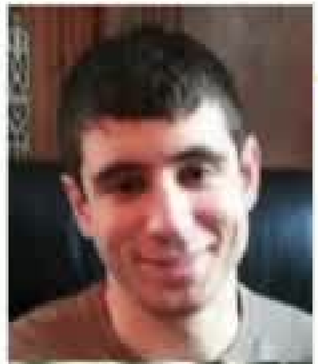
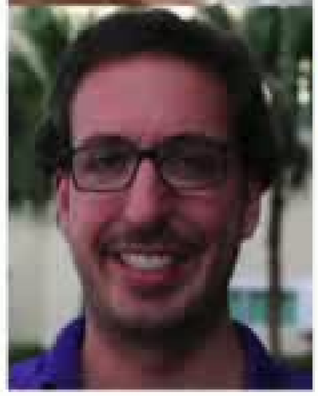
Homoclinic orbits in the Suspension bridge equation


Jean-Philippe Lessard


JB van den Berg


Maxime Breden


Maxime Murray

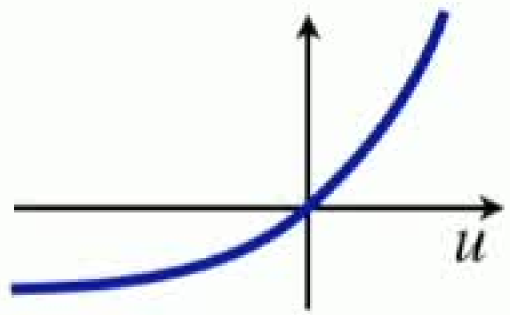
Suspension bridge equation

$$\frac{\partial^2 v}{\partial t^2} = -\frac{\partial^4 v}{\partial x^4} - e^v + 1$$

Some previous work

$$u'''' + c^2 u'' + e^u - 1 = 0$$

- periodic solutions: Peletier & Troy



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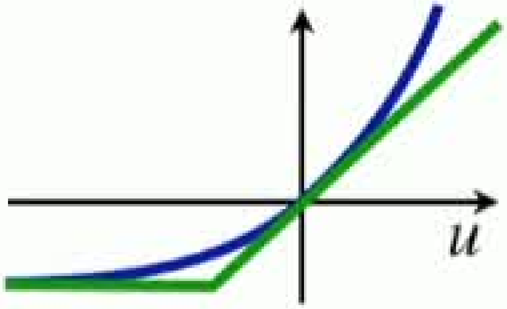
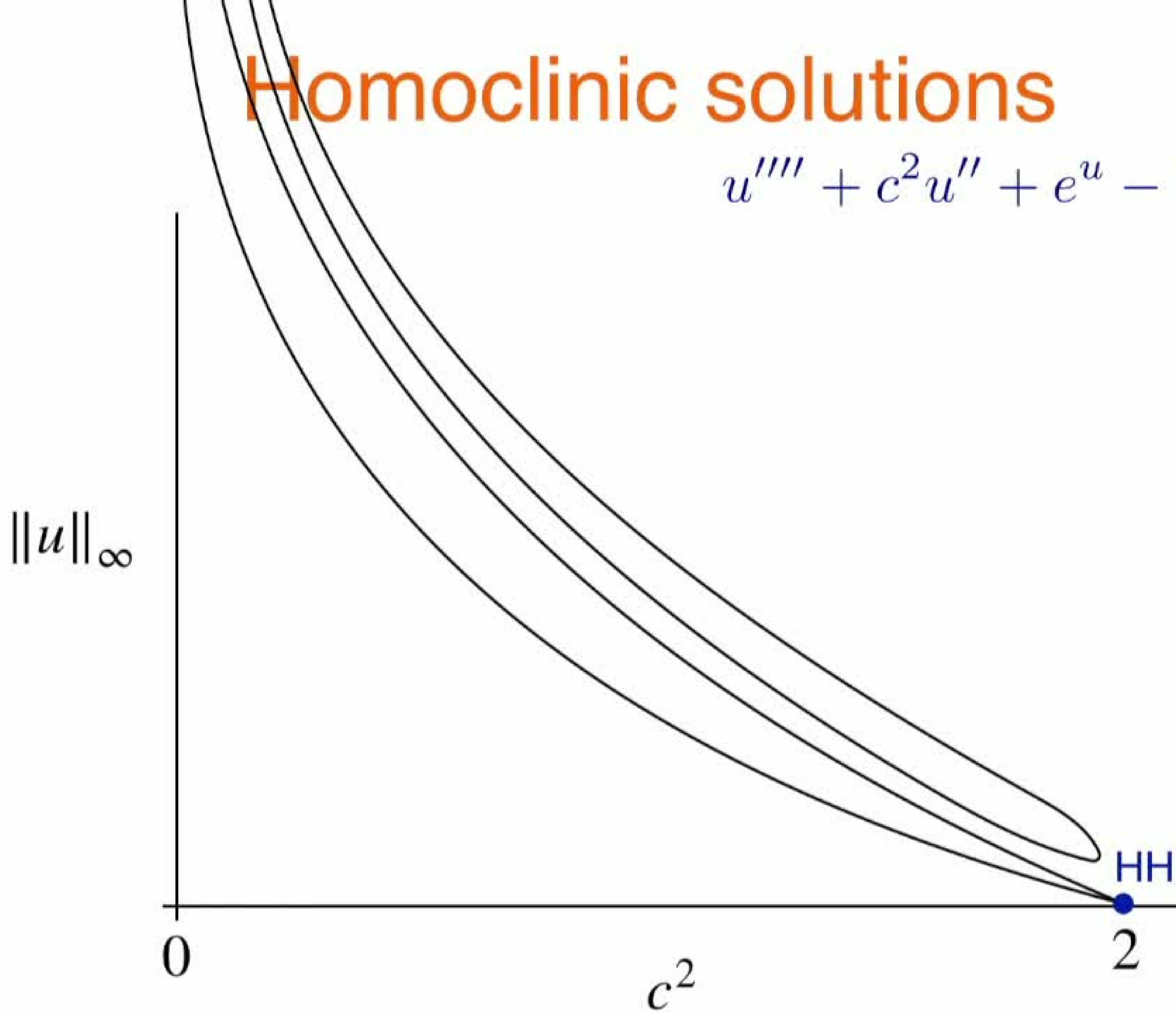


$$u'''' + c^2 u'' + \max\{u, -1\} = 0$$

- homoclinic solutions:
  Mc-Kenna, Chen, Champneys
  theorems, numerics, conjectures

Homoclinic solutions

$$u'''' + c^2 u'' + e^u - 1 = 0$$



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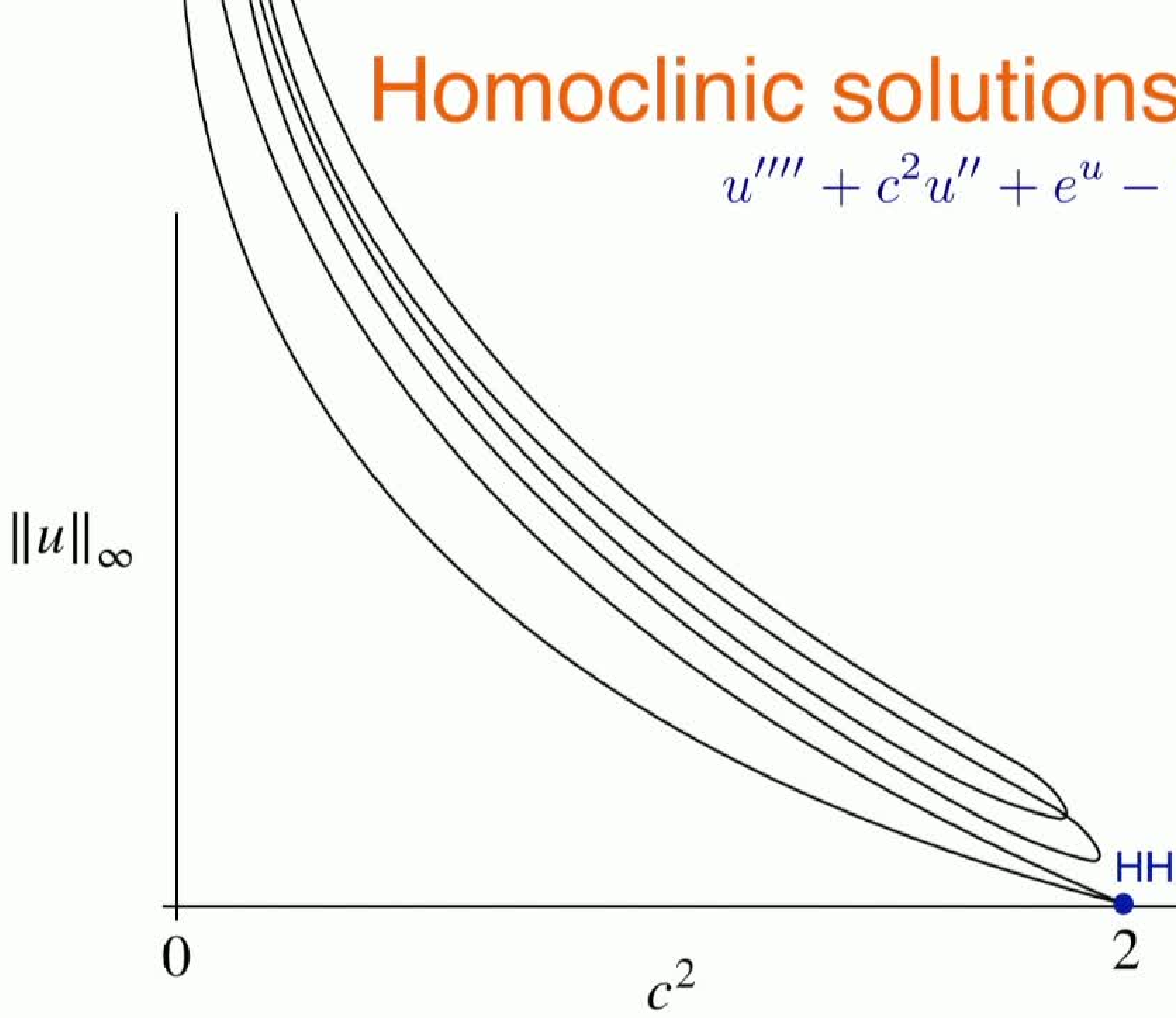

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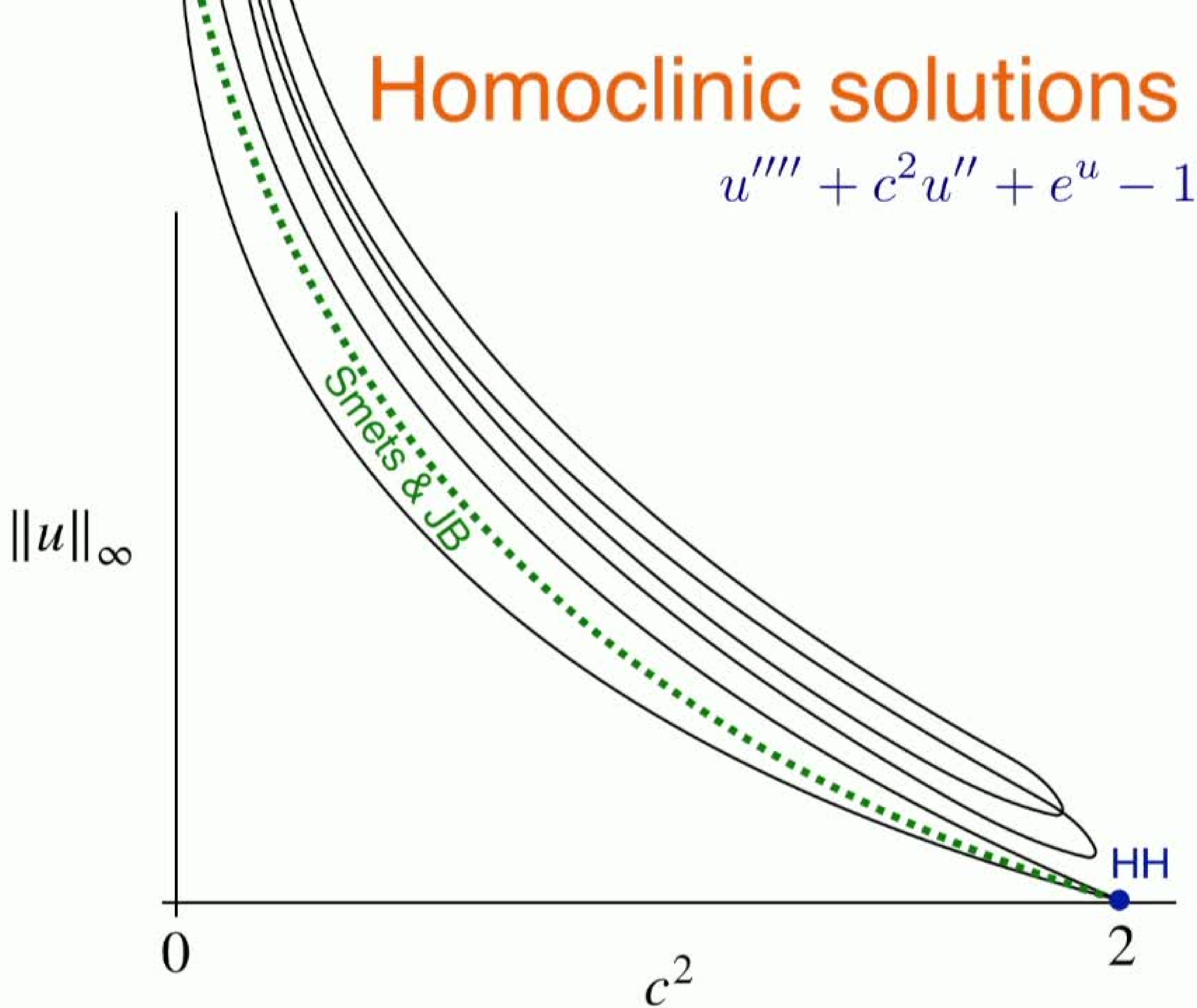

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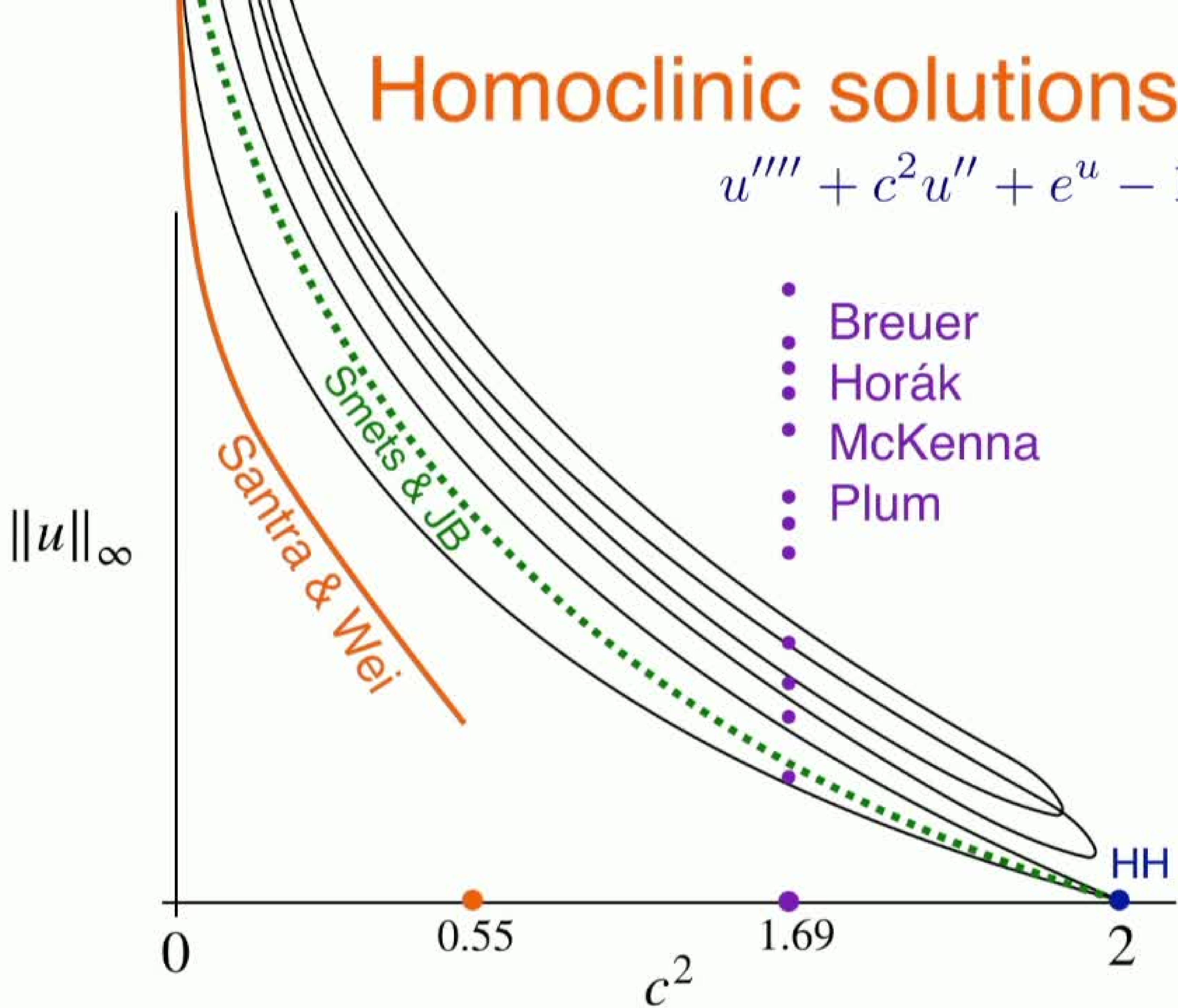

Write it as a system

$$u'''' + c^2 u'' + e^u - 1 = 0$$

$$\begin{cases} u_1 = e^u - 1 \\ u_2 = u' \\ u_3 = u'' \\ u_4 = u''' \end{cases} \qquad \begin{cases} u_1' = (u_1 + 1)u_2 \\ u_2' = u_3 \\ u_2' = u_4 \\ u_2' = -c^2 u_3 - u_1 \end{cases}$$

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$$\frac{du}{dt} = f(u)$$

Equilibria u_-, u_+

$$\lim_{t \to -\infty} u(t) = u_- \qquad \lim_{t \to +\infty} u(t) = u_+$$

Boundary Value Problem

$$\begin{cases} u'(t) = f(u(t)) \quad t \in [-L, L] \\ u(-L) \in W^u_{\mathrm{loc}}(u_-) \\ u(L) \in W^s_{\mathrm{loc}}(u_+) \end{cases}$$

- local manifolds $W^u_{\mathrm{loc}}(u_-)$ and $W^s_{\mathrm{loc}}(u_+)$
  - use parametrization method

    Cabré, Fontich, de la Llave

    Taylor series

- in transition $-L \leq t \leq L$
  - use Chebyshev series

    Chebyshev $\approx$ Fourier

Lessard-Reinhardt
Sheombarsing-JB

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Local (un)stable manifold

$P\colon \mathbb{R}^2 \to \mathbb{R}^4$

- parametrizes local unstable manifold
- analytic (non-resonance condition)
- conjugates linear flow
- similar to normal form
- compute P_N iteratively
- error estimate (work)
- control of derivatives (analyticity)

Mireles-James

Chebyshev polynomials

$$\begin{cases} u'(t) = Lf(u(t)) & t \in [-1,1] \\ u(-1) = P(\theta) \\ u(1) = Q(\phi) \end{cases}$$

- $u(t) = a_0 + 2\sum_{k=1}^{\infty} a_k T_k(t)$
- $T_k(\cos\theta) = \cos(k\theta)$

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$$\begin{cases} 2ka_k = L(\widehat{f}_{k-1}(a) - \widehat{f}_{k+1}(a)) & k = 1, 2, \ldots \\ a_0 + 2\sum(-1)^k a_k = P(\theta) \\ a_0 + 2\sum a_k = Q(\phi) \end{cases}$$

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Unknowns: $\begin{cases} \theta \in \mathbb{R}^2 \\ \phi \in \mathbb{R}^2 \\ a_k \in \mathbb{R}^4 & k = 0, 1, 2, \ldots \end{cases}$

a-posteriori Newton-Kantorovich

$$F(x) = 0$$

Infinite dimensions

infinite dimensional nonlinear problem

$$F(x) = 0$$

$F : X \to X'$ $\quad$ X, X' Banach spaces

Finite dimensional reduction

N-dimensional subspaces

$$X_N \subset X \qquad X'_N \subset X'$$

- truncated problem $\quad F_N : X_N \to X'_N$
- solve numerically $\quad F_N(\overline{x}_N) \approx 0$ 
- numerical “solution” $\quad \overline{x}_N = \overline{x} \in X_N \subset X$

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Fixed point operator

$$F(x) = 0 \qquad F : X \to X'$$

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$$T(x) = x - DF(x)^{-1}F(x)$$

Contraction mapping

- T maps $B_r(\overline{x}) \subset X$ into itself
- $\|T(x) - T(\tilde{x})\|_X \leq \kappa \|x - \tilde{x}\|_X \quad \kappa < 1$

Analytic estimates

$$\|T(\overline{x}) - \overline{x}\|_X \leq Y$$

$$\|DT(x)\|_{B(X)} \leq Z(r) \quad \forall x \in B_r(\overline{x})$$

Inequality $Y + \hat{r} Z(\hat{r}) < \hat{r}$

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x_{sol}

$\overline{x}$

Intlab

a-posteriori Newton-Kantorovich

Statement: assume bounds

$$\|T(\overline{x}) - \overline{x}\|_X \le Y$$

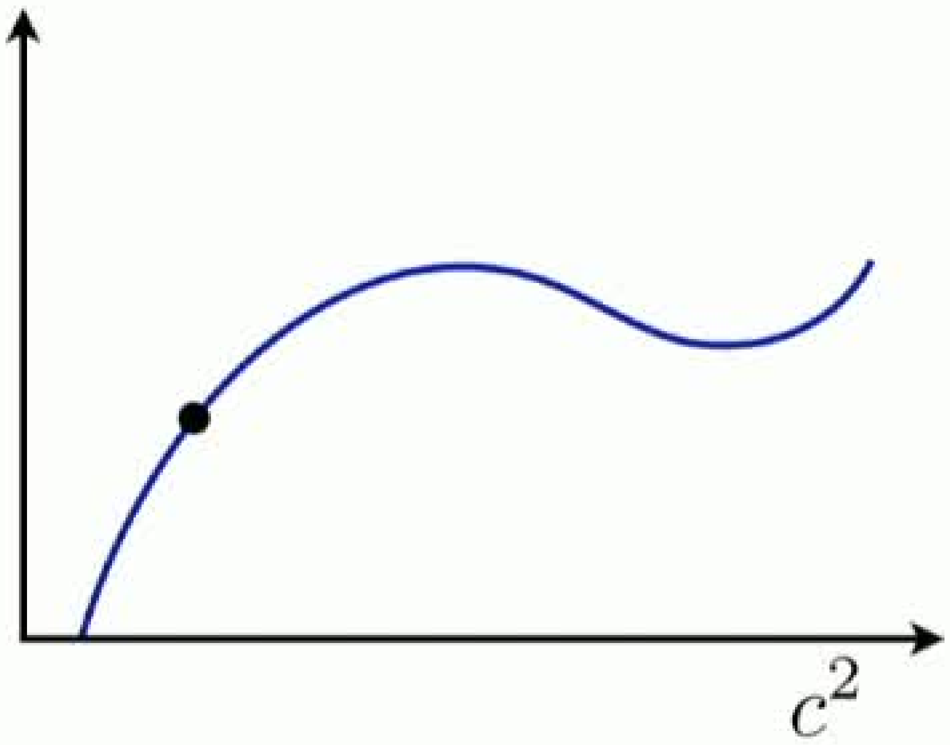
$$\|DT(x)\|_{B(X)} \le Z(r) \quad \forall x \in B_r(\overline{x})$$

If $Y + \hat{r}Z(\hat{r}) < \hat{r}$

Continuation

Continuation parameter c^2

Uniform contraction principle



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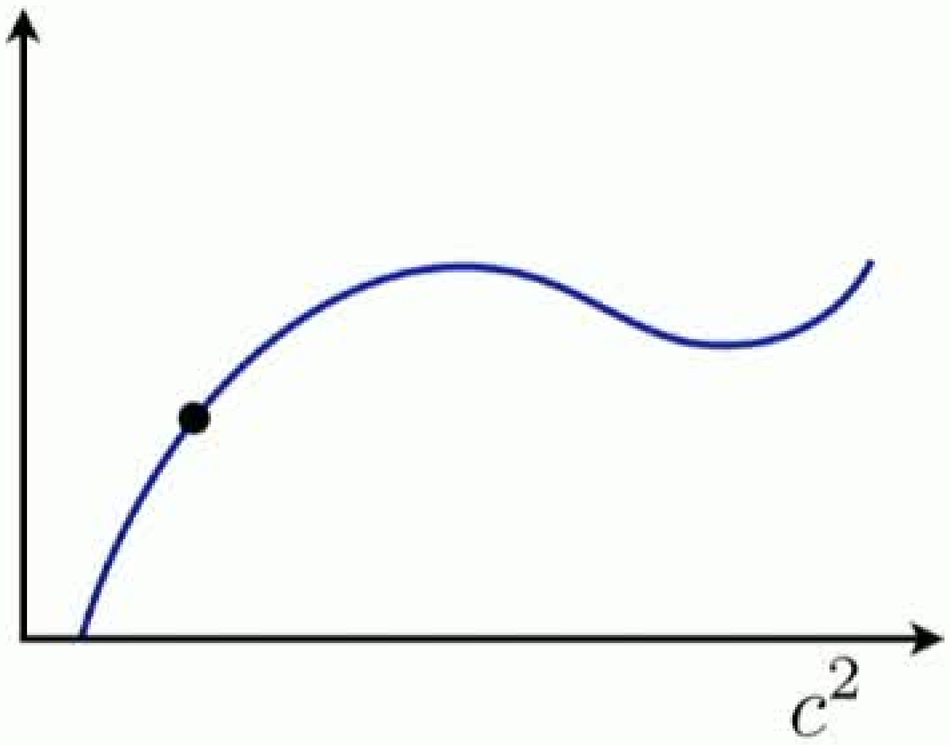

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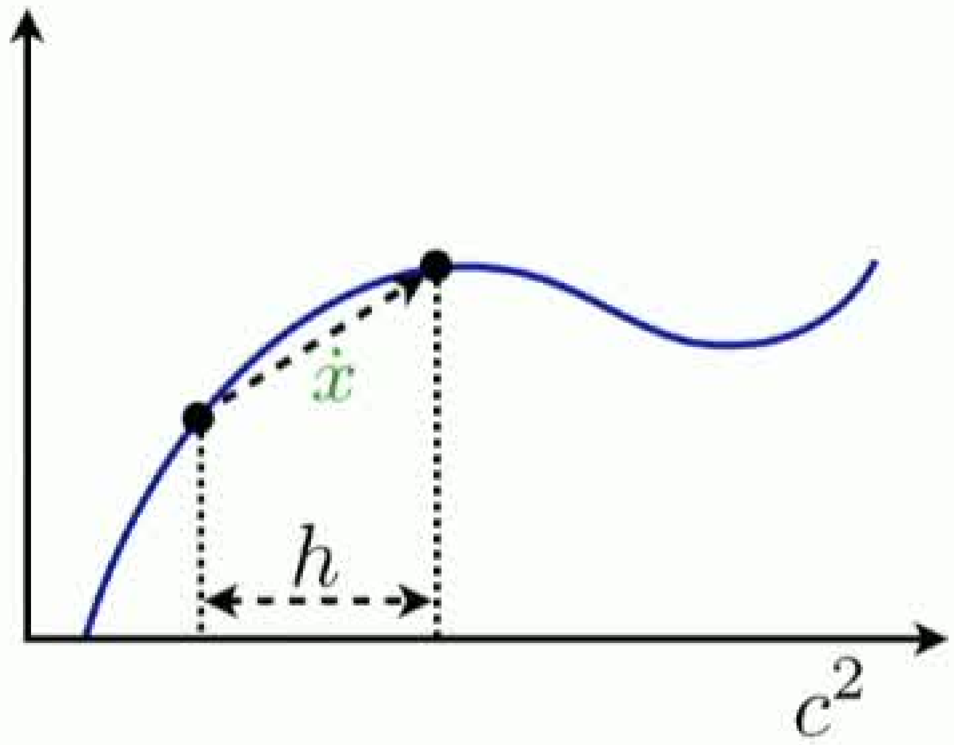
Predictor-corrector algorithm

- “tangent” vector $\dot{x}$
- replace $\overline{x}$ by $\overline{x} + h\dot{x}$
- h dependent estimates: $P(r,h) < 0$



$$P(r,h) = (\varepsilon_1 + \varepsilon_3 h) - (1 - \varepsilon_2) r + O(r^2, rh, h^2)$$

- ε_1 small due to choice of $\overline{x}$
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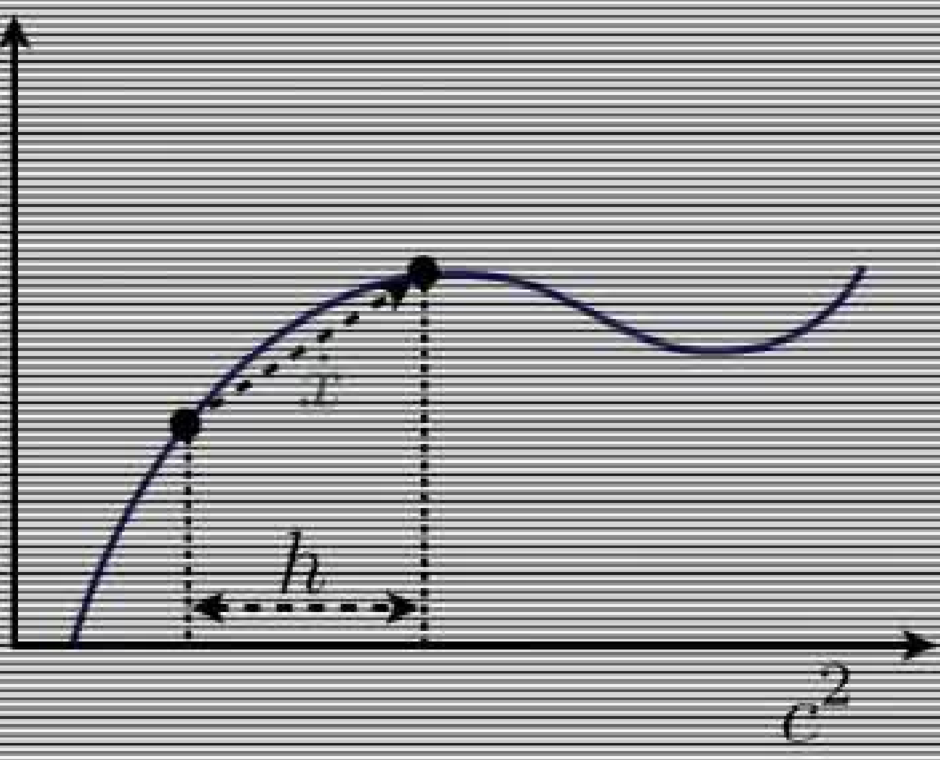

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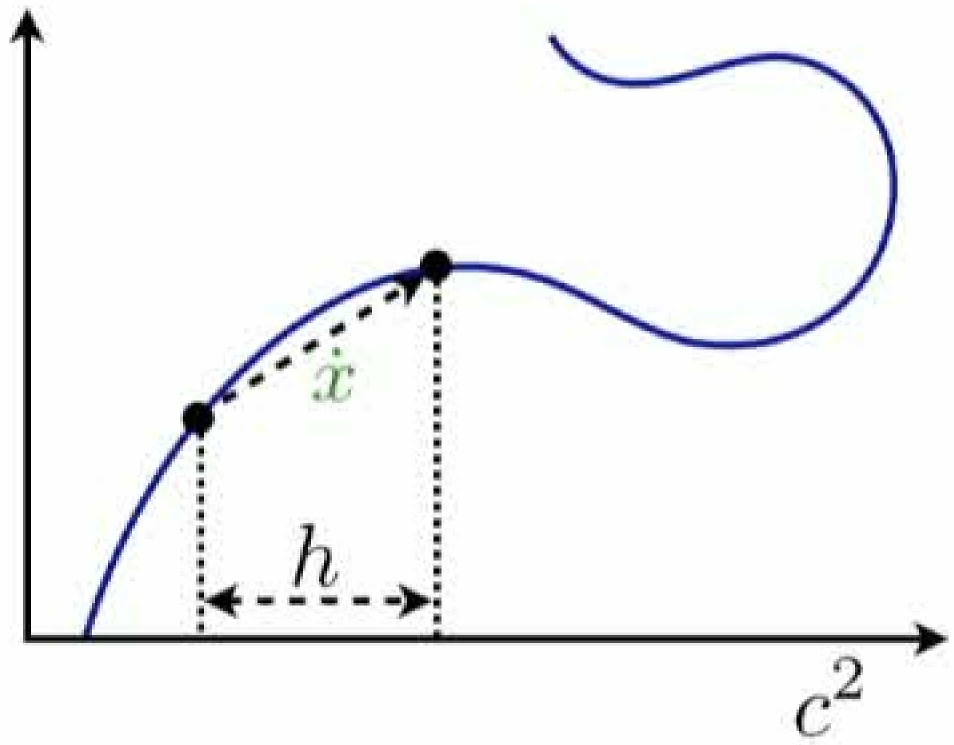

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- domain decomposition
- continuation of (un)stable manifolds through resonances (double eigenvalues)
- non-autonomous problems
- non-hyperbolic equilibria (center-stable manifolds)
- software for general connecting orbits (and continuation)
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- infinite dimensional problems (PDEs)

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Thank you

Supported by NWO
Netherlands Organisation for Scientific Research

JB van den Berg
VU Amsterdam