_{mh}^{n+1} - p_{fh}^{n}, \psi_{mh})_{\Omega_d} = 0.
\end{aligned}
$$

- Unconditional stability
- Optimal convergence

Example 1: test for convergence

- Domain: $\Omega = [0,1] \times [-0.25, 0.75]$ where $\Omega_c = [0,1] \times [-0.25, 0]$ and $\Omega_d = [0,1] \times [0, 0.75]$.
- Parameters:

$$\begin{array}{ll} \text{porosity :} & \phi_m = 1,\ \ \phi_f = 1, \\ \text{permeability :} & k_m = 0.01,\ \ k_f = 1, \\ \text{viscosity :} & \nu = 1, \\ \text{fluid density :} & \rho = 1, \\ \text{other parameter :} & \sigma = 1, \gamma_m = 1, \gamma_f = 1. \end{array}$$

Example 1: test for convergence

- Exact solution:

$$p_m = \sin(xy^2 - y^3)\cos(t), \ (x, y, t) \in \Omega_d \times [0, 1],$$

$$p_f = \Big(2 - \pi\sin(\pi x)\Big)\Big(\cos(\pi(1 - y)) - y\Big)\cos(2\pi t), \ (x, y, t) \in \Omega_d \times [0, 1],$$

$$\mathbf{u}_c = \begin{pmatrix} \Big(x^2y^2 + \exp(-y)\Big)\cos(2\pi t) \\ \Big(-\frac{2}{3}xy^3 + (2 - \pi\sin(\pi x))\Big)\cos(2\pi t) \end{pmatrix}, \ (x, y, t) \in \Omega_c \times [0, 1]$$

$$p = \Big(\pi\sin(\pi x) - 2\Big)\cos(2\pi y)\cos(2\pi t), \ (x, y, t) \in \Omega_c \times [0, 1]$$

Example 1: test for convergence

L^2 error and convergence order								
h	$\|\|p_m - p_{mh}\|\|_0$	rate	$\|\|p_f - p_{fh}\|\|_0$	rate	$\|\|\mathbf{u}_c - \mathbf{u}_{ch}\|\|_0$	rate	$\|\|p - p_h\|\|_0$	rate
1/8	2.033e-3	0.89	4.058e-2	0.59	1.423e-2	0.49	2.119e-1	0.66
1/16	1.136e-3	0.84	2.526e-2	0.68	8.557e-3	0.73	1.228e-1	0.79
1/32	5.515e-4	1.04	1.313e-2	0.94	4.272e-3	1.00	6.019e-2	1.03
1/64	2.838e-4	0.96	6.967e-3	0.91	2.217e-3	0.95	3.097e-2	0.96
H^1 error and convergence order								
h	$\|\|p_m - p_{mh}\|\|_1$	rate	$\|\|p_f - p_{fh}\|\|_1$	rate	$\|\|\mathbf{u}_c - \mathbf{u}_{ch}\|\|_1$	rate	$\|\|p - p_h\|\|_1$	rate
1/8	2.125e-2	0.84	2.046e-1	0.92	1.308e-1	0.55	2.692e+00	0.73
1/16	1.188e-2	0.84	1.191e-1	0.78	7.854e-2	0.74	1.508e+00	0.84
1/32	5.756e-3	1.05	6.038e-2	0.98	3.930e-2	1.00	8.148e-1	0.89
1/64	2.959e-3	0.96	3.170e-2	0.93	2.043e-2	0.94	4.827e-1	0.76

Table: The L^2 and H^1 errors and convergence orders for all variables at T=1 with time step size $dt = h$.

Example 1: test for convergence

L^2 error and convergence order								
h	$\|\|p_m - p_{mh}\|\|_0$	rate	$\|\|p_f - p_{fh}\|\|_0$	rate	$\|\|\mathbf{u}_c - \mathbf{u}_{ch}\|\|_0$	rate	$\|\|p - p_h\|\|_0$	rate
1/8	3.968e-4	1.92	9.816e-3	1.84	3.076e-3	1.84	4.728e-2	1.91
1/16	1.002e-4	1.99	2.523e-3	1.96	7.830e-4	1.97	1.185e-2	2.00
1/32	2.510e-5	2.00	6.352e-4	1.99	1.967e-4	1.99	2.962e-3	2.00
1/64	6.278e-6	2.00	1.591e-4	2.00	4.927e-5	2.00	7.403e-4	2.00
H^1 error and convergence order								
h	$\|\|p_m - p_{mh}\|\|_1$	rate	$\|\|p_f - p_{fh}\|\|_1$	rate	$\|\|\mathbf{u}_c - \mathbf{u}_{ch}\|\|_1$	rate	$\|\|p - p_h\|\|_1$	rate
1/8	4.283e-3	1.87	8.332e-2	1.90	3.392e-2	1.83	1.425e+00	1.12
1/16	1.083e-3	1.98	2.124e-2	1.97	8.720e-3	1.96	6.518e-1	1.13
1/32	2.714e-4	2.00	5.341e-3	1.99	2.200e-3	1.99	3.165e-1	1.04
1/64	6.787e-5	2.00	1.338e-3	2.00	5.520e-4	1.99	1.570e-1	1.01

Table: The L^2 and H^1 errors and convergence orders for all variables at T=1 with time step size $dt = h^2$.

Example 1: test for convergence

L^2 error and convergence order								
h	$\|\|p_m - p_{mh}\|\|_0$	rate	$\|\|p_f - p_{fh}\|\|_0$	rate	$\|\|\mathbf{u}_c - \mathbf{u}_{ch}\|\|_0$	rate	$\|\|p - p_h\|\|_0$	rate
1/8	7.176e-5	2.92	2.169e-3	2.97	6.629e-4	2.91	2.036e-2	2.05
1/16	9.034e-6	2.99	2.665e-4	3.02	8.351e-5	2.99	4.710e-3	2.11
1/32	1.133e-6	3.00	3.292e-5	3.02	1.047e-5	3.00	1.143e-3	2.04
1/64	1.418e-7	3.00	4.090e-6	3.01	1.309e-6	3.00	2.833e-4	2.01
H^1 error and convergence order								
h	$\|\|p_m - p_{mh}\|\|_1$	rate	$\|\|p_f - p_{fh}\|\|_1$	rate	$\|\|\mathbf{u}_c - \mathbf{u}_{ch}\|\|_1$	rate	$\|\|p - p_h\|\|_1$	rate
1/8	1.373e-3	2.37	7.129e-2	1.96	2.045e-2	2.05	1.274e+00	0.87
1/16	3.047e-4	2.17	1.799e-2	1.99	5.027e-3	2.02	6.310e-1	1.01
1/32	7.352e-5	2.05	4.513e-3	1.99	1.253e-3	2.00	3.137e-1	1.01
1/64	1.821e-5	2.01	1.130e-3	2.00	3.134e-4	2.00	1.566e-1	1.00

Table: The L^2 and H^1 errors and convergence orders for all variables at T=1 with time step size $dt = h^3$.

Example 2: multi-stage fractured horizontal wellbore

- The parameters of the model are chosen as: $\phi_m = 10^{-2}$, $\phi_f = 10^{-5}$, $k_m = 10^{-9}$, $k_f = 10^{-3}$, $\mu = 10^{-3}$, $\nu = 10^{-6}$, $\sigma = 0.5$, $\Gamma_m = 10^{-4}$, $\Gamma_f = 10^{-4}$.
- A horizontal cross-section is considered: the simulation domain is the square $[0, 6]^2$ and the horizontal wellbore is simplified as a rectangle $[1.8, 4.2] \times [2.8, 3.2]$ in this domain.
- The interface conditions are applied on the boundary of the hydraulic macro-fractures.

Example 2: multi-stage fractured horizontal wellbore

- The vertical wellbore is connected to the right end of the horizontal wellbore. Therefore, the no-exchange boundary condition $-\frac{k_f}{\mu}\nabla p_f \cdot (-\mathbf{n}_{cd}) = 0$ and $-\frac{k_m}{\mu}\nabla p_m \cdot (-\mathbf{n}_{cd}) = 0$ and the outflow boundary condition $\mathbb{T}(\mathbf{u}_c, p)\mathbf{n}_{cd} = 0$ are imposed on the right end of the horizontal wellbore.
- With cased hole completion, the following boundary conditions are imposed on rest of the boundary of the horizontal wellbore:

$$-\frac{k_f}{\mu}\nabla p_f \cdot (-\mathbf{n}_{cd}) = 0,\ -\frac{k_m}{\mu}\nabla p_m \cdot (-\mathbf{n}_{cd}) = 0,\ \mathbf{u}_c \cdot \mathbf{n}_{cd} = 0.$$

- We impose the constant pressure boundary condition for p_m and p_f on the left, top, and bottom boundaries: $p_m = 10^5,\ p_f = 10^4$.

Example 2: multi-stage fractured horizontal wellbore

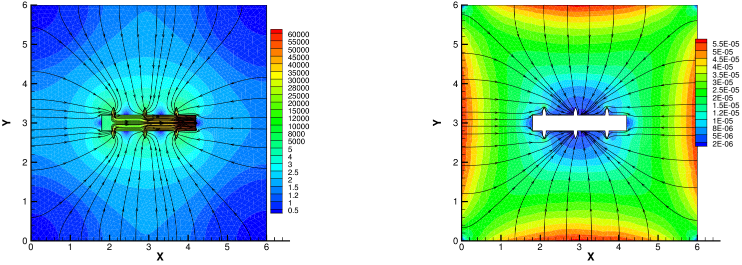


Figure: Example 4: Flow speed and streamlines around a multi-stage hydraulic fractured horizontal wellbore with cased hole completion. Left figure: the flow in micro-fractures and the multi-stage hydraulic fractured horizontal wellbore; Right figure: the flow in matrix.

Example 2: multi-stage fractured horizontal wellbore

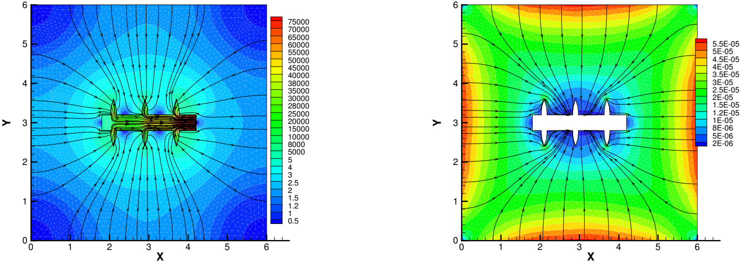


Figure: Example 4: Flow speed and streamlines around a multi-stage hydraulic fractured horizontal wellbore with cased hole completion and larger macro-fractures. Left figure: the flow in micro-fractures and the multi-stage hydraulic fractured horizontal wellbore; Right figure: the flow in matrix.

Example 3: horizontal wellbore with vertical injection and production wellbore

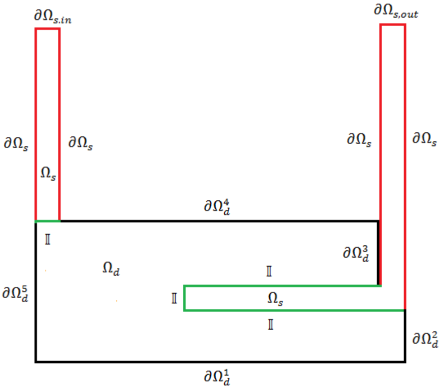


Figure: The computational domain for an injection well and a horizontal wellbore with a vertical production wellbore.

Example 3: horizontal wellbore with vertical injection and production wellbore

- The parameters of the model are chosen as: $\phi_m = 10^{-2}$, $\phi_f = 10^{-4}$, $k_m = 10^{-8}$, $k_f = 10^{-3}$, $\mu = 10^{-3}$, $\nu = 10^{-6}$, $\sigma = 0.9$, $\Gamma_m = 10^{-4}$, $\Gamma_f = 10^{-4}$.
- The interface conditions are applied on the interface between the wellbore and the dual-porosity media.
- On the top of the left vertical well, the Dirichlet inflow boundary condition is applied: $\mathbf{u}_c = [0, -32x(0.5 - x)]$.
- On the top of the right vertical well, free outflow boundary condition is applied: $-p\mathbf{I} + \nu\nabla\mathbf{u}_c = 0$.
- On the vertical boundary of the vertical well outside of the dual porosity media: $\mathbf{u}_c = (0, 0)$.
- On the boundary of the dual-porosity media, the velocity is set to be along the normal vector pointing to the inside of the dual-porosity media with the magnitudes $|\mathbf{u}_f| = 1$ and $|\mathbf{u}_m| = 0.001$.

Example 3: horizontal wellbore with vertical injection and production wellbore

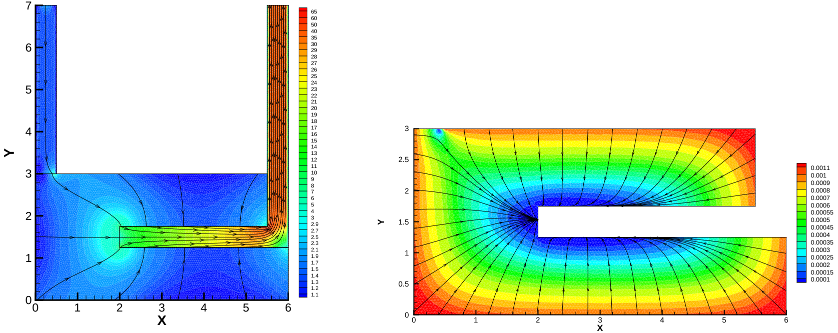


Figure: Left: the flow in the microfractures and conduits; Right: the flow in the matrix.

Ongoing & future work

- Dual-porosity-Navier-Stokes model with data assimilation
- Stochastic Dual-porosity-Navier-Stokes model
- Multi-phase dual-porosity-Stokes (or Stokes-Darcy) model by using phase field model (diffuse interface model: Cahn-Hilliard or Allen-Cahn equations)
- Multi-physics domain decomposition method for physically valid decoupling and efficient parallel computation
- Nonlinear dual-porosity models
- ....