Results on the Stabilization of Fingering Instabilities in Porous Media Flows

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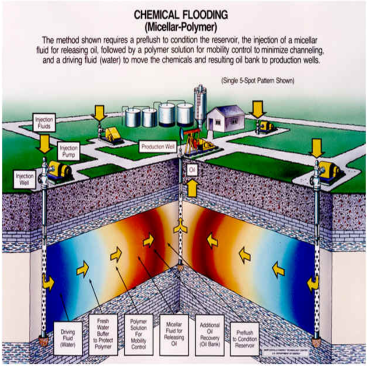
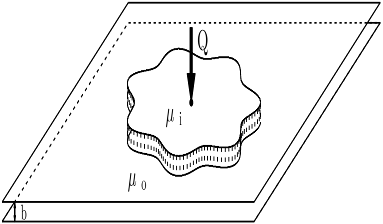
Enhanced Oil Recovery

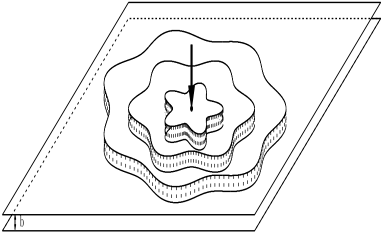
- Multi-layer flow
- Stability of such multi-layer flows is an important issue.
  - Groundwater flows
  - CO_2 sequestration
  - Geology
- Study the effect of various parameters on the stability
  $\Rightarrow$ help in the design of efficient injection policies



Drawing courtesy of the Department of Energy National Energy Technology Laboratory

Radial Hele-Shaw Flow





Governing Equations:

- Continuity equation for incompressible flow

$$\nabla \cdot \mathbf{u} = 0$$

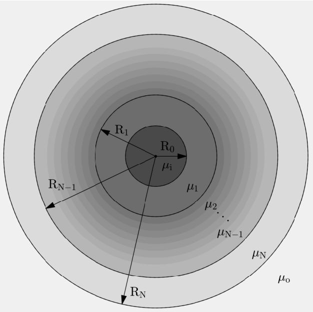
- Darcy's Law

$$\nabla p = -\frac{\mu}{\kappa}\mathbf{u}$$

- Interface conditions

$$\frac{d\eta}{dt} = \mathbf{u} \cdot \hat{n}, \qquad [p] = T\nabla \cdot \hat{n}$$

Multi-Layer Flow



Notation:

$\mathbf{u} := (u_r, u_\theta)$
Q - injection rate
R_0 - radius of innermost interface
R_j - radius of jth interface
R_N - radius of outermost interface
μ_i - viscosity of innermost (injected) fluid
μ_j - viscosity of intermediate fluids
μ_o - viscosity of outermost (displaced) fluid

Dimensionless Formulation

$$Q_{ref} = \frac{12\pi\kappa T_N}{R_N(0)\mu_o},$$

$$Q^* = \frac{Q}{Q_{ref}},$$

$$r^* = \frac{r}{R_N(0)},$$

$$\mu^* = \frac{\mu}{\mu_o},$$

$$t^* = \frac{12\kappa T_N}{R_N^3(0)\mu_o}t,$$

$$\mathbf{u}^* = \frac{R_N^2(0)\mu_o}{12\kappa T_N}\mathbf{u},$$

$$p^* = \frac{R_N(0)}{12T_N}p,$$

$$T^* = \frac{T}{T_N}$$

Original Equations

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla p = -\frac{\mu}{\kappa}\mathbf{u}$$

Dimensionless Equations

$$\nabla^* \cdot \mathbf{u}^* = 0$$

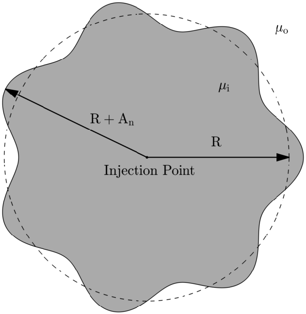
$$\nabla^* p^* = -\mu^* \mathbf{u}^*$$

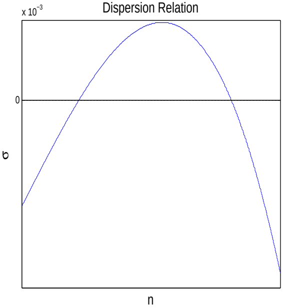
Two-Layer Flow (Previous results)

Disturbance: $A_n(t)e^{in\theta}$ $\qquad$ $Q =$ Injection rate

$$\sigma := \frac{A_n'(t)}{A_n(t)} = \frac{Qn}{2R^2}\frac{1-\mu_i}{1+\mu_i} - \frac{Q}{2R^2} - \frac{1}{1+\mu_i}\frac{n(n^2-1)}{12R^3}$$

$$n_{max} = \sqrt{2QR(1-\mu_i)+1/3}$$





L. Paterson, Radial fingering in a Hele Shaw cell. *J. Fluid Mech.*, 113:513-529, 1981.

Variable injection rate (Previous results)

- Maximum injection rate for a stable flow

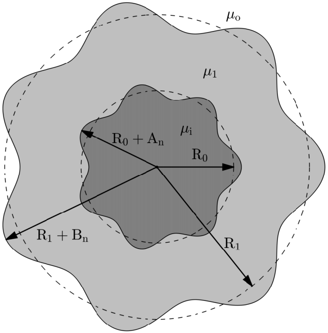
$$Q_M(n) = \frac{1}{6R}\frac{n\left(n^2-1\right)}{n(1-\mu_i)-(1+\mu_i)}$$

- Maximum injection rate for a stable flow

$$Q_M \propto t^{-\frac{1}{3}}, \qquad \text{for} \quad t >> 1$$

T. Beeson-Jones and A. Woods, On the selection of viscosity to suppress the saffman-taylor instability in a radially spreading annulus. *J. Fluid Mech.*, 782:127-143, 2015.

Three-Layer Flow



Notation:

$\mathbf{u} := (u_r, u_\theta)$
Q - injection rate
R_0 - radius of inner interface
R_1 - radius of outer interface
μ_i - viscosity of innermost (injected) fluid
μ_o - viscosity of the outermost fluid
μ_1 - viscosity of the intermediate fluid

Disturbances:

Inner Interface: $A_n(t)e^{in\theta}$ Outer Interface: $B_n(t)e^{in\theta}$

Three-Layer Flow

$$\frac{d}{dt}\begin{pmatrix}A_n(t)\\B_n(t)\end{pmatrix} = \mathbf{M}(t)\begin{pmatrix}A_n(t)\\B_n(t)\end{pmatrix}$$

where $\mathbf{M}$ is the 2×2 matrix given by:

$$\mathbf{M_{1,1}} = \frac{\left\{(1+\mu_1)-(1-\mu_1)\left(\frac{R_0}{R_1}\right)^{2n}\right\}F_0}{(\mu_1-\mu_i)(1-\mu_1)\left(\frac{R_0}{R_1}\right)^{2n}+(\mu_1+\mu_i)(1+\mu_1)} - \frac{Q}{2R_0^2}$$

$$\mathbf{M_{1,2}} = \frac{2\mu_1\left(\frac{R_0}{R_1}\right)^{n-1}F_1}{(\mu_1-\mu_i)(1-\mu_1)\left(\frac{R_0}{R_1}\right)^{2n}+(\mu_1+\mu_i)(1+\mu_1)}$$

$$\mathbf{M_{2,1}} = \frac{2\mu_1\left(\frac{R_0}{R_1}\right)^{n+1}F_0}{(\mu_1-\mu_i)(1-\mu_1)\left(\frac{R_0}{R_1}\right)^{2n}+(\mu_1+\mu_i)(1+\mu_1)}$$

$$\mathbf{M_{2,2}} = \frac{\left\{(\mu_1+\mu_i)+(\mu_1-\mu_i)\left(\frac{R_0}{R_1}\right)^{2n}\right\}F_1}{(\mu_1-\mu_i)(1-\mu_1)\left(\frac{R_0}{R_1}\right)^{2n}+(\mu_1+\mu_i)(1+\mu_1)} - \frac{Q}{2R_1^2}$$

and

$$F_0 = \frac{Qn}{2R_0^2}(\mu_1-\mu_i) - \frac{T_0}{12}\frac{n^3-n}{R_0^3}, \qquad F_1 = \frac{Qn}{2R_1^2}(1-\mu_1) - \frac{1}{12}\frac{n^3-n}{R_1^3}.$$

(N+2)-Layer

$$\frac{d}{dt}\begin{pmatrix} A_n^0(t) \\ \vdots \\ A_n^N(t) \end{pmatrix} = \mathbf{M}_N(t)\begin{pmatrix} A_n^0(t) \\ \vdots \\ A_n^N(t) \end{pmatrix},$$

$$\mathsf{M}_N(t) = \widetilde{\mathbf{M}}_N^{-1}(t)\begin{pmatrix} F_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & F_N \end{pmatrix} - \frac{Q}{2}\begin{pmatrix} \frac{1}{R_0^2} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{1}{R_N^2} \end{pmatrix}$$

$$\left(\widetilde{\mathbf{M}}_N(t)\right)_{11} = \mu_i - \mu_1 \frac{\left(\frac{R_0}{R_1}\right)^{2n} + 1}{\left(\frac{R_0}{R_1}\right)^{2n} - 1}, \qquad \left(\widetilde{\mathbf{M}}_N(t)\right)_{12} = 2\mu_1 \frac{\left(\frac{R_0}{R_1}\right)^{n-1}}{\left(\frac{R_0}{R_1}\right)^{2n} - 1},$$

$$\left(\widetilde{\mathbf{M}}_N(t)\right)_{j+1,j} = 2\mu_j \frac{\left(\frac{R_{j-1}}{R_j}\right)^{n+1}}{\left(\frac{R_{j-1}}{R_j}\right)^{2n} - 1}, \qquad \left(\widetilde{\mathbf{M}}_N(t)\right)_{j+1,j+2} = 2\mu_{j+1} \frac{\left(\frac{R_j}{R_{j+1}}\right)^{n-1}}{\left(\frac{R_j}{R_{j+1}}\right)^{2n} - 1},$$

$$\left(\widetilde{\mathbf{M}}_N(t)\right)_{j+1,j+1} = -\mu_j \frac{\left(\frac{R_{j-1}}{R_j}\right)^{2n} + 1}{\left(\frac{R_{j-1}}{R_j}\right)^{2n} - 1} - \mu_{j+1} \frac{\left(\frac{R_j}{R_{j+1}}\right)^{2n} + 1}{\left(\frac{R_j}{R_{j+1}}\right)^{2n} - 1},$$

$$\left(\widetilde{\mathbf{M}}_N(t)\right)_{N+1,N} = 2\mu_N \frac{\left(\frac{R_{N-1}}{R_N}\right)^{n+1}}{\left(\frac{R_{N-1}}{R_N}\right)^{2n} - 1}, \qquad \left(\widetilde{\mathbf{M}}_N(t)\right)_{N+1,N+1} = 1 - \mu_N \frac{\left(\frac{R_{N-1}}{R_N}\right)^{2n} + 1}{\left(\frac{R_{N-1}}{R_N}\right)^{2n} - 1}.$$

Condition to Stabilize Three-Layer Flow

$$Q \leq \min\{G_0, G_1\},$$

where

$$G_0 = \frac{T_0}{6R_0}\frac{(n^3-n)D_0}{nD_1 - D_2},$$

$$G_1 = \frac{1}{6R_1}\frac{(n^3-n)D_3}{nD_4 - D_2},$$

$$D_0 = (1+\mu_1) + 2\mu_1 T_0^{-1}\left(\frac{R_0}{R_1}\right)^{n+2} - (1-\mu_1)\left(\frac{R_0}{R_1}\right)^{2n},$$

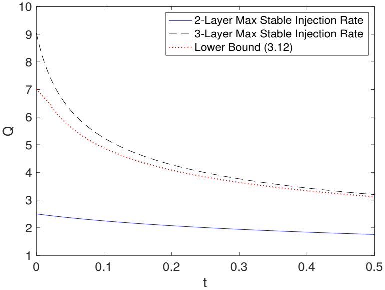
$$D_1 = (1+\mu_1)(\mu_1-\mu_i) + 2\mu_1(1-\mu_1)\left(\frac{R_0}{R_1}\right)^{n+1} - (1-\mu_1)(\mu_1-\mu_i)\left(\frac{R_0}{R_1}\right)^{2n},$$

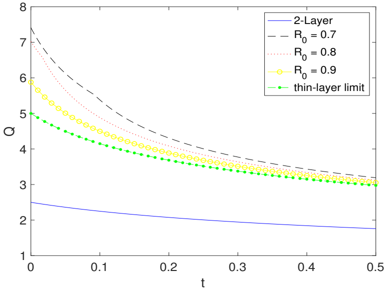
$$D_2 = (1+\mu_1)(\mu_1+\mu_i) + (1-\mu_1)(\mu_1-\mu_i)\left(\frac{R_0}{R_1}\right)^{2n},$$

$$D_3 = (\mu_1+\mu_i) + 2\mu_1 T_0\left(\frac{R_0}{R_1}\right)^{n-2} + (\mu_1-\mu_i)\left(\frac{R_0}{R_1}\right)^{2n},$$

$$D_4 = (1-\mu_1)(\mu_1+\mu_i) + 2\mu_1(\mu_1-\mu_i)\left(\frac{R_0}{R_1}\right)^{n-1} + (1-\mu_1)(\mu_1-\mu_i)\left(\frac{R_0}{R_1}\right)^{2n}.$$

Condition to Stabilize Three-Layer Flow





$\mu_i = 0.2,\ \mu_o = 1,\ T = 1,\ R(0) = 1$ for two-layer flow

$\mu_i = 0.2, \mu_1 = 0.6, \mu_o = 1, T_0 = T_1 = 1, R_0(0) = 0.8, R_1(0) = 1$ for three-layer flow

Condition to stabilize (N+2)-Layer flow

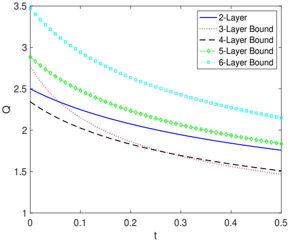
$$Q \le \min_{j=0,\ldots,N} H_j$$

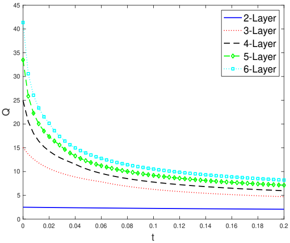
where

$$H_0 = \frac{1}{6R_0}\frac{T_0 n(n^2-1)}{n(\mu_1-\mu_i)-\mu_i},$$

$$H_j = \frac{1}{6R_j}\frac{T_j(n^2-1)}{\mu_{j+1}-\mu_j},$$

$$H_N = \frac{1}{6R_N}\frac{n(n^2-1)}{n(1-\mu_N)-1}.$$

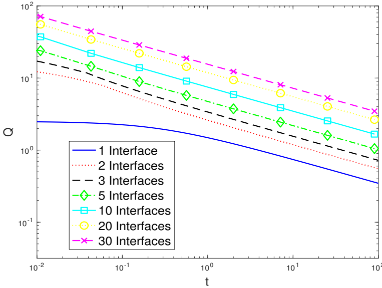




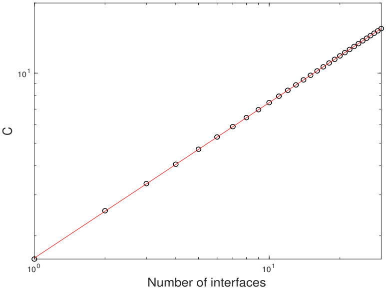
Maximum injection rate for stable flow

Examine the long time behavior of the maximum injection rate for a stable flow

$$Q_M(t) = Ct^{\alpha}$$

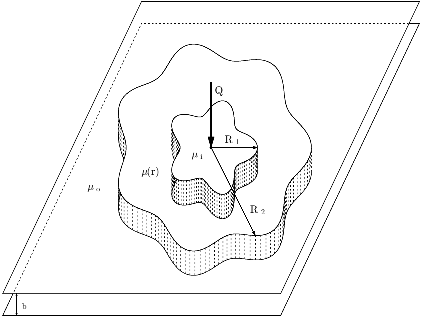


$$Q_M \propto t^{-1/3}$$



$$C \propto N_I^{2/3}$$

Variable Viscosity Radial Flow



Notation:

Q - injection rate

R_1 - radius of inner interface

R_2 - radius of outer interface

μ_i - viscosity of innermost (injected) fluid

μ_o - viscosity of the outermost fluid

$\mu(r)$ - viscosity of the intermediate fluid

Advection Equation

$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = 0$

Challenges of Radial Flow:

1. The solution depends on the curvature of the interfaces which evolve with time.
2. The thickness of the middle layer changes with time.
3. The viscous profile is time-dependent.

Variable Viscosity Radial Flow

In order to overcome the time-dependence, we use a change of variables

$$\zeta = \frac{r^2 - R_0^2(t)}{R_2^2(t) - R_0^2(t)} = \frac{r^2 - R_0^2(t)}{R_2^2(0)}.$$

where $R_2(t)$ is the position of the outermost interface and $R_0^2(t) = Qt/\pi$. Then the basic solution is constant in time. If we perturb the velocity in the ζ direction and write the disturbance as

$$\widetilde{u_\zeta} = f(\zeta)e^{in\theta+\sigma t},$$

we get the following eigenvalue problem:

$$\Big(\left(\zeta R_2^2(0) + R_0^2(\tau)\right) \mu f'(\zeta)\Big)' - \frac{n^2 R_2^4(0)}{4(\zeta R_2^2(0) + R_0^2(\tau))}\mu f(\zeta) = -\frac{Qn^2 R_2^2(0)}{4\pi(\zeta R_2^2(0) + R_0^2(\tau))}\frac{1}{\sigma(\tau)}\frac{d\mu}{d\zeta}f(\zeta)$$

which holds within the intermediate layer.

Variable Viscosity Radial Flow

In the new coordinates, the interfaces are located at $\zeta = \zeta_1$ and $\zeta = 1$. The interface conditions are

$$\frac{2R_1^2(\tau)}{nR_2^2(0)}\mu(\zeta_1)f'(\zeta_1) = \left(\mu_i - \frac{F_1}{\sigma(\tau)}\right) f(\zeta_1)$$
$$-\frac{2R_2^2(\tau)}{nR_2^2(0)}\mu(1)f'(1) = \left(\mu_o - \frac{F_2}{\sigma(\tau)}\right) f(1).$$

where

$$F_1 = \frac{Qn}{2\pi R_1^2(\tau)}\Big(\mu(\zeta_1) - \mu_i\Big) - T_1\frac{n^3 - n}{R_1^3(\tau)},$$
$$F_2 = \frac{Qn}{2\pi R_2^2(\tau)}\Big(\mu_o - \mu(1)\Big) - T_2\frac{n^3 - n}{R_2^3(\tau)}.$$

Upper Bound

The modal upper bound is:

$$\sigma < \max\left\{\frac{F_1}{\mu_i}, \frac{F_2}{\mu_o}, \frac{Q}{\pi R_2^2(0)}\frac{1}{\mu_i}\sup_{\zeta\in(\zeta_1,1)}\mu'(\zeta)\right\}.$$

By taking the maximum over all n, the absolute upper bound is:

$$\begin{aligned}\sigma < \max\Bigg\{&\frac{2T_1}{\mu_i R_1^3(\tau)}\left(\frac{QR_1(\tau)}{6\pi T_1}(\mu(\zeta_1)-\mu_i)+\frac{1}{3}\right)^{\frac{3}{2}},\\ &\frac{2T_2}{\mu_o R_2^3(\tau)}\left(\frac{QR_2(\tau)}{6\pi T_2}(\mu_o-\mu(1))+\frac{1}{3}\right)^{\frac{3}{2}},\\ &\frac{Q}{\pi R_2^2(0)}\frac{1}{\mu_i}\sup_{\zeta\in(\zeta_1,1)}\mu'(\zeta)\Bigg\}.\end{aligned}$$

Eigenvalues and Eigenfunctions

Using $\lambda = 1/\sigma$,

$$\left.\begin{array}{l}\left(\left(\zeta R_2^2(0) + R_0^2(\tau)\right)\mu f'(\zeta)\right)' - \left(\frac{n^2 R_2^4(0)}{4(\zeta R_2^2(0)+R_0^2(\tau))}\mu - \frac{Q n^2 R_2^2(0)}{4\pi(\zeta R_2^2(0)+R_0^2(\tau))}\mu'\lambda\right) f(\zeta) = 0,\\ \left(\mu_i - \lambda F_1\right) f(\zeta_1) - \frac{2R_1^2(\tau)}{R n_2^2(0)}\mu(\zeta_1) f'(\zeta_1) = 0,\\ \left(\mu_o - \lambda F_2\right) f(1) + \frac{2R_2^2(\tau)}{R n_2^2(0)}\mu(1) f'(1) = 0.\end{array}\right\} \quad (1)$$

Theorem

Let F_1, F_2, Q, n, μ_i, $\mu_o > 0$. Let $\mu(\zeta)$ be a positive, strictly increasing function in $C^1([\zeta_1, 1])$. Then the eigenvalue problem (1) *has a countably infinite number of real eigenvalues that can be ordered*

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots$$

with the property that for the corresponding eigenfunctions, $\{f_i\}_{i=0}^{\infty}$, f_i has exactly i zeros in the interval $(\zeta_1, 1)$. Additionally, the eigenfunctions are continuous with a continuous derivative.

Eigenvalues and Eigenfunctions

We can write equation (1) in the form $Af = \lambda f$.

Define the measure

$$\nu(M) := \begin{cases} \frac{nR_2^2(0)F_1}{2}, & \text{for } M = \{\zeta_1\} \\ \int_M r(\zeta)d\zeta, & \text{for } M \subset (\zeta_1, 1) \\ \frac{nR_2^2(0)F_2}{2}, & \text{for } M = \{1\}. \end{cases}$$

where

$$r(\zeta) = \frac{Qn^2 R_2^2(0)\mu'(\zeta)}{4\pi(\zeta R_2^2(0) + R_0^2(\tau))}$$

Consider the Hilbert space $H := L^2([\zeta_1, 1]; \nu)$

J. Walter, Regular eigenvalue problems with eigenvalue parameter in the boundary condition. *Math. Z.*, 133, 1973.

Self-adjoint Operator

Theorem

Let F_1, F_2, Q, n, μ_i, $\mu_o > 0$. Let $\mu(\zeta)$ be a positive, strictly increasing function in $C^1([\zeta_1, 1])$.

A is a self-adjoint operator on H and for any $u \in H$,

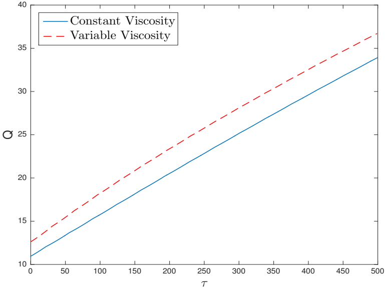
$$u = \sum_{k=0}^{\infty} f_k \int_{\zeta_1}^{1} u(\zeta) f_k(\zeta) d\nu,$$

where the f_k are the eigenfunctions of A.

J. Walter, Regular eigenvalue problems with eigenvalue parameter in the boundary condition. *Math. Z.*, 133, 1973.

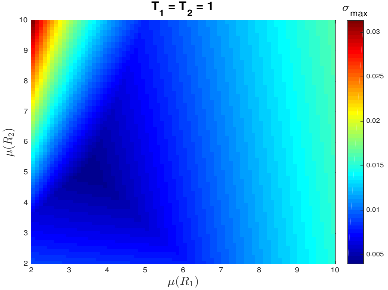
Variable Injection Rate

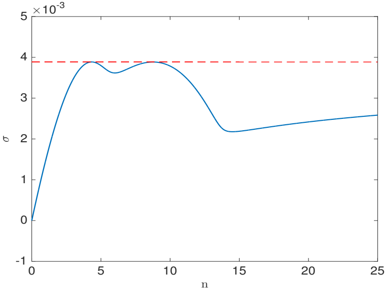
Compare the injection rate that maintains a fixed value of σ.

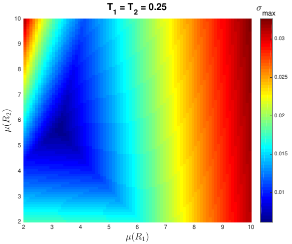


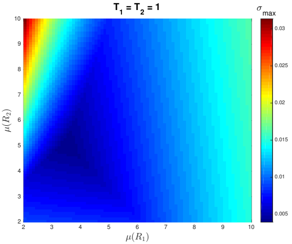
The variable viscosity flow has the same average viscosity as the constant viscosity flow

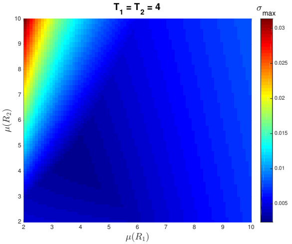
Optimal Linear Viscous Profile



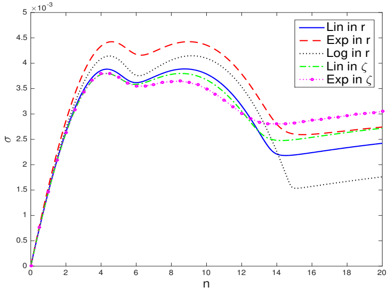


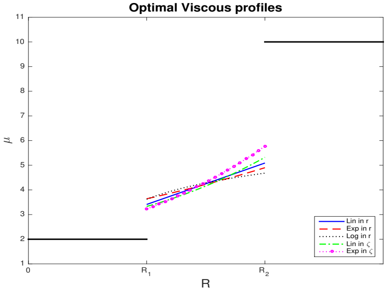






Optimal Viscous Profile





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Thank you

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