A Green's Function Approach to Efficient Shallow Water Uncertainty Quantification

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Outline

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Harrison Ko[1]

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[1] SRI International

[2] Oregon State University

MOTIVATION

"Rising sea level has worldwide consequences because of its potential to alter ecosystems and the vulnerability of coastal regions by increasing the prevalence of recurrent tidal flooding events and life-threatening storm surge events.[1]"

[1] NOAA. The Ecological Effects of Sea Level Rise Program. coastalscience.noaa.gov/. Access 3/19.

A GREEN'S FUNCTION APPROACH

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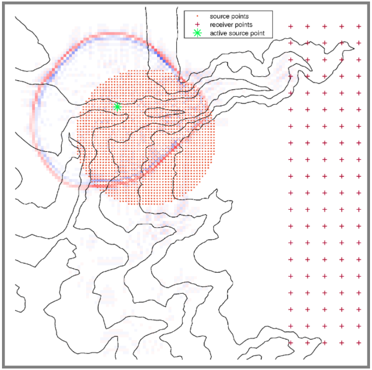

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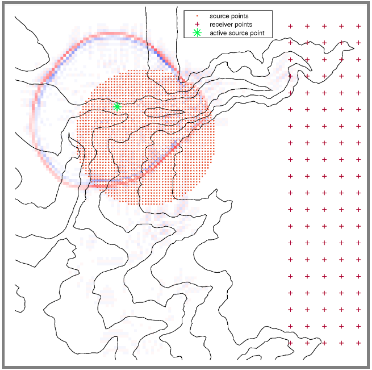


Green's Functions

Given an inhomogeneous **linear** system,

$$\mathcal{L}[q(x,t)] = f(x,t),$$

the Green's function solves the system perturbed by an impulse (Dirac delta):

$$\mathcal{L}[G(x,t;x',t')] = \delta(x-x')\delta(t-t').$$

The "**magic rule**" property of the Green's function recovers the solution:

$$q(x,t) = \int_0^T \int_\Omega f(x',t')G(x,t;x',t')dx'dt'.$$

Numerically, we can use a unit impulse (Kronecker delta) and solve for Green's Functions on a spacetime grid ($\{x_i\}$, $\{t_n\}$):

$$\mathcal{L}[G(x,t;x_i,t_n)] = \delta(x-x_i)\delta(t-t_n).$$

The solution is

$$q(x,t) = \sum_{n=1}^{N}\sum_{i=1}^{S} f(x_i,t_n)G(x,t;x_i,t_n).$$

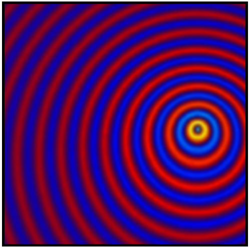
GREEN'S FUNCTIONS—SOME ANALYTIC EXAMPLES

2D Wave impulsive force

- $\nabla^2\phi - \frac{1}{c^2}\phi_{tt} = \delta(\mathbf{x})\delta(t)$
- $\phi(\mathbf{x}, t) = \frac{c}{2\pi}\frac{H(ct-r)}{\sqrt{(ct)^2-r^2}}, r = \|\mathbf{x}\|_2$

2D Wave Time-Harmonic force

- $\nabla^2\phi - \frac{1}{c^2}\phi_{tt} = \delta(\mathbf{x})e^{i\omega t}$
- $\phi(\mathbf{x}, t) = \frac{i}{4}e^{i\omega t}H_0^{(2)}(\frac{\omega r}{c})$



GREEN'S FUNCTIONS

For the initial condition problems,

$$q(x,t) = \sum_{i=1}^{S} f(x_i) G(x,t;x_i).$$

Or, in a matrix form,

$$\underbrace{\begin{bmatrix} G_{11} & G_{12} & \dots & G_{1S} \\ G_{21} & G_{22} & \dots & G_{2S} \\ G_{31} & G_{32} & \dots & G_{3S} \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1} & G_{N2} & \dots & G_{NS} \end{bmatrix}}_{Green's\ Functions} \cdot \underbrace{\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_S \end{bmatrix}}_{Initial\ Condition} = \underbrace{\begin{bmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \\ \tilde{\eta}_3 \\ \vdots \\ \tilde{\eta}_N \end{bmatrix}}_{Solution}$$

For a time-dependent forcing, add an extra dimension to this calculation.

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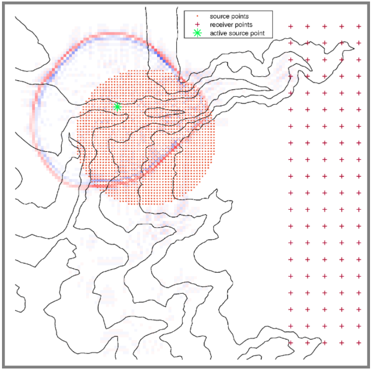

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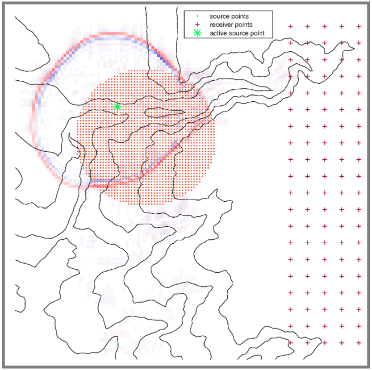


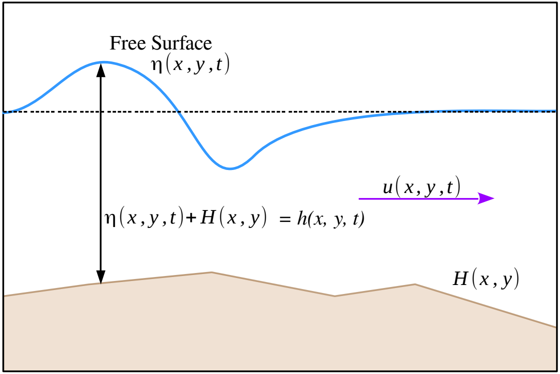
Shallow Water Equations

Free surface elevation: $\eta(\mathbf{x}, t)$
Velocities: $\mathbf{u}(\mathbf{x}, t)$
Bathymetry: $H(\mathbf{x})$

$$\eta_t + \nabla \cdot ((H + \eta)\mathbf{u}) = 0$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + g\nabla\eta = 0$$

SHALLOW WATER EQUATIONS



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...but we need a linear model to use Green's functions, right?

Shallow Water Perturbation Equations

Instead, we try to get better results within the previously stated goal to perturb **around** some kind of "reference solution."

- Assume a given solution to SWE: $[E, \mathbf{U}]^T$.
  - This is the **mean** of our forecast.

SHALLOW WATER PERTURBATION EQUATIONS

Instead, we try to get better results within the previously stated goal to perturb **around** some kind of "reference solution."

- Assume a given solution to SWE: $[E, \mathbf{U}]^T$.
  - This is the **mean** of our forecast.
- Perturb it: $[E + \tilde{\eta}, \mathbf{U} + \tilde{\mathbf{u}}]^T$
- Assumption: $\tilde{\eta}, \tilde{\mathbf{u}}, \sim \mathcal{O}(\epsilon)$

LINEARIZED *Perturbation* EQUATIONS

$$\tilde{\eta}_t + (\mathbf{U} \cdot \nabla)\tilde{\eta} + (H + E)(\nabla \cdot \tilde{\mathbf{u}}) = -(\nabla \cdot \mathbf{U})\tilde{\eta} - (\tilde{\mathbf{u}} \cdot \nabla)(H + E)$$

$$\tilde{\mathbf{u}}_t + (\mathbf{U} \cdot \nabla)\tilde{\mathbf{u}} + g\nabla\tilde{\eta} = -(\tilde{\mathbf{u}} \cdot \nabla)\mathbf{U}$$

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$$\tilde{\mathbf{u}}_t + (\mathbf{U}\cdot\nabla)\tilde{\mathbf{u}} + g\nabla\tilde{\eta} = -(\tilde{\mathbf{u}}\cdot\nabla)\mathbf{U}$$

or, to emphasize the linearity in the perturbations,

$$\begin{bmatrix}\tilde{\eta}\\ \tilde{u}\\ \tilde{v}\end{bmatrix}_t + \begin{bmatrix}U & (H+E) & 0\\ g & U & 0\\ 0 & 0 & U\end{bmatrix}\begin{bmatrix}\tilde{\eta}\\ \tilde{u}\\ \tilde{v}\end{bmatrix}_x + \begin{bmatrix}V & 0 & (H+E)\\ 0 & V & 0\\ g & 0 & V\end{bmatrix}\begin{bmatrix}\tilde{\eta}\\ \tilde{u}\\ \tilde{v}\end{bmatrix}_y$$
$$= -\begin{bmatrix}(U_x+V_y) & (H+E)_x & (H+E)_y\\ 0 & U_x & U_y\\ 0 & V_x & V_y\end{bmatrix}\begin{bmatrix}\tilde{\eta}\\ \tilde{u}\\ \tilde{v}\end{bmatrix}$$

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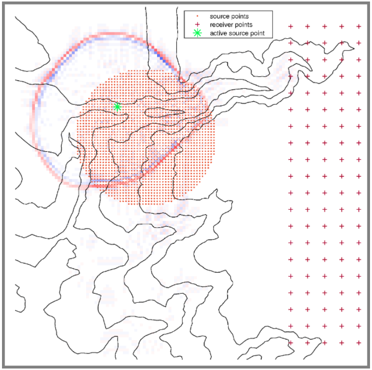
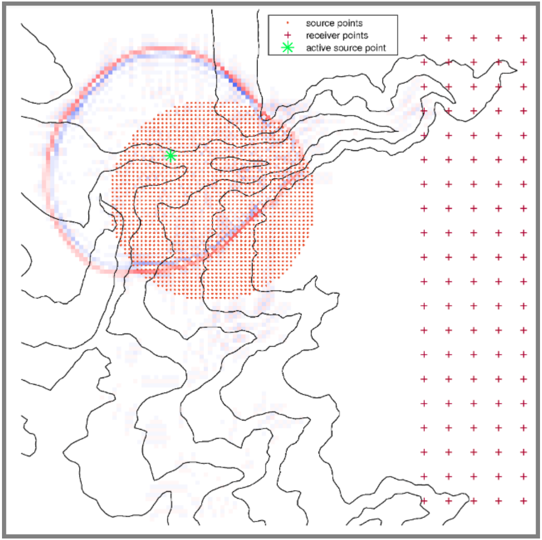


Domain Snapshot—Sources and Receivers



Efficiency

Parameters affecting computation and storage of the Green's functions:

- Source region size (S)
- Receiver region size (R)
- Timesteps (T)

2 stages:

- Pre-computation of Green's functions (expensive—model runs)
- Re-combination of Green's functions (effectively instantaneous—matrix-vector multiply)

MONTE CARLO VS GREEN'S FUNCTIONS

So what's the difference?

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- In a straightforward Monte Carlo approach, we calculate many model runs.
  - Convergence is slow, and model runs are long.
- In the GF approach, we must pre-compute:
  - For Initial Condition / Tsunami: S model runs.
  - For time-dependent forcing / Storm surge: $S \cdot T$ model runs.

Reducing the Problem

We need to keep the number of Green's functions reasonable.

For the spatial dimension (S):

- Coarsen the Green's Functions grid
- Other basis representations

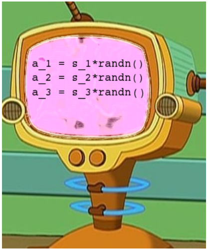
For the temporal dimension (T):

- Time-harmonic Green's functions

So they're pre-computed—what now?

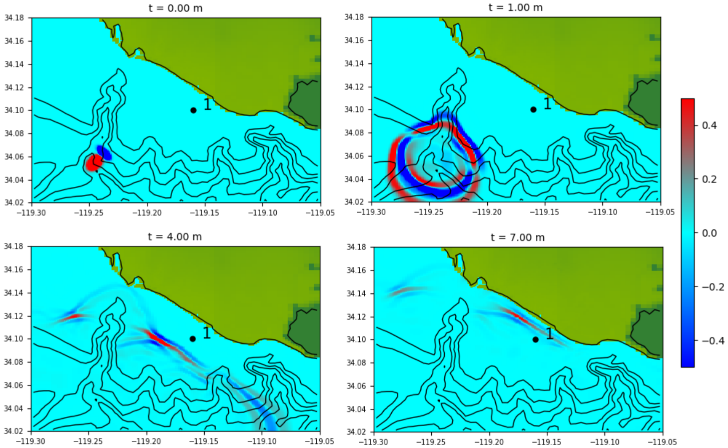
After pre-computing Green's functions, the world of parameter perturbation is open.

- What-if scenarios
- Look at individual parameter effects



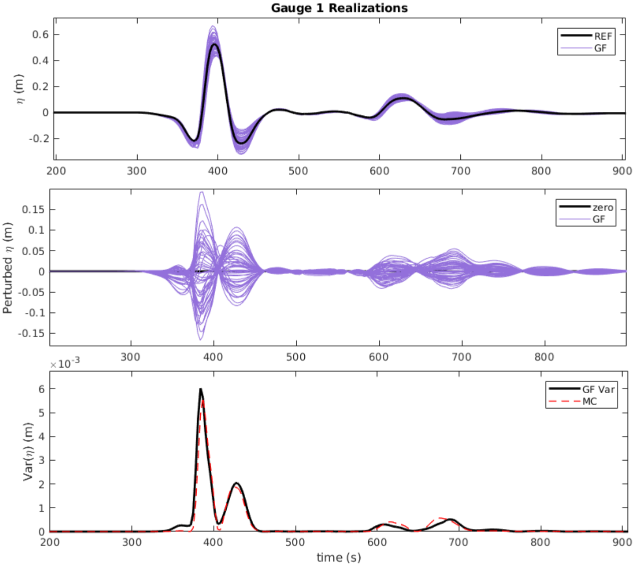
DEMONSTRATION

The mean forecast:



DEMONSTRATION

Results:



Thank you.