Plant invasion dynamics: a birth-jump approach

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- Understanding the mechanisms behind plant invasion is important.
- Invasion patterns depend on many factors, *e.g.* scale, dispersal mechanism, environment, etc.
- Motivation: How does **dispersal mechanisms** affect the advance of a species, and an ecotone, on an **environmental gradient**?

Figure: Mikania Micrantha

- An aquatic invasive plant, Eurasian watermilfoil, reduced Vermont lakeside property values by 16% and Wisconsin lakeside property by 13%.
- Salt Cedar, an invasive tree, costs the Western states $420-2,800 annual per 2.5 acres in water loss, as well as flood control losses.
- Annually, non-native species borne in the ballast or the hulls of ships cost the Great Lakes Regions $ 200 million to control.

U.S. Fish & Wildlife Service

The Cost of Invasive Species

Zebra mussels invaded U.S. waters and have caused millions of dollars of damage by occluding pipes in municipal and industrial raw-water systems

The negative consequences of invasive species are far-reaching, costing the United States billions of dollars in damages every year. Compounding the problem is that these harmful invaders spread at astonishing rates. Such infestations of invasive plants and animals can negatively affect property values, agricultural productivity, public utility operations, native fisheries, tourism, outdoor recreation, and the overall health of an ecosystem.

The most widely referenced paper (Pimental et al. 2005) on this issue reports that invasive species cost the United States more than $120 billion in damages every year.

In 2011 alone, the Department of the Interior will spend $100 million on invasive species prevention, early detection and rapid response, control and management, research, outreach, international cooperation and habitat restoration.

The Environmental Impacts

In Executive Order 13112, invasive species is defined as an alien species whose introduction does or is likely to cause economic or environmental harm or harm to human health. Invasive species typically harm native species through predation, habitat degradation and competition for shared resources.

Invasive species are a leading cause of population decline and extinction in animals. For example:

- More than 400 of the over 1,300 species currently protected under the Endangered Species Act, and more than 180 candidate species for listing are considered to be at risk at least partly due to displacement by, competition with, and predation by invasive species.
- Invasive species are a leading factor in freshwater fish extinctions and endangerments.
- Brown tree snakes have been implicated in the precipitous decline in native forest birds and the modern extinction of at least 10 species in Guam.

More Facts about the Cost of Invasives:

- If zebra and quagga mussels invade the Columbia River, they could cost hydroelectric facilities alone up to $250-300 million annually. This does not inclued costs associated with environmental damages or increased operating expenses to hatcheries and water diversions.
- Annually, the Massachusetts Department of Conservation and Recreation spends $250,000 on staff, $30, 000 on equipment and $25,000 on publications related to zebra mussel prevention and control. The state will spend an additional $71,000 over 5 months to install new boat ramp monitors for zebra mussels.
- An aquatic invasive plant, Eurasian watermilfoil, reduced Vermont lakefront property values up to 16 percent and Wisconsin lakefront property values by 13 percent.
- From 2010 to 2020, an invasive forest pathogen (*Phytophthora ramorum*), called sudden oak death, is projected to cost $7.5 million in tree treatment, removal and replacement costs, corresponding to a $135 million loss in residential property values for California.
- Salt cedar (*Tamarisk* spp.), an invasive tree, costs the western states $450-2,800 annually per 2.5 acres (1 hectare) in water loss (municipal, agricultural and hydropower) as well as flood control losses. Eradication and re-vegetation projects are estimated to be $7,400 per 2.5 acres.
- Annually, black and Norway rats consume stored grains and destroy other property valued over $19 billion.
- Annually, nonnative species borne in the ballast or hulls of ships cost the Great Lakes Region $200 million to control.
- U.S. agriculture loses $13 billion annually in crops from invasive insects, such as vine mealybugs.

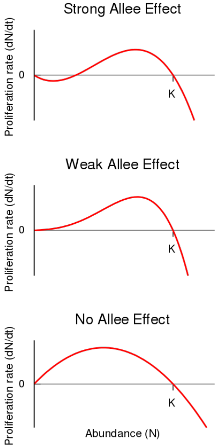
Models for Invasion

- Classical Reaction Diffusion: $u_t = \Delta u + f(u)$
  - Logistic: $f(u) = u(1-u)$.
  - Allee Effect: $f(u) = u(1-u)(1-\theta)$.
- Position-Jump Process:

$$u_t = \int \mathcal{J}(x,y)u(y,t)\,dy - \int \mathcal{J}(y,x)u(x,t)\,dy + f(u).$$

  - $\mathcal{J}(x,y)$ probability that a particle in y will jump to x.
- **Invasion**: Solutions of the form $U(x-ct)$ that satisfy the ODE:

$$\begin{cases} -cU' = U'' + f(U) \\ U(-\infty) = 1 \text{ and } U(+\infty) = 0. \end{cases}$$



Birth-jump Process

- Process where reproduction and dispersal are inextricably linked:
  - Ovarian cancer (T. Hillen, H. Enderling, P. Hahnfeld 13')
  - Spread of wildfires (T. Hillen, B. Greese, J. Martin, G. de Vries 15')
  - Plant dynamics (N. Rodríguez and G. Malanson)

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$$u_t(x,t) = \underbrace{\int_{\mathbb{R}^2} S(x,y)\beta(u(y,t),y)g(u(x,t),x)u(y,t)\,dy}_{\text{Birth-jump process}} - \delta(u(x,t),x)u(x,t),$$

- Notation
  - $\mathbb{R}^2$: a lattice with $\vec{x}_{ij} = (i\ell, j\ell)$ with $i, j \in \{0, \pm 1, \pm 2, \ldots, \pm n, \ldots\}$.
  - Discretize time into periods of δt time.
  - Let $n_{ij}(t)$ be the number of plants at location $\vec{x}_{ij}$:

$$\begin{aligned}\#\text{ of plants at time } t + \delta t = &\text{ arriving seeds that germinate} \\ &- \text{dying plants from time } t.\end{aligned}$$

- Ignore time-delay.

- Proliferation: production of seeds that end in location $\vec{x}_{ij}$ and germinate.
- Let $p_\beta(\vec{y})$ be probability that a plant at location $\vec{y}$ will produce a seed during the period $(t, t+\delta t)$.
- Let $s_{\vec{x}\vec{y}}$ be the relocation probabilities: the probability that a seed from a plant at location $\vec{y}$ will land in location $\vec{x}$.
- Let $p_g(\vec{x})$ be the probability that a seed at location $\vec{x}$ will germinate.

Individual Based Model

- Proliferation term at $\vec{x}_{ij}$:

$$\sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} p_\beta(\vec{x}_{km}) p_g(\vec{x}_{ij}) s_{\vec{x}_{km}\vec{x}_{ij}} n_{km}(t).$$

- Survival term:

$$(1 - p_\delta) n_{ij}(t),$$

- Final equation:

$$\boxed{n_{ij}(t+\delta t) - n_{ij}(t) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} p_\beta(\vec{x}_{km}) p_g(\vec{x}_{ij}) s_{\vec{x}_{km}\vec{x}_{ij}} n_{km}(t) - p_\delta(\vec{x}_{ij}) n_{ij}(t).}$$

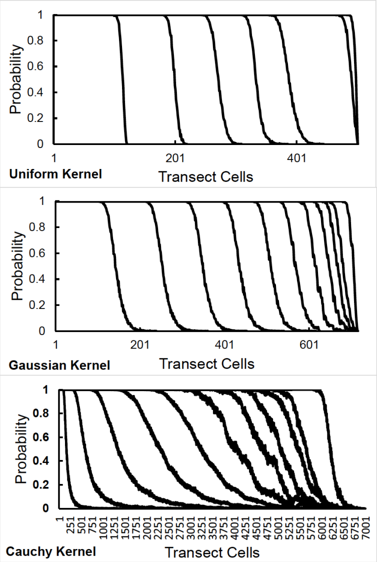
Proliferation, Germination Probabilities, and Relocation Kernel

- Assume a Poisson process for proliferation, germination, and decay:

$$p_\beta(\vec{x}_{ij}) = 1 - e^{-\tilde{\beta}(\vec{x}_{ij}, n_{ij})\delta t}.$$

- Remark: density-dependence.
- Relocation Kernel:
  - Wind
  - Animals
  - Gravity
- Dispersal via each method p_i with $\sum_{k=1}^{3} p_k = 1$.

ABM Result



Probability
1
0.8
0.6
0.4
0.2
0
1
201
401
Uniform Kernel
Transect Cells
Probability
1
0.8
0.6
0.4
0.2
0
1
201
401
601
Gaussian Kernel
Transect Cells
Probability
1
0.8
0.6
0.4
0.2
0
1 251 501 751 1001 1251 1501 1751 2001 2251 2501 2751 3001 3251 3501 3751 4001 4251 4501 4751 5001 5251 5501 5751 6001 6251 6501 6751 7001
Cauchy Kernel
Transect Cells

Continuum Derivation

- Density:

$$u(\mathbf{x}, t) = n_{ij}/\ell^2.$$

- Rewrite:

$$\frac{u(\mathbf{x}, t+\delta t) - u(\mathbf{x}, t)}{\delta t} = \frac{1}{\delta t} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} p_\beta(\vec{x}_{km}) p_g(\vec{x}_{ij}) s_{\vec{x}_{ij}\vec{x}_{ij}} u(\vec{y}_{km}, t)$$
$$- \frac{1}{\delta t} p_\delta(\vec{x}_{ij}) u(\mathbf{x}, t).$$

- A Taylor series expansion for $p_\beta(\vec{x}_{km})p_g(\vec{x}_{ij})$ and p_δ then yields:

$$u_t(x, t) = \sum_{n=1}^{3} p_n \int_{\mathbb{R}} S_n(x, y)\beta(u(y, t), y)g(u(x, t), x)u(y, t)\, dy$$
$$- \delta(u(x, t), x)u(x, t).$$

Diffusion Limit ($\mathbb{R}$)

- Approximate the proliferation term:

$$\int_{\mathbb{R}} S(x,y)g(u(x,t),x)\beta(u(y,t),y)u(y,t)\,dy$$

- Let $h(y,t) = \beta(u(y,t),y)u(y,t)$ and β is regular enough:

$$\begin{aligned}
\int_{\mathbb{R}} S(x,y)h(y,t)g(u(x,t))dy &= \int_{\mathbb{R}} S(x,y)g(u(x,t),x)\sum_{k=0}^{\infty}\frac{1}{k!}\partial_x^k h(x,t)(y-x)^k dy \\
&= g(u(x,t),x)\sum_{k=0}^{\infty} M_k(x)\partial_x^k h(x,t) \\
&= g(u(x,t),x)\sum_{k=0}^{\infty} M_k(x)\partial_x^k[\beta(u(x,t),x)u(x,t)],
\end{aligned}$$

- Moments

$$M_k(x) = \frac{1}{k!}\int_{\mathbb{R}} S(x,y)(y-x)^k\,dy$$

- If the moments exist and form an asymptotic sequence one can safely truncate after the first few moments.

Symmetric Potential with Finite Second Moment

- Obtaining:

$$u_t = dg(u)(\beta(u)u(x,t))_{xx} + (g(u)\beta(u) - \delta(u))u(x,t),$$

where $d = \sum_{n=1}^{3} p_n M_2^n$ and $\sum_{n=1}^{3} p_n M_0^n = 1$.

- Remarks:
  - g and β appear in both reaction and diffusion
  - Asymmetric potentials

Symmetric Potential with Finite Second Moment

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- Remarks:
  - g and β appear in both reaction and diffusion
  - Asymmetric potentials
- Rewrite equation:

$$\frac{1}{g(u)} u_t = (D(u)u_x)_x + f(u),$$

with

$$D(u) := \beta'(u)u + \beta(u) \quad \text{and} \quad f(u) := \left(\beta(u) - \frac{\delta(u)}{g(u)}\right) u.$$

C^r Traveling Wave Solutions

- Non-degenerate case: $D(u) > 0$.
- Take as examples:

$$g(u) = \frac{1}{1+u}, \ \beta(u) = \mu(\gamma + u), \ \delta(u) = \mu u,$$

with $\mu, \gamma > 0$.

- Regularity: for $r > 0$

(H1) $D \in C^r([0,\infty)), \ D(z) > 0$ for $z \in [0,1]$.

(H2) $g \in C^r([0,\infty)), \ g(z) > 0$ and $g'(z) < 0$ for $z \in [0,1]$.

(H3) $f \in C^r([0,\infty)), \ f'(1) < 0$.

Non-degenerate case: C^r Traveling Wave Solutions

- Let $V = D(U)U'$ then rewrite equation system of two ODEs:

$$\begin{cases} U' &= \frac{V}{D(U)}, \\ V' &= -\frac{cV}{g(U)D(U)} - f(U). \end{cases} \tag{1}$$

- Minimum speed:

$$c^* = 2g(0)\sqrt{f'(0)D(0)}.$$

Theorem

- *There exists traveling wave solutions for all $c \geq c^*$.*
- *No traveling wave solution exists for $c < c^*$.*

- Degenerate: $D(0) = 0$, $D(U) > 0, D'(U) > 0$ and $D''(U) \neq 0$.
    - Monostable Case: $f'(0) \neq 0$
    - Bistable Degenerate Case: $f'(0) = 0$, $f''(0) > 0$
- Example

$$g(u) = \frac{1}{1+u}, \ \beta(u) = \mu u, \ \delta(u) = \bar{\mu} u,$$

with $0 < \bar{\mu} < \mu$.

- To maintain a logarithmic-type growth chose δ appropriately:

$$f(u) = u^2[\mu - \bar{\mu} - \bar{\mu} u] \quad \text{and} \quad D(u) = 2\mu u,$$

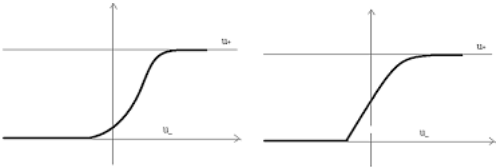
- The degeneracy of $D(u)$ leads to our system becoming an ODE when $u = 0$ and a parabolic PDE for $u > 0$.

Sharp Traveling Wave Solutions

Sharp traveling wave: If there exists a value c and $z^* \in \mathbb{R}$ such that $U(x-ct)$ satisfies the traveling wave equation for all $z \in (-\infty, z^*)$ and

$$
\begin{aligned}
&U(-\infty) = 1, \quad U(z^{*-}) = U(z^{*+}) = 0, \quad \text{and} \quad U(z) = 0 \text{ for } z \in (z^*, \infty); \\
&U'(z^{*-}) = -\frac{c}{g(0)D'(0)}, \quad U'(z^{*+}) = 0, \quad U'(z) < 0 \text{ for } z \in (z^*, \infty),
\end{aligned}
\tag{2}
$$

then $U(x-ct)$ is a traveling wave solution with speed c of the **sharp-type**.



smooth traveling wave   sharp type traveling wave

- Variational formula for speed:

$$c = \frac{\int_0^{U(-\infty)} D(w)h(w)dw}{\int_{-\infty}^{z^*} \frac{D(U)}{g(U)}(U')^2 \; dz}.$$

- Existence:
  - (i) has no traveling wave solutions for speed $c < c^*$.
  - (ii) has a traveling wave solution $U(x - c^* t)$ of the sharp-type satisfying (2).
  - (iii) for $c > c^*$ has a strictly monotone continuously differentiable traveling wave solution $U(x - ct)$ satisfying $U(-\infty) = 1$ and $U(+\infty) = 0$.

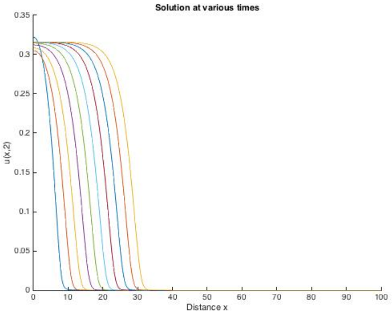


Figure: $c > c^*$: C^r traveling wave.

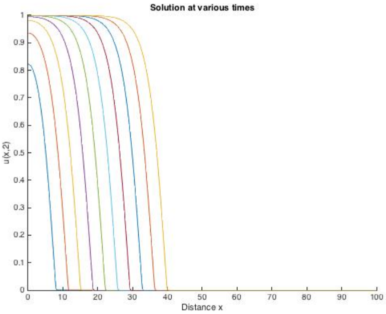


Figure: $c = c^*$: Sharp traveling wave.

Idea of Proof

- Traveling wave solutions satisfy:

$$D(U)U'' + D'(U)(U')^2 + f(U) + \frac{c}{g(U)}U' = 0,$$

- ODE form:

$$\begin{cases} U' = V, \\ D(U)V' = -D'(U)V^2 - f(U) - \frac{c}{g(U)}V. \end{cases}$$

- Change of variables:

$$\tau = \int_0^z \frac{ds}{D(u(s))} \Rightarrow \frac{d\tau}{dz} = \frac{1}{D(u(z))},$$

to get

$$\begin{cases} U' = D(U)V, \\ V' = -D'(U)V^2 - f(U) - \frac{c}{g(U)}V. \end{cases}$$

Idea of Proof

- System:

$$\begin{cases} U' = \underbrace{D(U)V}_{F(U,V)}, \\ V' = \underbrace{-D'(U)V^2 - f(U) - \frac{c}{g(U)}V}_{G(U,V)}. \end{cases}$$

- Steady-states are $P_0 := (0,0)$, $P_1 := (1,0)$, $P_c := \left(0, -\frac{c}{g(0)D'(0)}\right)$.
- P_c depends on the speed c
- Saddle-node bifurcation for $c = 0$

- At P_0 we have:

$$J[F,G]_{(0,0)} = \begin{bmatrix} 0 & 0 \\ -f'(0) & -\frac{c}{g(0)} \end{bmatrix}$$

- Eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -\frac{c}{g(0)} < 0$, with respective eigenvectors $(c/g(0), -f'(0))^T$ and $(0,1)^T$.
- Hence P_0 is a non-hyperbolic equilibrium and we need to do a second-order approximation of that system.
- Following the techniques of Andronov *et al.* to get that P_0 is a saddle node.
- P_1, P_c a saddle point.

- Monotonicity of trajectories
- Uniqueness of sharp wave speed
- Non-existence for small speeds
- Existence of a wave for large enough speeds for $c \geq \sqrt{M}$ with

$$M := \max_{s \in (0,1)} 4D'(u)f(u)g^2(u).$$

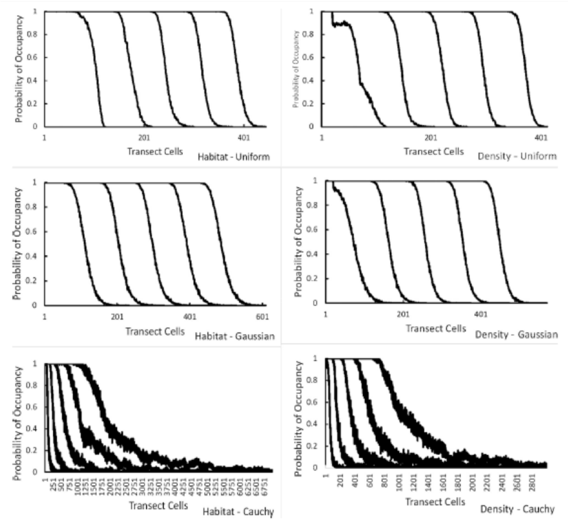
- Existence of a sharp wave for speed c^*.
  - Monotonicity of trajectory with respect to c.
  - The critical speed

$$c^* = \inf\{c > 0 : U_c = 1, v_c < 0\}.$$

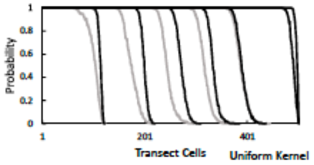
  is well defined and $V_{c^*} = 0$.

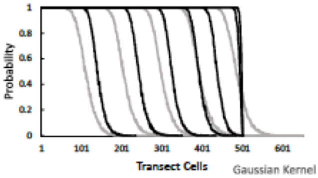
Dispersal Kernels

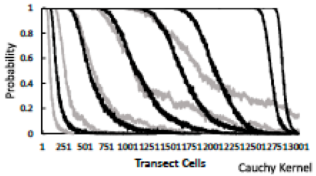
- Kernel effect: Uniform, Gaussian, and Cauchy.



- Linear gradient in direction of movement.
- Affects birth, establishment, and death.







Discussion

- Fatter tails lead to faster advance.
- Environmental gradients will slow the advance.
- Analysis on the nonlocal equation is still at a very infant stage
- Cauchy problem
- Planar traveling wave solutions
  - Existence and uniqueness
  - Qualitative behavior: asymptotic rates, monotonicity
  - Stability

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- George Malanson
- NSF
- You all!