Visualizing Chaotic and Regular Orbits in Three and Four Dimensional Maps

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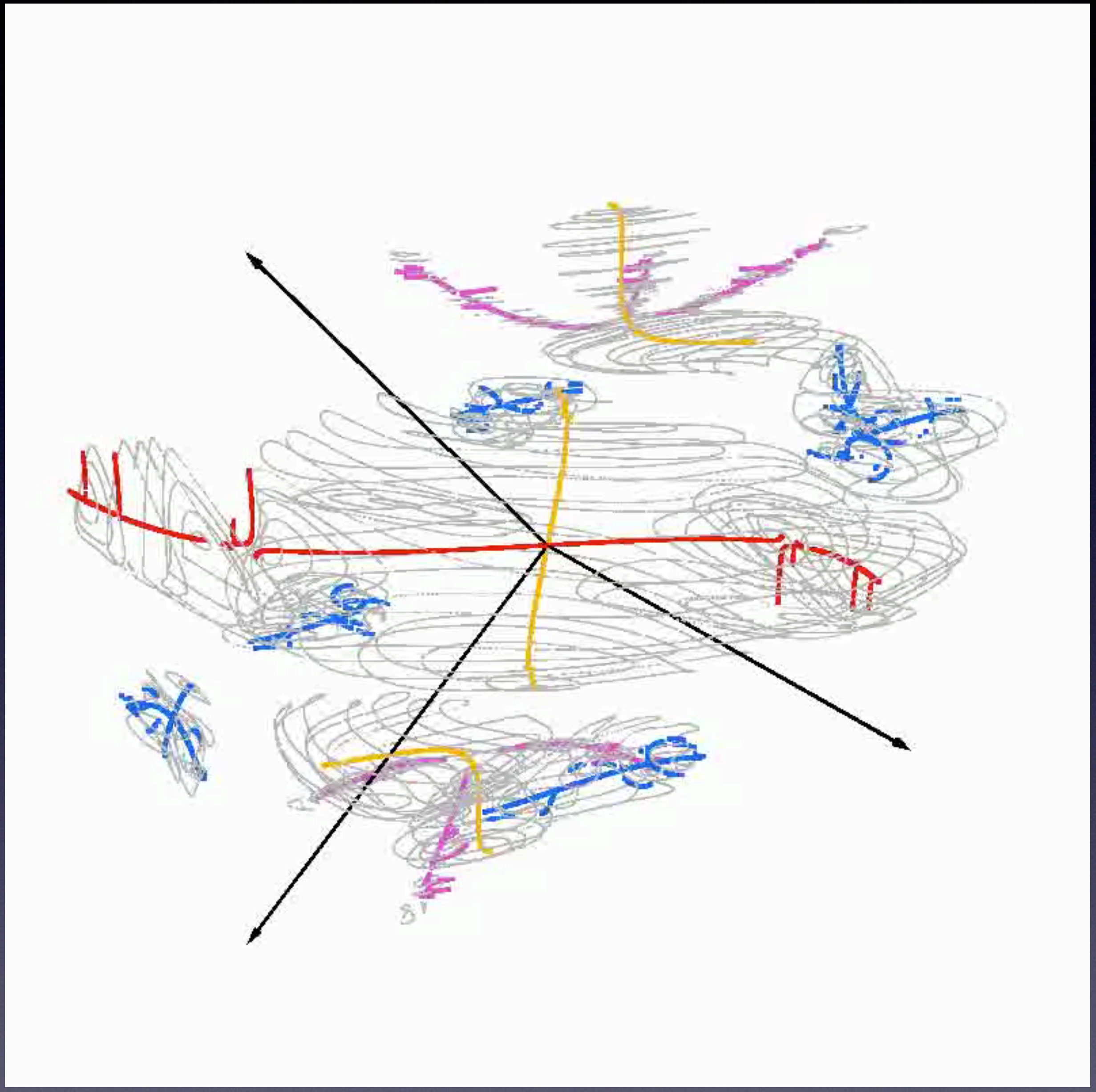
Visualization: Geometry

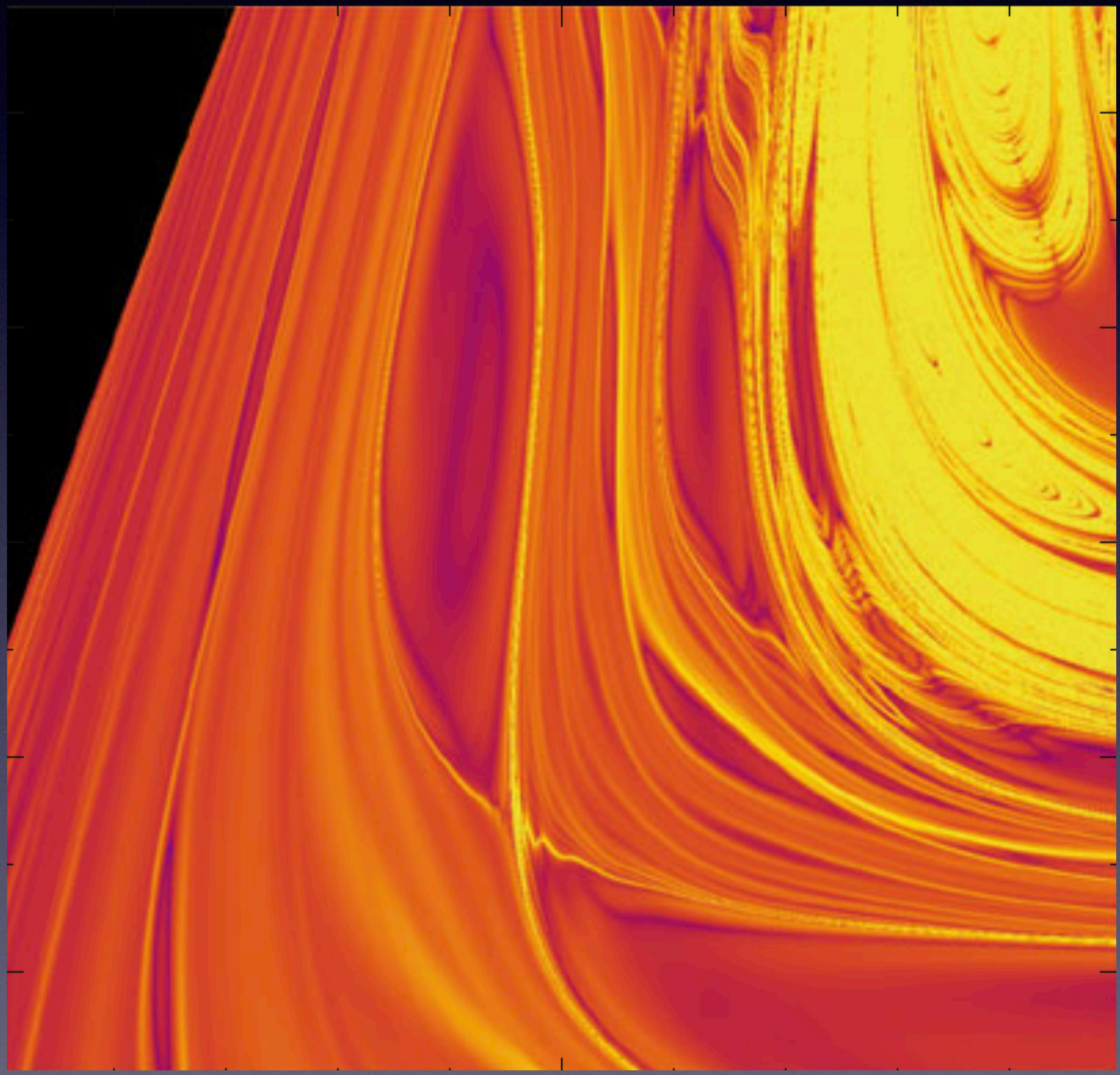
- Phase Portraits (2D & 3D)
- Projections
  - Color for a missing dimension
- Slices
  - 4D ⇒ 2D: Kaneko-Bagley (1985), Kook-Meiss (1989)
  - 3D ⇒ 2D: Artuso-Casati- Shepelyansky (1992), Cartwright-Feingold-Piro (1994)
  - 4D ⇒ 3D: Richter-Lange-Bäcker-Ketzmerick (2014)
- Cube Plots
  - Assembled Slices: Lega-Guzzo-Froeschlé (2016)



Lange, S., M. Richter, et al. (2014). "Global structure of regular tori in a generic 4D symplectic map." Chaos 24: 024409.

Visualization: Indicators

- Chaos
  - Fast Lyapunov Indicator (FLI): Froeschlé-Gonczi-Lega (1997)
  - Finite Time Lyapunov (FTLE): Haller (2000)
  - Smaller Alignment Index (SALI): Skokos (2001)
  - Lagrangian Coherent Structures (LCS): Shadden-Leiken -Marsden (2005)
  - Lagrangian Descriptors: Madrid-Mancho (2009)
- Regularity
  - Resonance Partitions: MacKay-Meiss-Percival (1987)
  - Frequency Maps: Laskar (1992)
  - Width (a Lagrangian Indicator): Easton-Meiss-Carver (1993)
  - Ergodic Partition: Mezic-Wiggins (2001)
  - Parameterization Method (tori): Cabré-Fontich-de la Llave (2003)
- Transport
  - Drift/Diffusion: Chirikov (1979)
  - Exit Time Distributions: Karney (1982)
  - Action Diameter: Meiss-Guillery (2017)



Lega, E., M. Guzzo and C. Froeschle (2016). Theory and Applications of the FLI Method. Chaos Detection and Predictability Springer. 915: 35-54.

Angle-Action Maps

- Standard (angle-action) form on $\mathbb{T}^n \times \mathbb{R}^m$

Angle-Action Form

$$\begin{aligned} x' &= x + \Omega(y') \mod 1 \\ y' &= y + F(x) \end{aligned}$$

insert "new" *y* here!

m actions

- Note: number of angles and actions can be different, $n \neq m$

- Frequency map

$$\Omega : \mathbb{R}^m \to \mathbb{T}^n$$

- Force

$$F : \mathbb{T}^n \to \mathbb{R}^m$$

Angle-Action Maps

- Symplectic Examples: $(n = m)$
  - Chirikov's Standard map (1,1) $(x', y') = (x + y', y - \varepsilon \sin(2\pi x))$
  - Froeschlé map (2,2) $(x', y') = (x + y', y - \varepsilon \nabla V(x))$
- Volume-Preserving Normal form (2,1)

$\Omega = (\Omega_1(y), \Omega_2(y))$

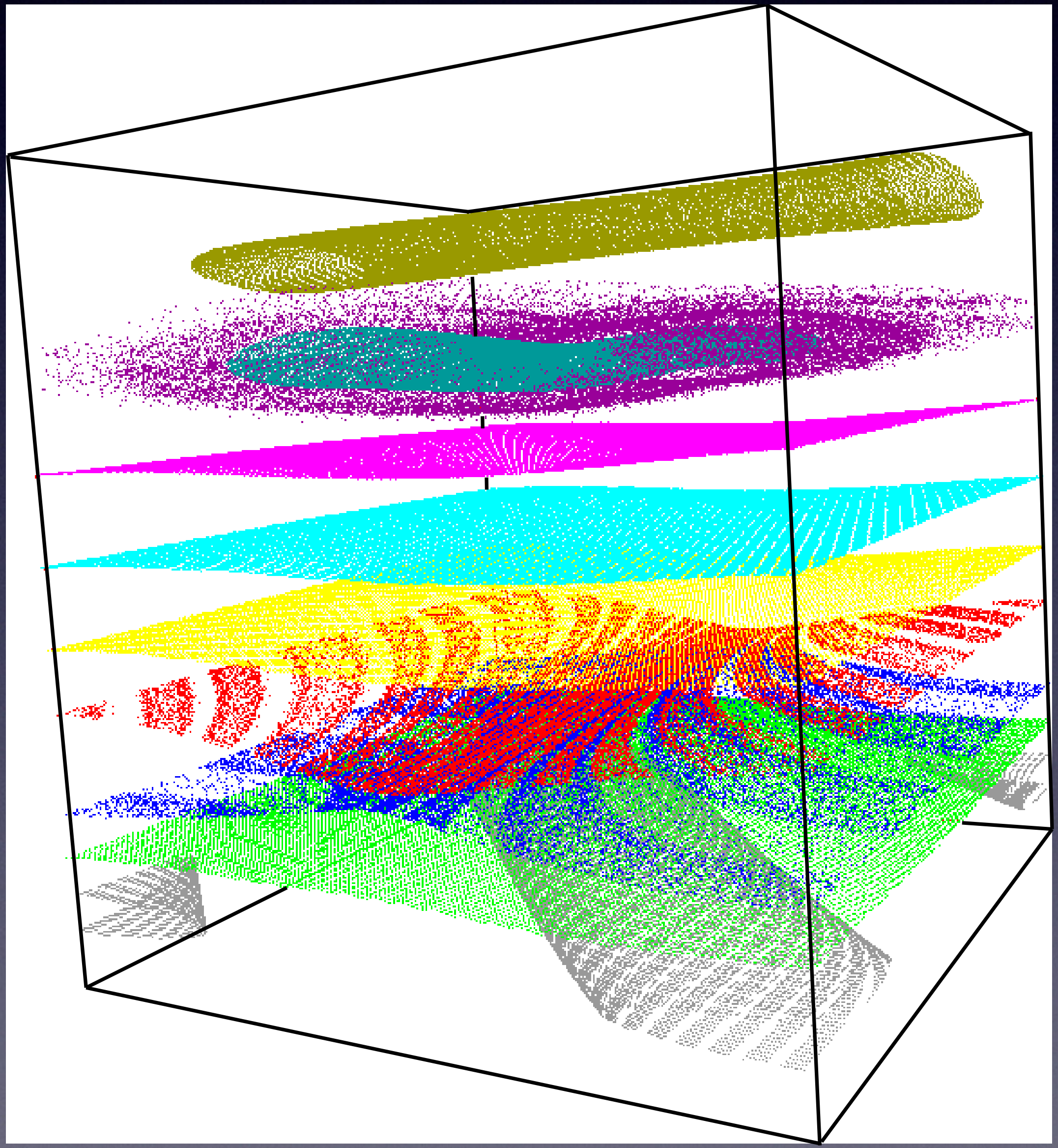
$$x'_1 = x_1 + y' + \gamma$$
$$x'_2 = x_2 + \beta y'^2 - \delta$$
$$y' = y - \varepsilon[\sin(2\pi x_1) + \sin(2\pi x_2) + \sin(2\pi(x_1 - x_2))]$$

Dullin, H. R. and J. D. Meiss (2012). "Resonances and Twist in Volume-Preserving Mappings." SIAM J. Appl. Dyn. Sys. 11: 319-349.

3D Dynamics:
Invariant Tori
&
Frequency Maps

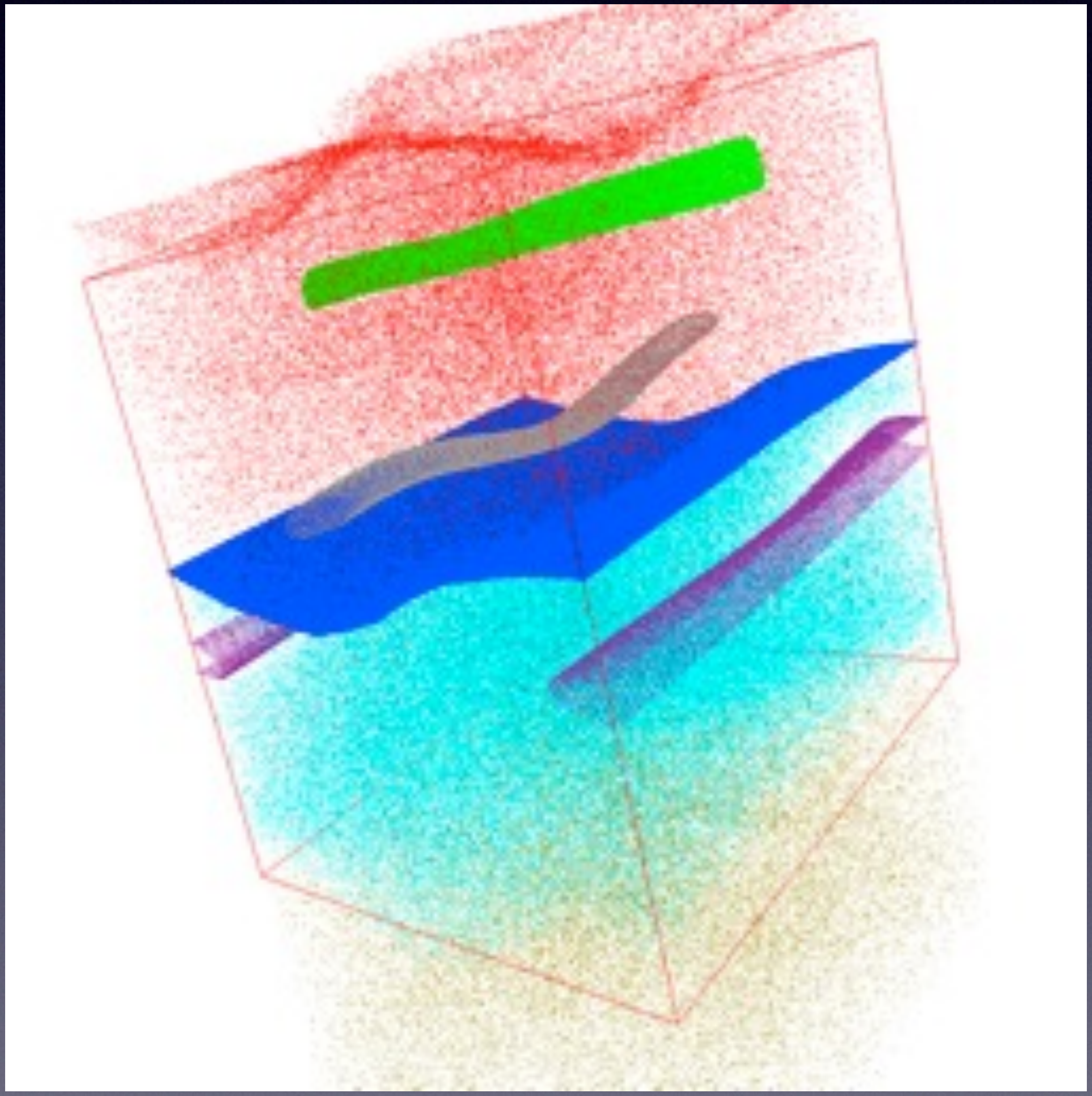
$$x_1' = x_1 + y' + \gamma$$
$$x_2' = x_2 + \beta y'^2 - \delta$$
$$y' = y - \varepsilon[\sin(2\pi x_1) + \sin(2\pi x_2) + \sin(2\pi(x_1 - x_2))]$$

Many rotational tori



ε = 0.005

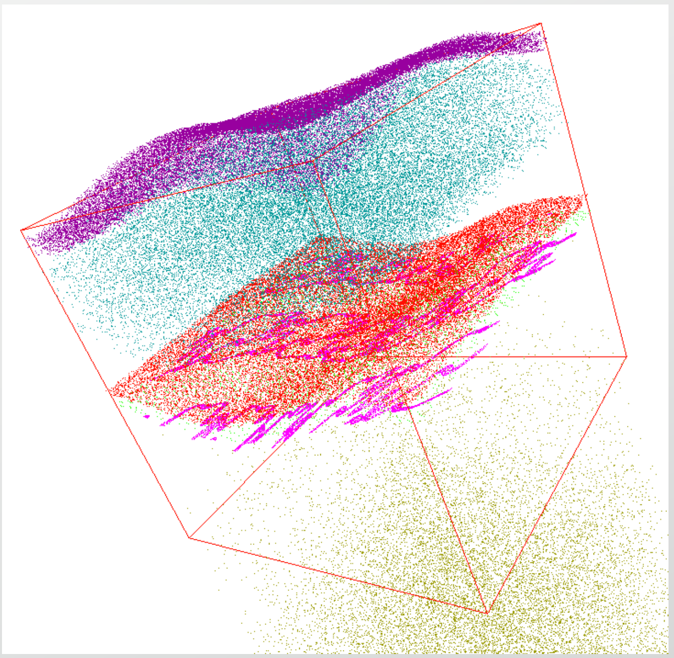
Only one rotational torus



ε = 0.02725

β = 2, δ = 0.1, γ = ½(1+√5)

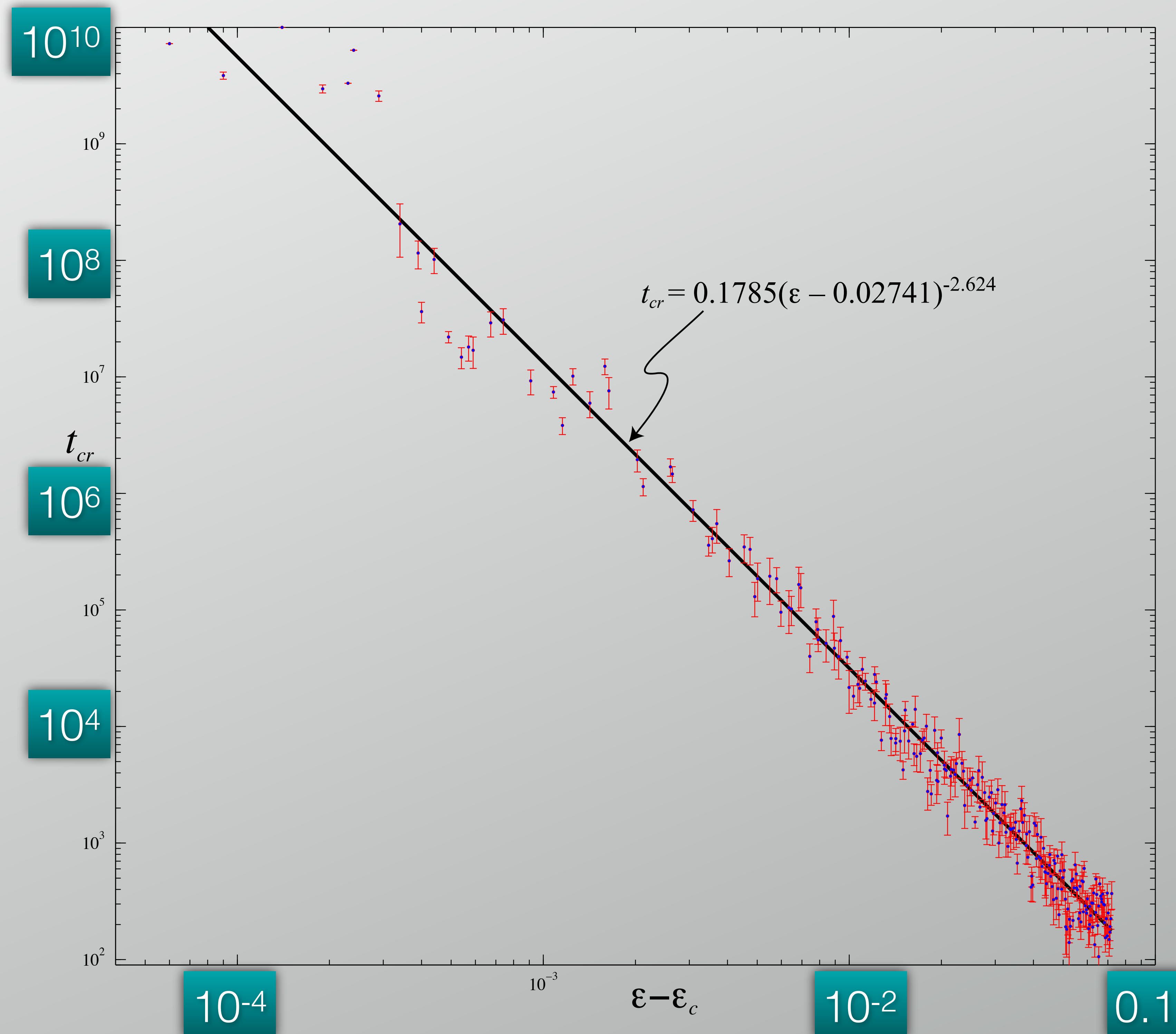
Destruction ⇒ Transport



Crossing Time vs. $\varepsilon - \varepsilon_{cr}$

$\delta = 0.1$

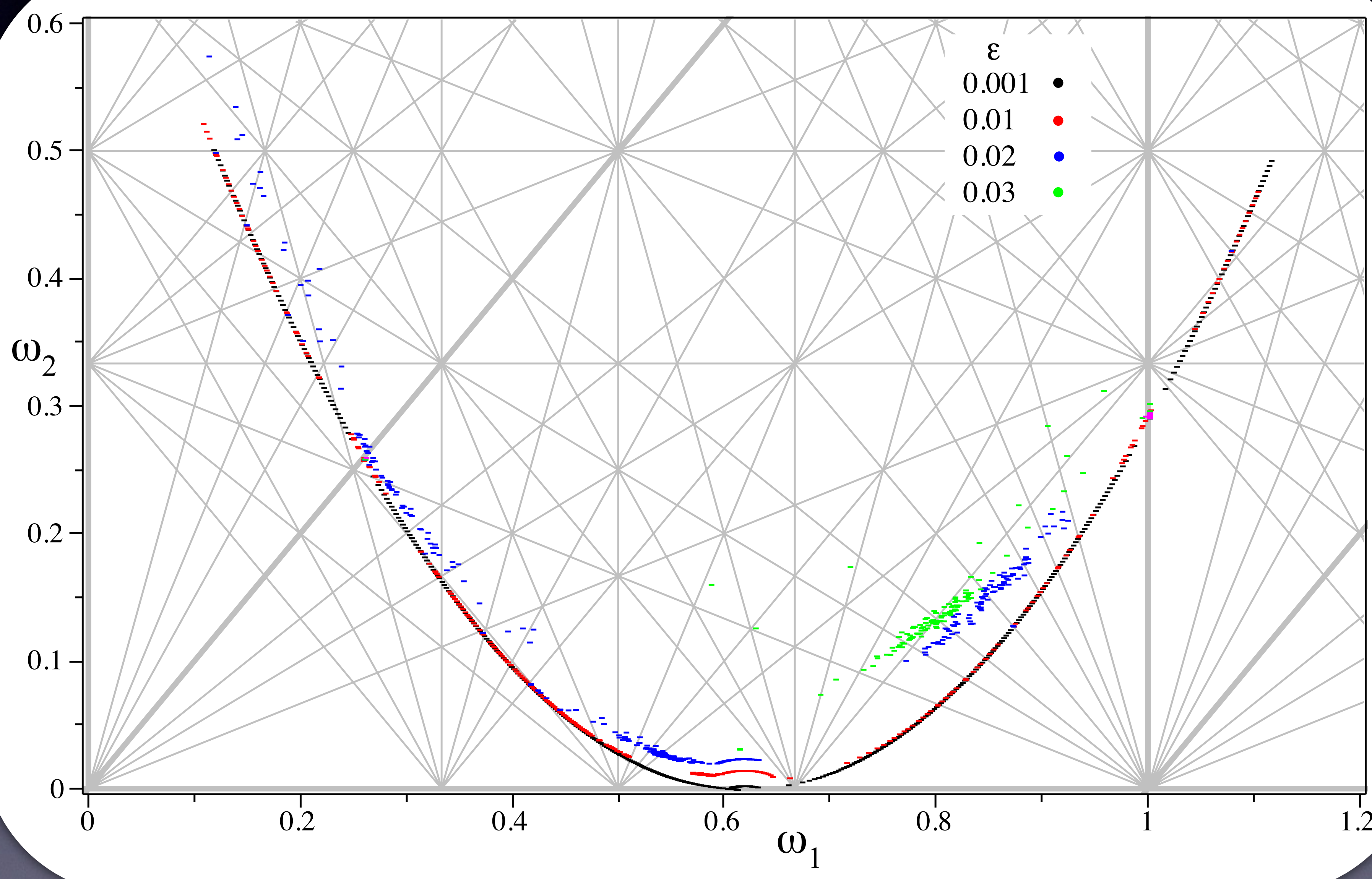
$\varepsilon_{cr} \approx 0.02741$



Meiss, J. D. (2012). "The Destruction of Tori in Volume-Preserving Maps." Comm. Nonl. Sci and Num. Sim. 17: 2108-2121.

Frequency Maps

- Varying Force Amplitude, ε.



$$x_1' = x_1 + y' + \gamma$$
$$x_2' = x_2 + \beta y'^2 - \delta$$
$$y' = y - \varepsilon[\sin(2\pi x_1) + \sin(2\pi x_2) + \sin(2\pi(x_1 - x_2))]$$

$$\Omega : \mathbb{R} \to \mathbb{T}^2$$

$$\gamma = \frac{\sqrt{5}-1}{2}, \beta = 2, \delta = 0.1$$

4D Dynamics
Projections & Slices

Fast Lyapunov Indicator

- Iterate arbitrarily chosen initial deviation v_0
- Compute the supremum up to time T

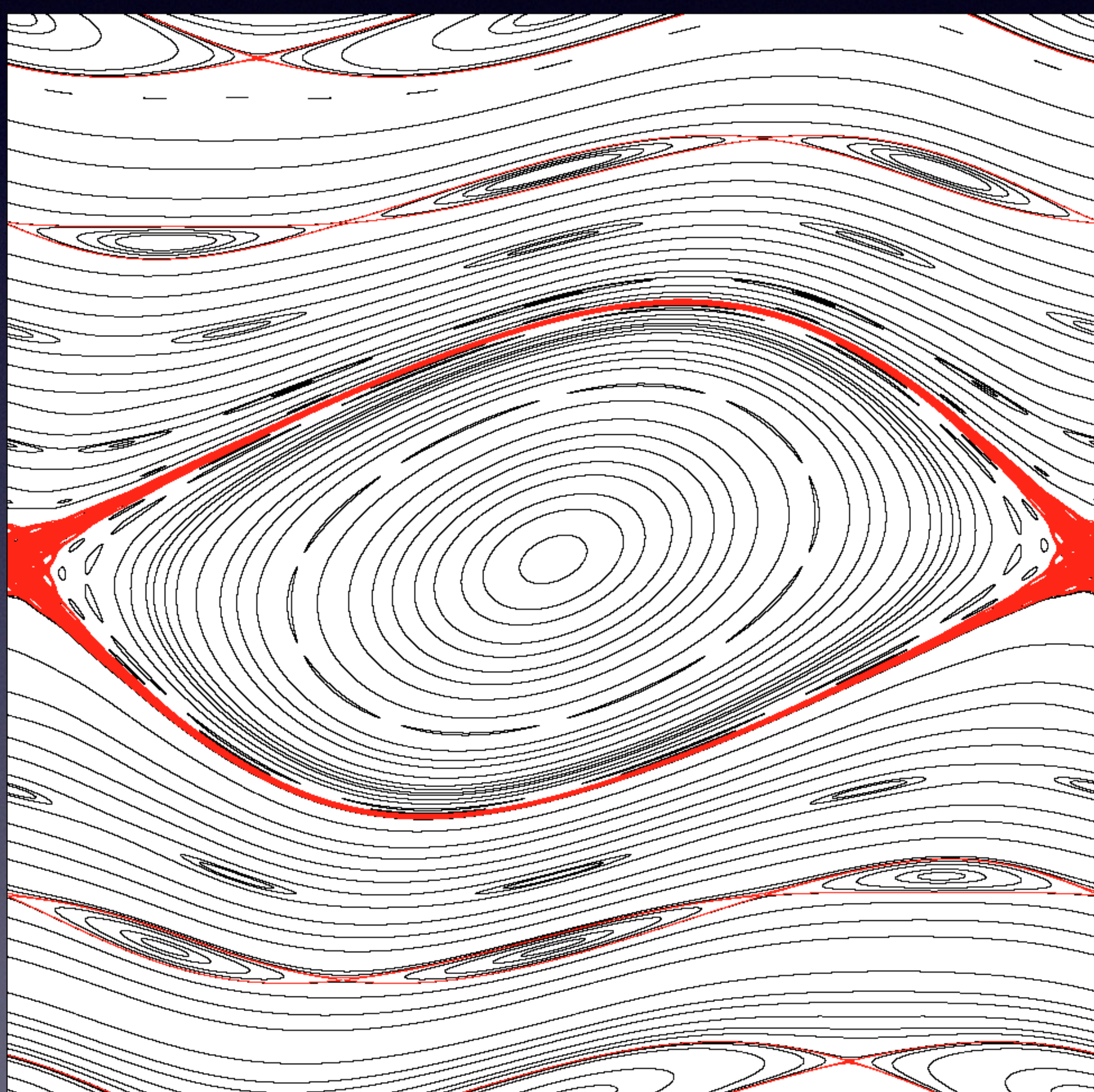
$$FLI = \sup_{t<T} \left(\log_{10} \|Df^t(z_0)v_0\| \right)$$

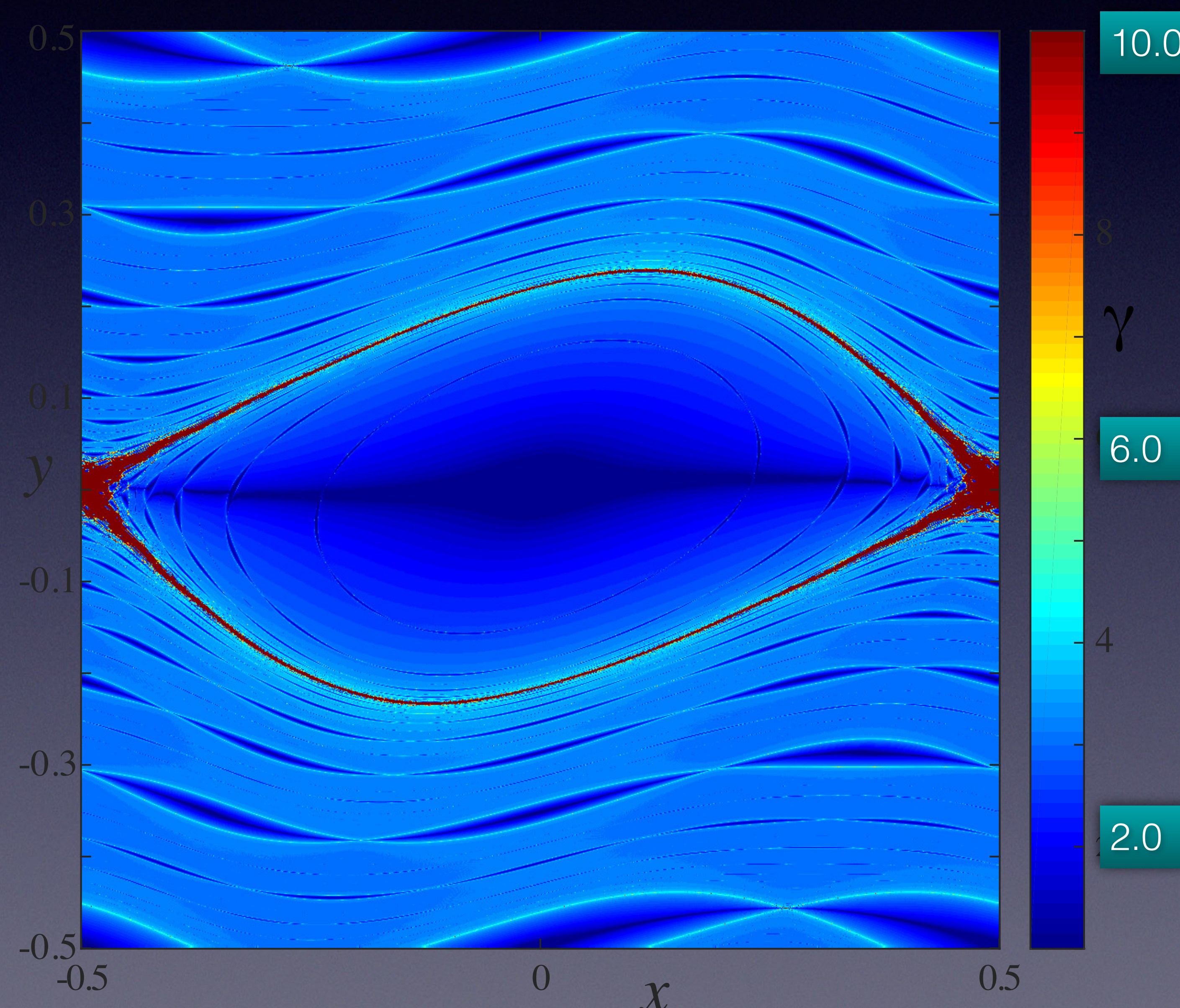
  - similar to FTLE, but supremum reduces oscillations

Froeschle, C., R. Gonczi and E. Lega (1997). “The fast Lyapunov indicator: a simple tool to detect weak chaos. Planetary and Space Science 45(7): 881-886.

FLI: 2D Standard Map

$$FLI = \sup_{t<T} \left(\log_{10} \| Df^t(z_0) v_0 \| \right)$$





- $a = 0.52$, $T = 10^3$, grid of 10^6 points

Froeschlé 4D Map

$$x' = x + y' \mod 1$$
$$y' = y + F(x)$$

two angles

two actions

$m = n = 2$

- Froeschlé-like forces

$$F = -\frac{1}{2\pi}\begin{pmatrix} a\sin(2\pi x_1) & + & c\sin(2\pi(x_1+x_2)) \\ b\sin(2\pi x_2) & + & c\sin(2\pi(x_1+x_2+\varphi)) \end{pmatrix}$$

  - $\varphi = 0$: Symplectic since $F = -\nabla V$
  - $\varphi = \frac{1}{2}$: "maximally non-symplectic" coupling

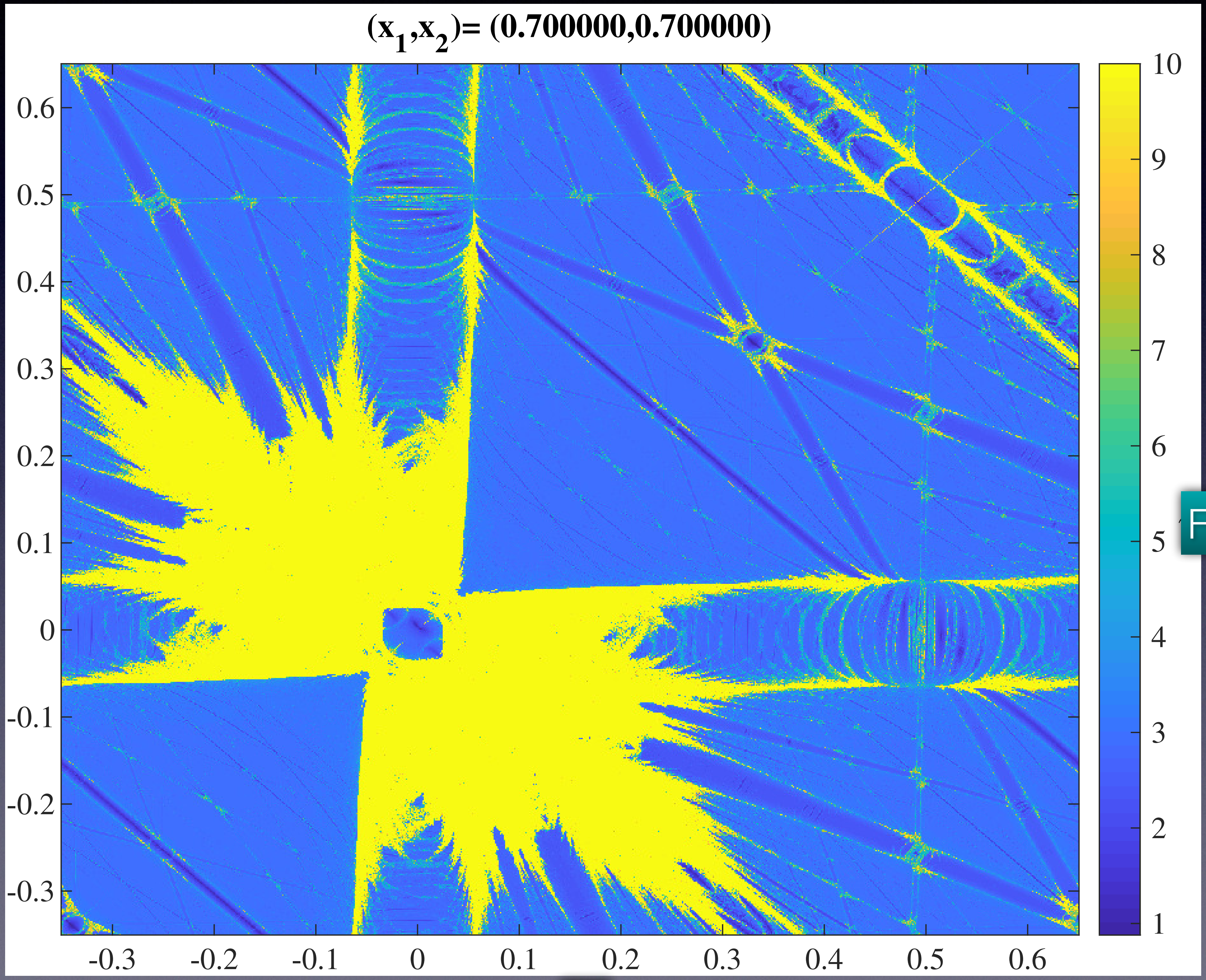
- Full Spectrum Force:

$$F_{fs} = -\frac{1}{2\pi}\frac{d}{(2.1+\cos(2\pi x_1)+\cos(2\pi x_2))^2}\begin{pmatrix} \sin(2\pi x_1) \\ \sin(2\pi(x_2+\varphi)) \end{pmatrix}$$

2D Slices: Action Plane

Varying ($x_1 = x_2$) Slices

$$FLI = \sup_{t<T} \left(\log_{10} \| Df^t(z_0)v_0 \| \right)$$



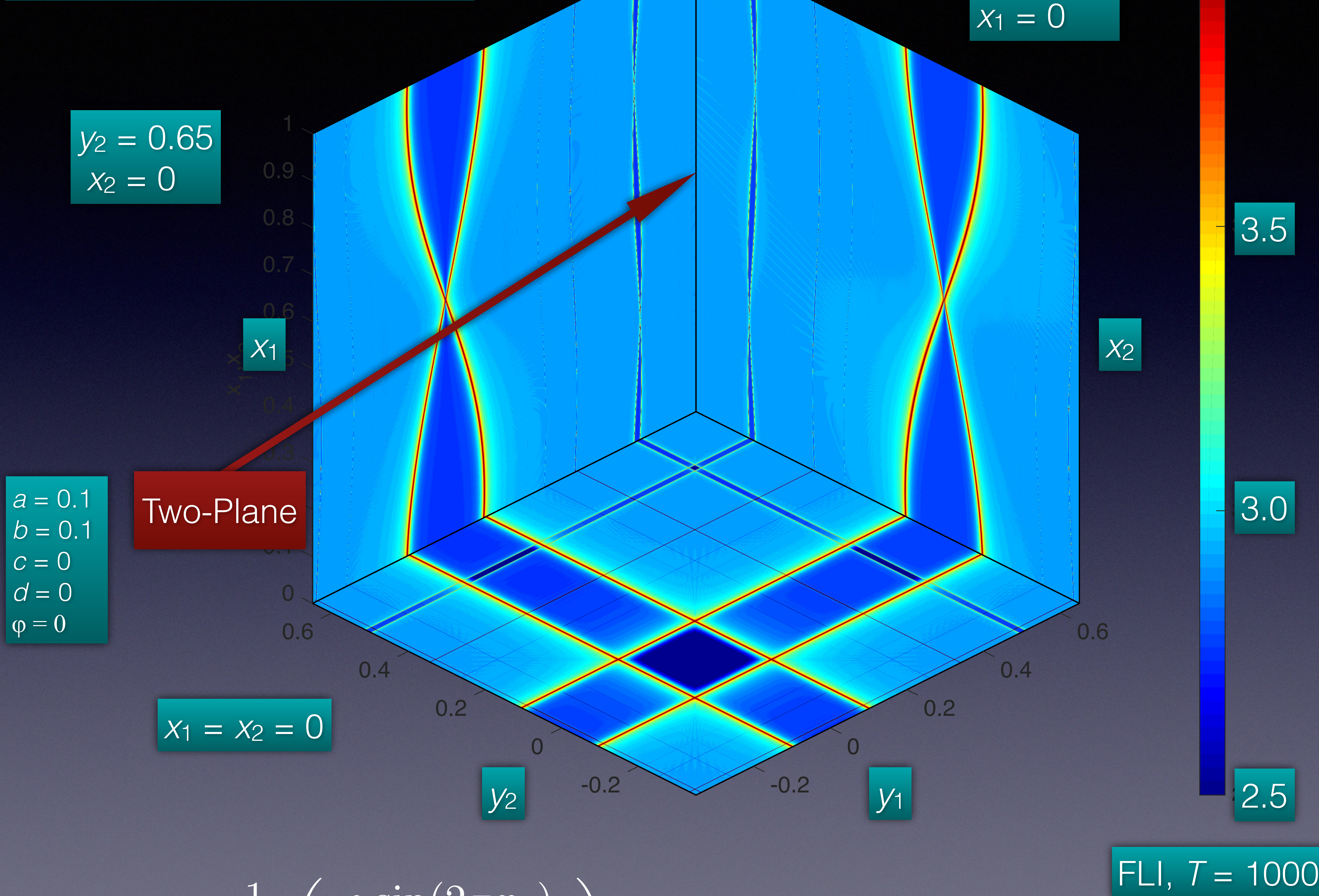
y_2

FLI, T = 1000

y_1

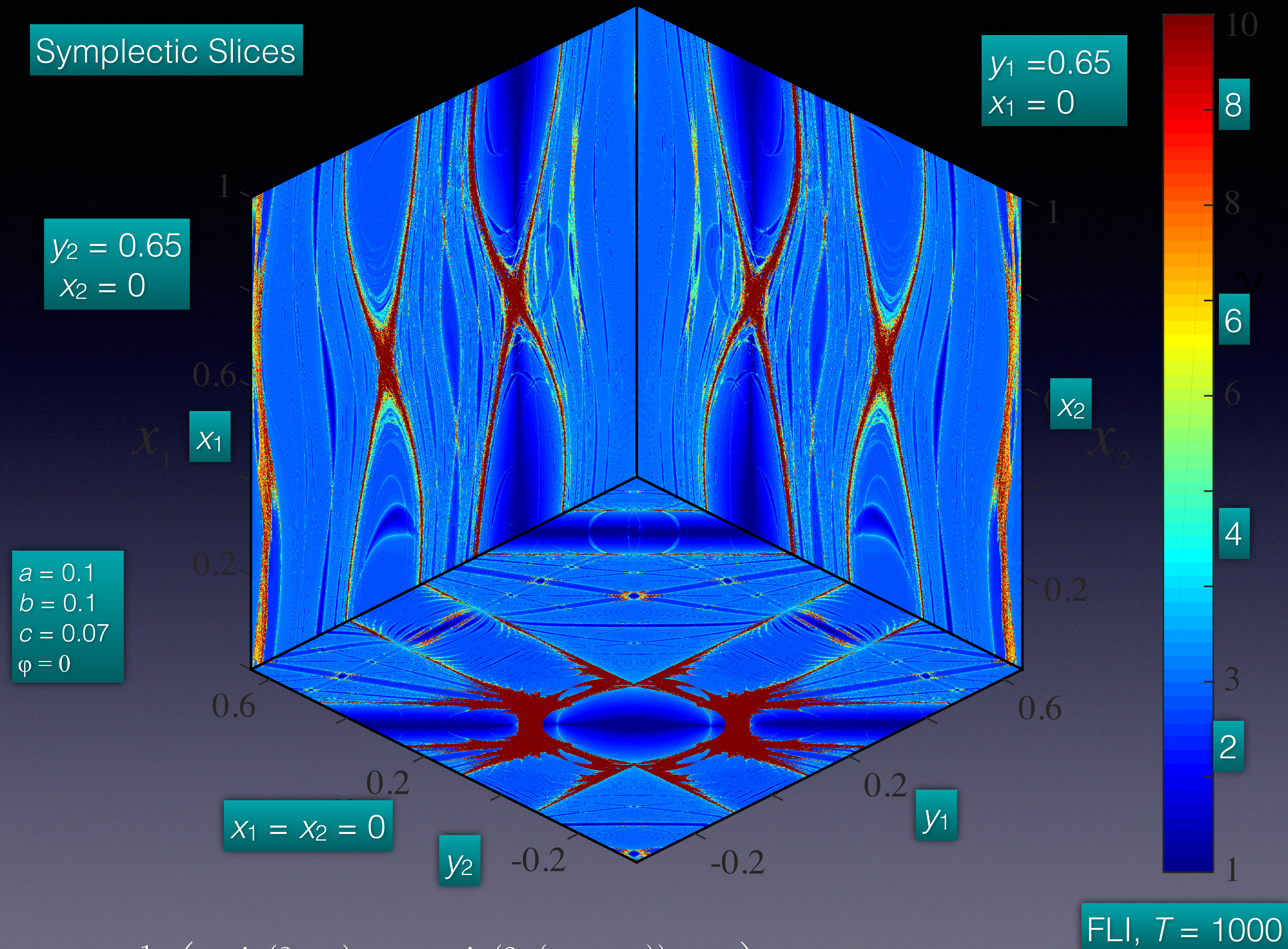
a = 0.1
b = 0.1
c = 0.07
φ = 0

Cube Plots: 3 Slices



$$F = -\frac{1}{2\pi}\begin{pmatrix} a\sin(2\pi x_1) \\ b\sin(2\pi x_2) \end{pmatrix}$$

Symplectic Slices

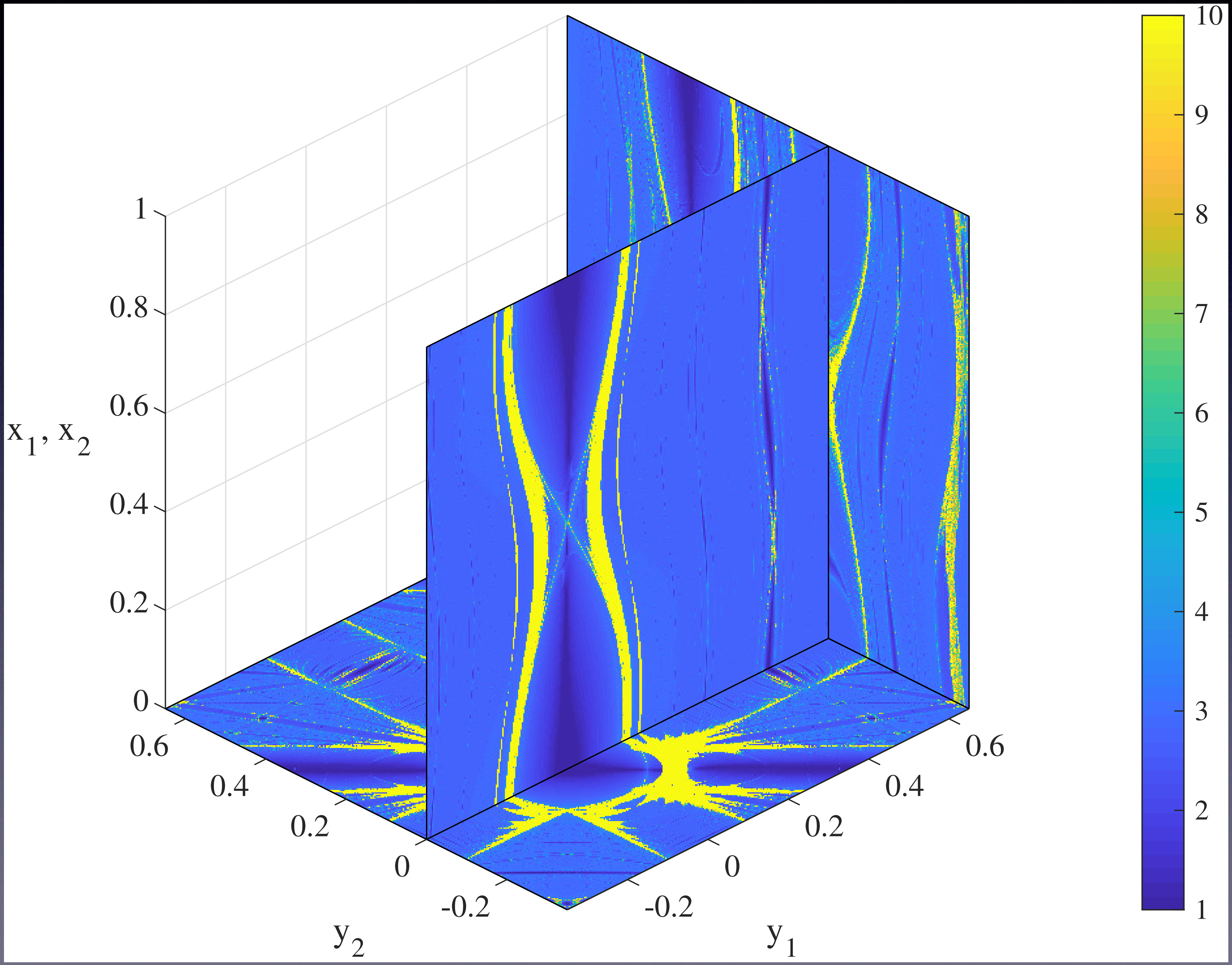


$$F = -\frac{1}{2\pi}\left(\begin{array}{ccc} a\sin(2\pi x_1) & + & c\sin(2\pi(x_1+x_2)) \\ b\sin(2\pi x_2) & + & c\sin(2\pi(x_1+x_2+\varphi)) \end{array}\right)$$

Varying Slices

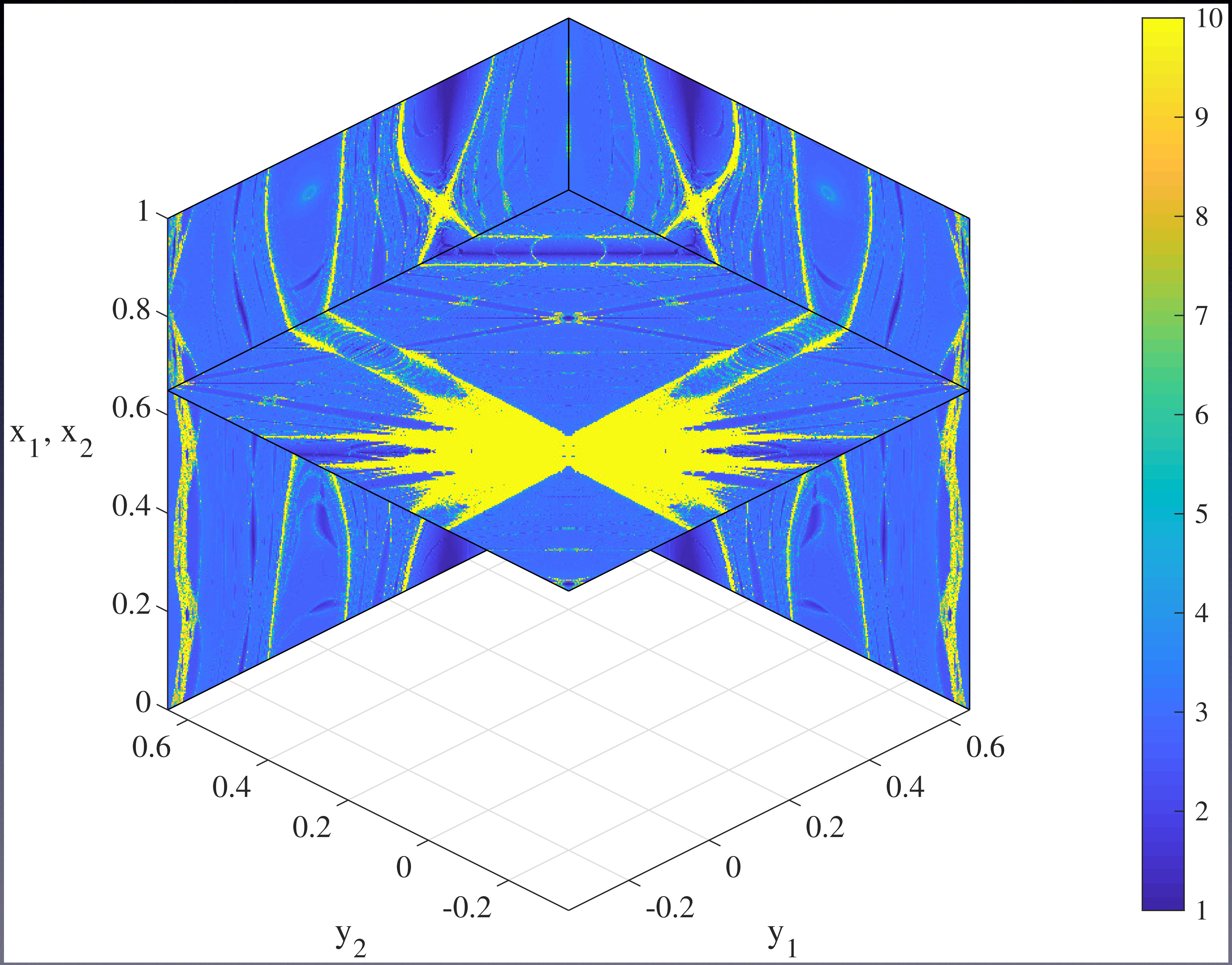
(x_2=0,y_2 varies) Slices

a = 0.1
b = 0.1
c = 0.07
φ = 0



Varying Slices

$x_1 = x_2$ Slices



$a = 0.1$
$b = 0.1$
$c = 0.07$
$\varphi = 0$

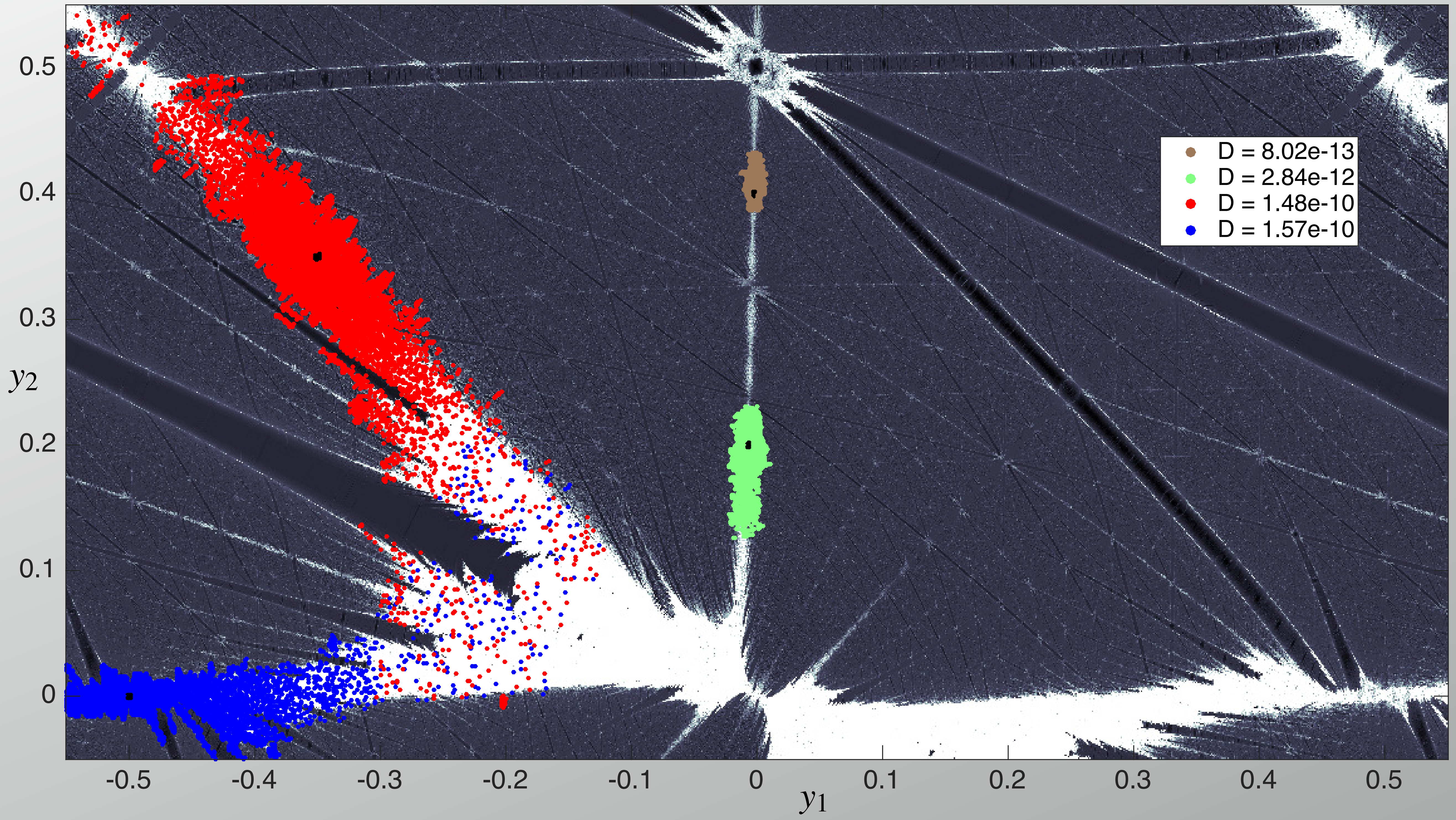
Transport

Symplectic: Diffusion

$T = 10^8$

$x = (0,0.5)$ slice

(a,b,c,d) = 0.0,0.1,0.07, 0.0001

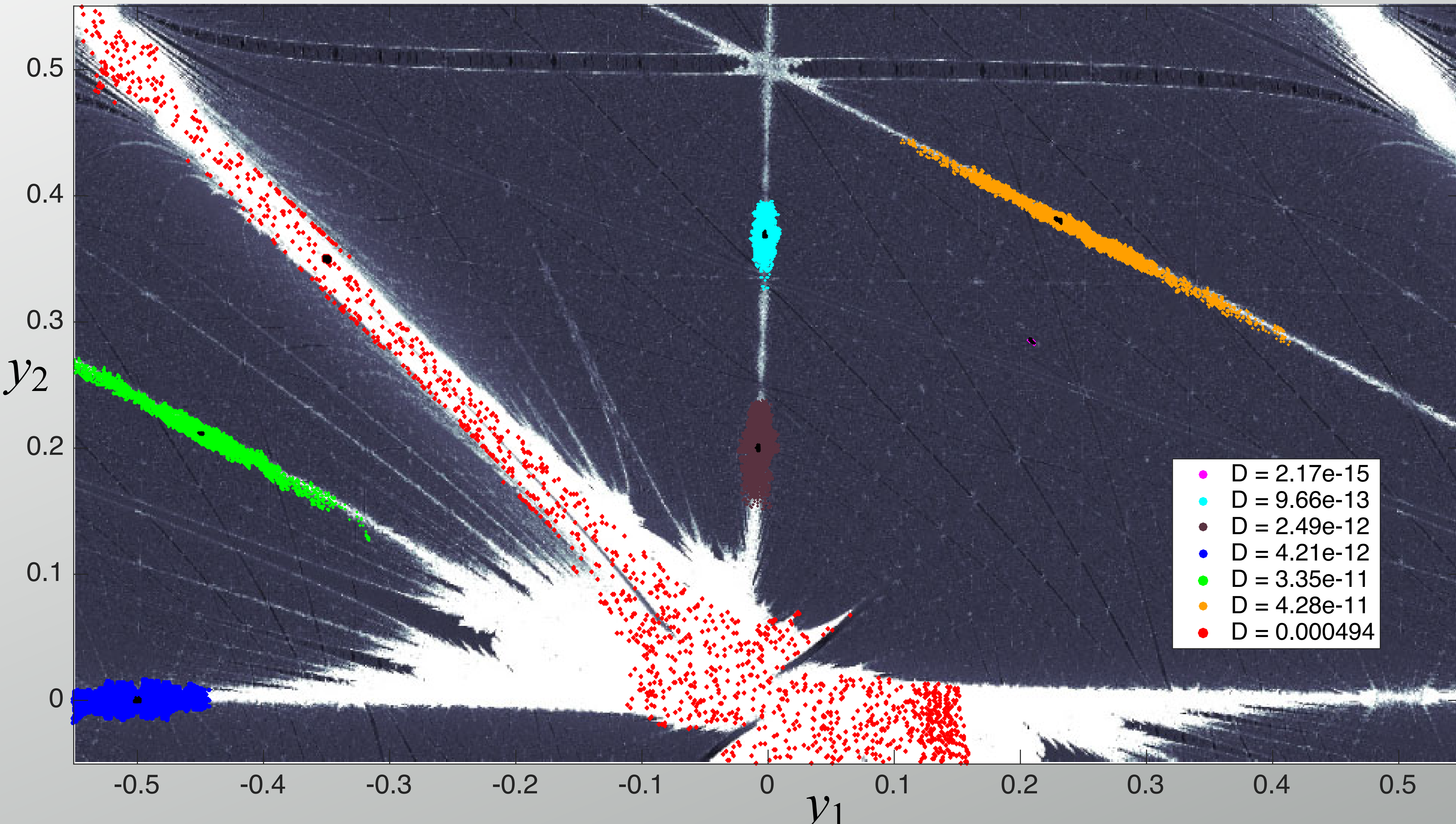


Volume Preserving: Drift

x = (0,0.5) slice

T = 10^8

(a,b,c,d) = 0.0,0.1,0.07, 0.0001 φ = (0.5,0)



Structure of 4D Resonances

- Moser’s Map
  - 4D Quadratic Symplectic map—generalizes Hénon’s map
  - Local Approximation near a periodic orbit

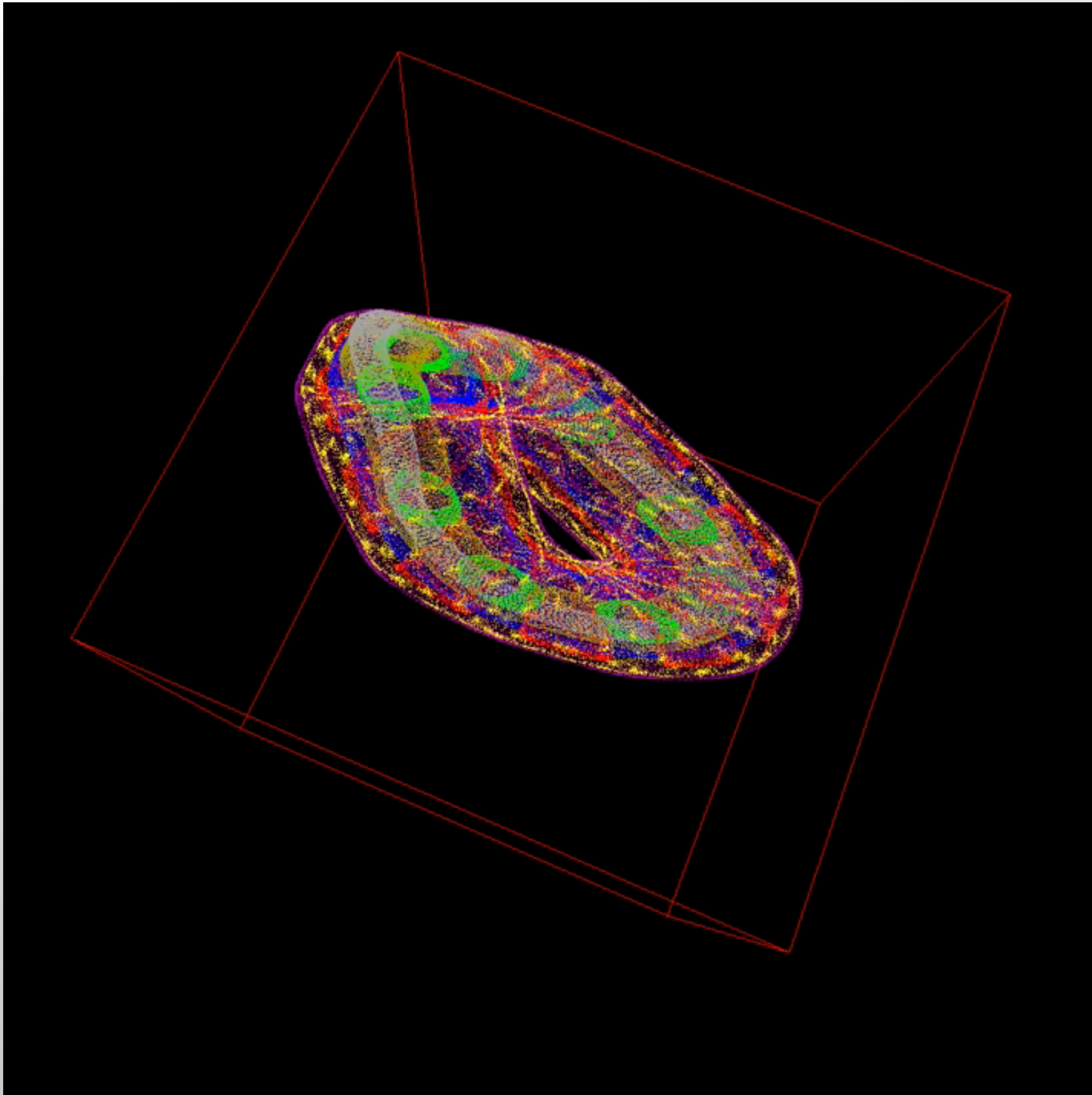
$$M(x, y) = (x + C^{-T}(-y + Cx + \nabla U(x)), Cx)$$

$$C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \det(C) = \alpha\delta - \beta\gamma = \varepsilon_1 = \pm 1$$

$$U = ax_1 + bx_2 + \tfrac{1}{2}cx_1^2 + \varepsilon_2 x_1^3 + x_1 x_2^2 \qquad \varepsilon_2 = \pm 1 \text{ or } 0$$

- 6 parameters + two discrete

Bäcker, A. and J. D. Meiss (2018). “Moser's Quadratic Symplectic Map.” Reg. and Chaotic Dyn. 23(6): 654-664.



3D Projection

(x_1, x_2, y_2)

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

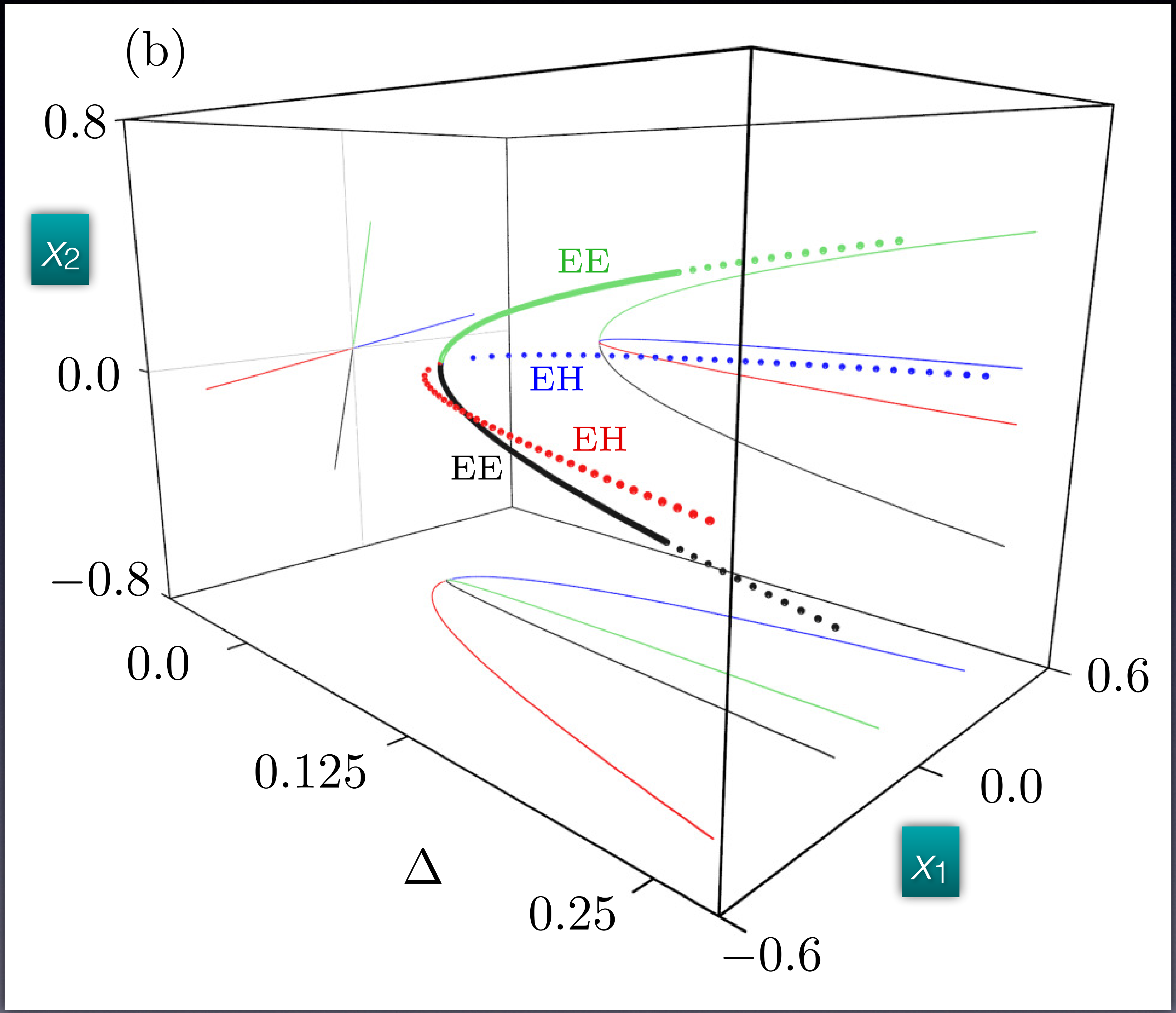
$$(a, b, c) = (-1.0, 0.1, 0.1)$$

$$\varepsilon_1 = -1, \varepsilon_2 = 1$$

Fixed Points born in a Quadfurcation*

$$M(x,y) = (x + C^{-T}(-y + Cx + \nabla U(x)), Cx)$$

- When $\varepsilon_2 = 1$, a a path through a=b=c=0 can result in a quadfurcation in a sector in parameter space.



$(a, b, c) = -\Delta(1.5, 0.3, 1)$

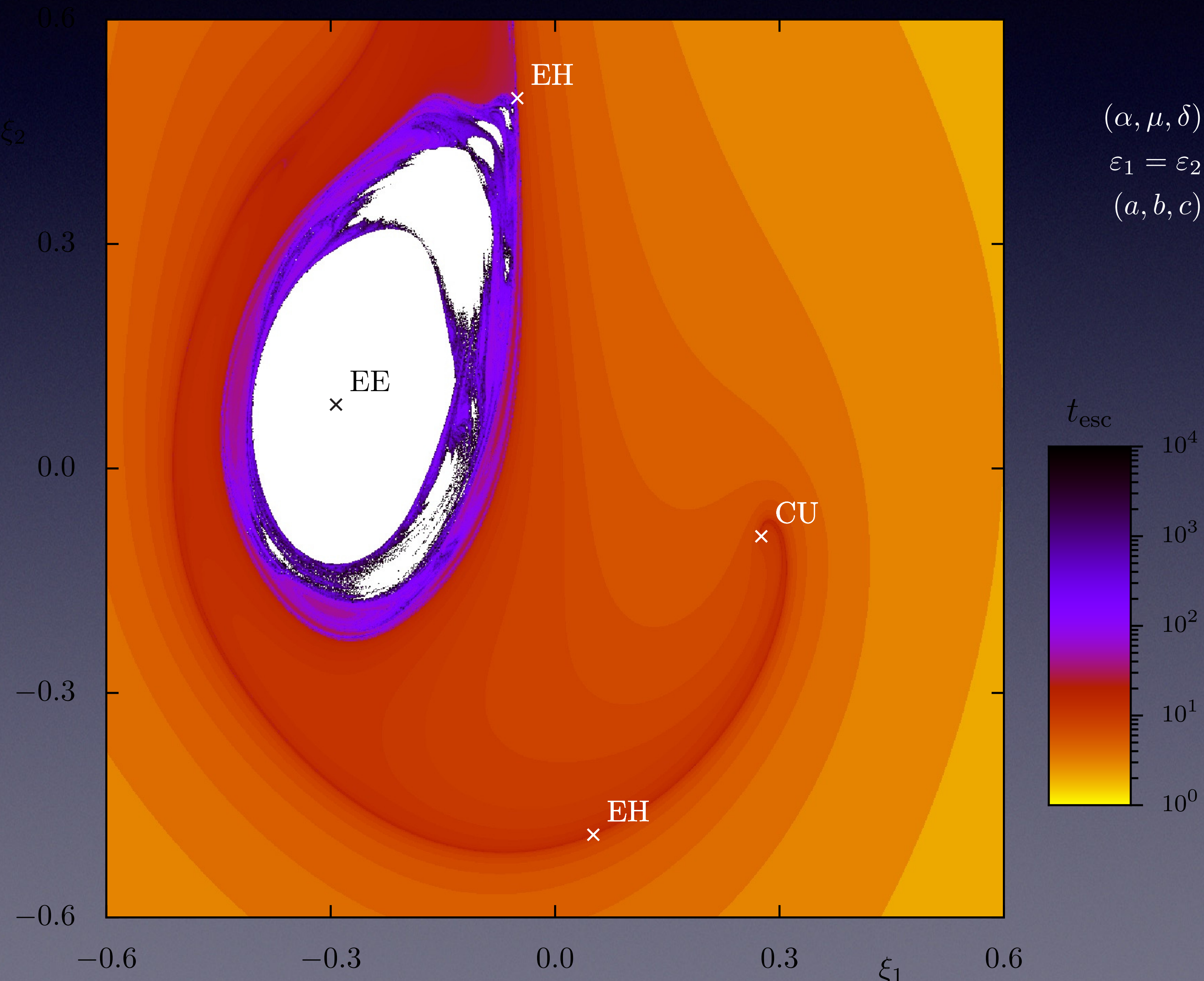
*As coined by Steve Strogatz in "*Nonlinear Dynamics and Chaos*"

4D Moser: Bounded Orbits

$$M(x,y) = (x + C^{-T}(-y + Cx + \nabla U(x)), Cx)$$

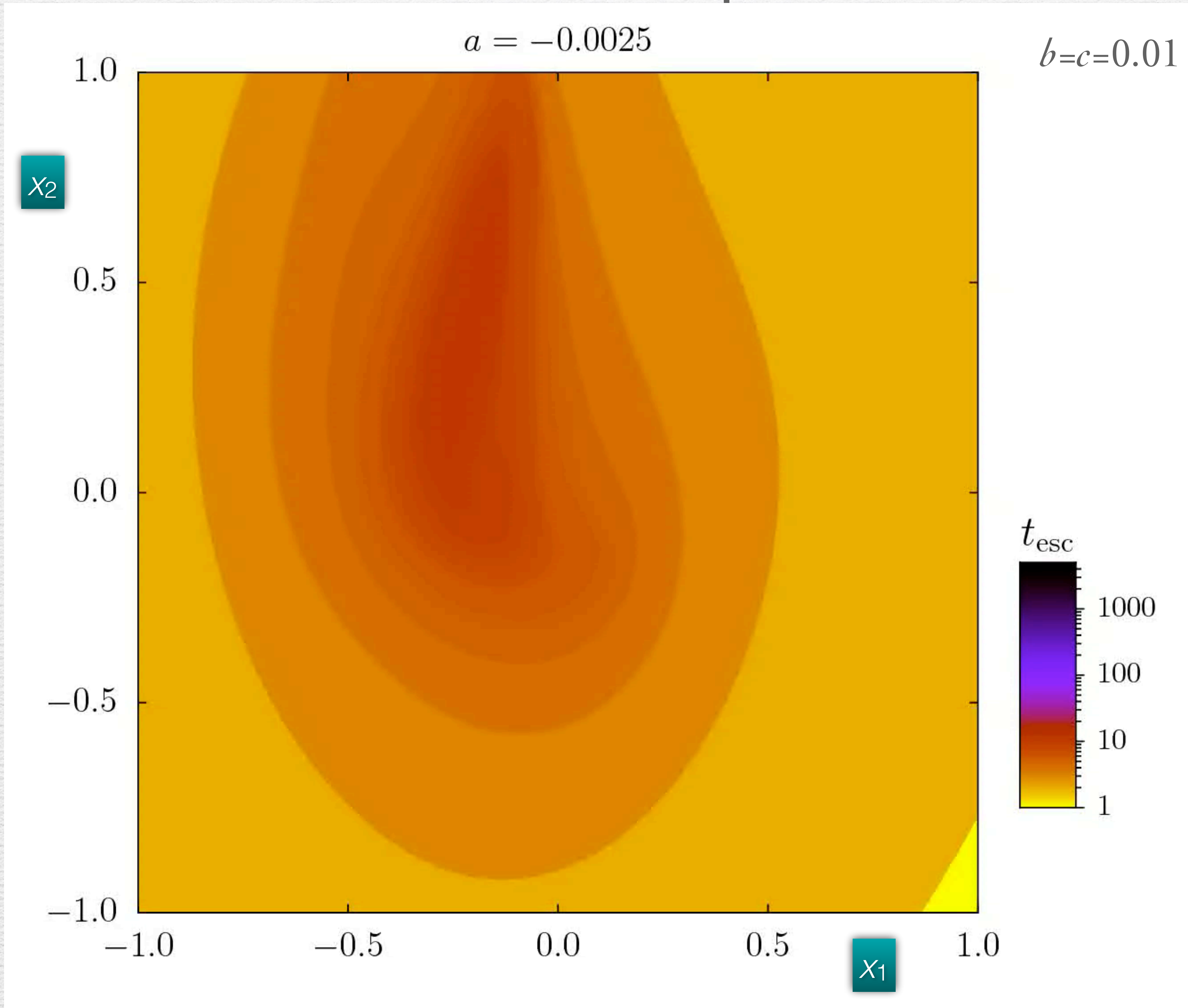
- Exit Time Distributions

Two-plane
$(x, y = Cx)$



$(\alpha, \mu, \delta) = (1, 0.1, 0.5)$
$\varepsilon_1 = \varepsilon_2 = 1$
$(a, b, c) = (-0.25, 0.05, 0.05)$

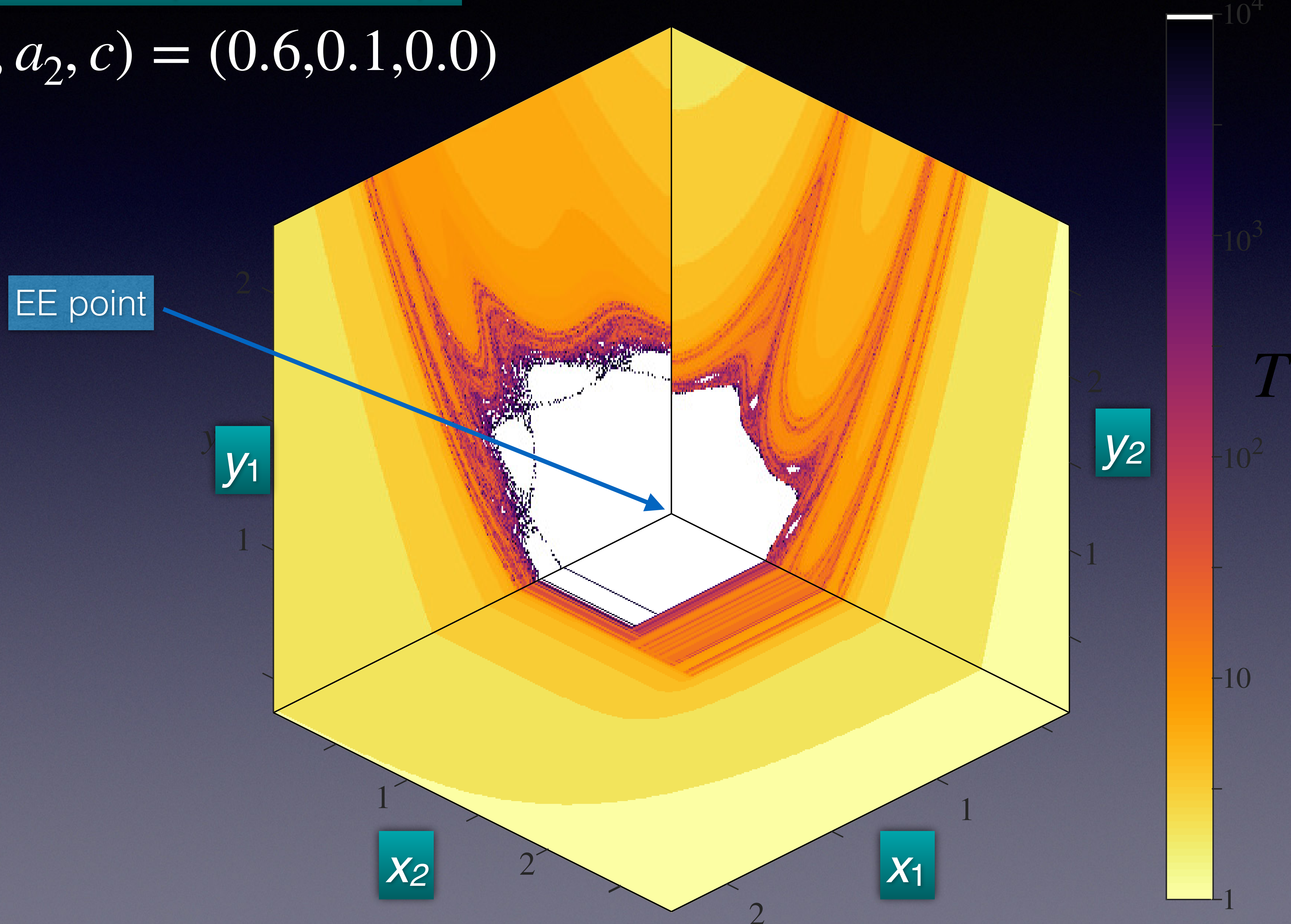
4D Moser: Escape Time



Escape Times

Pair of Uncoupled Hénon maps

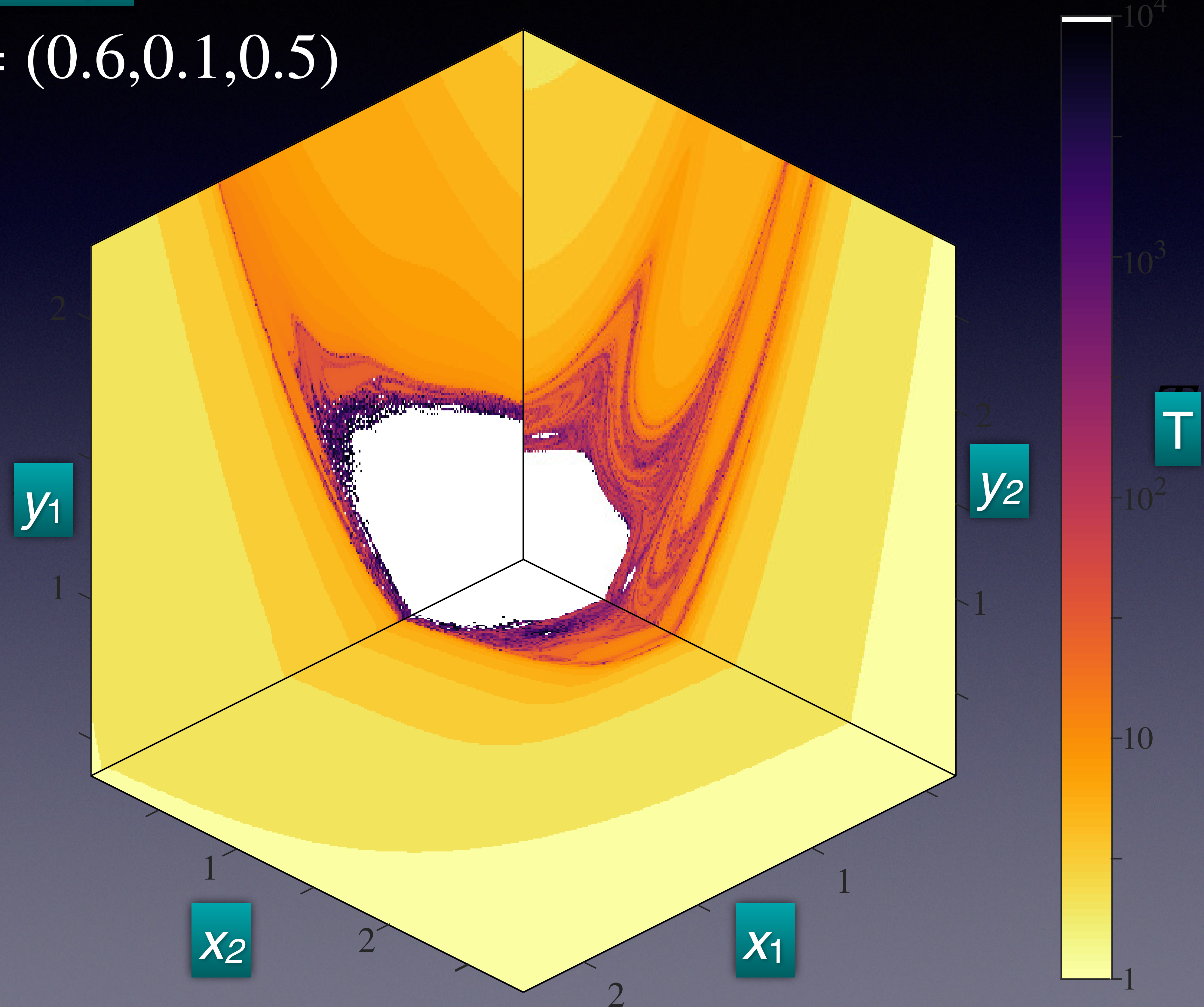
$(a_1, a_2, c) = (0.6, 0.1, 0.0)$



Escape Times

Coupled Hénon maps

$(a_1, a_2, c) = (0.6, 0.1, 0.5)$

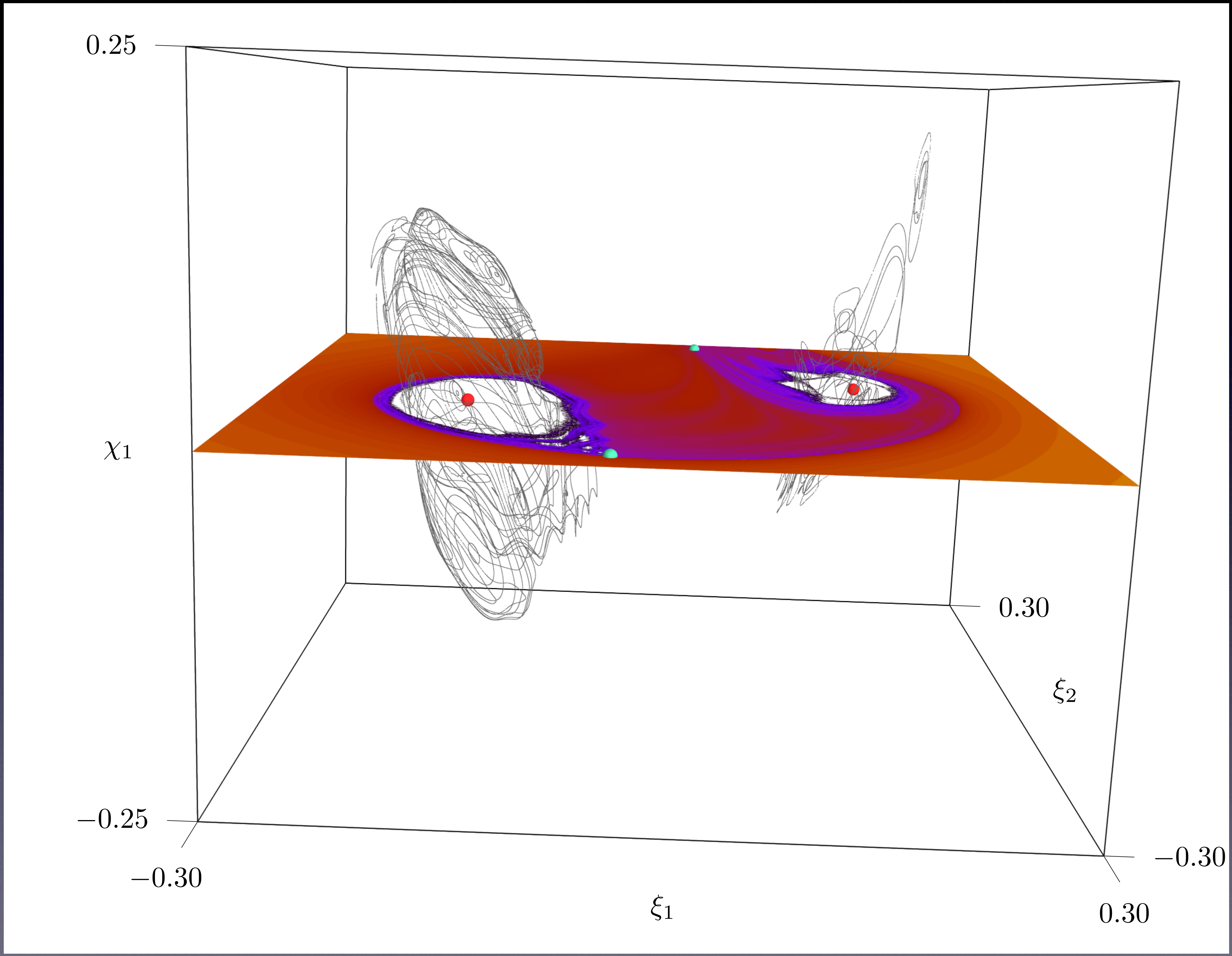


2D Escape + 3D Slices

EE + EE + EH + EH

$\chi = C\xi$

$|\chi_2| < 10^{-6}$



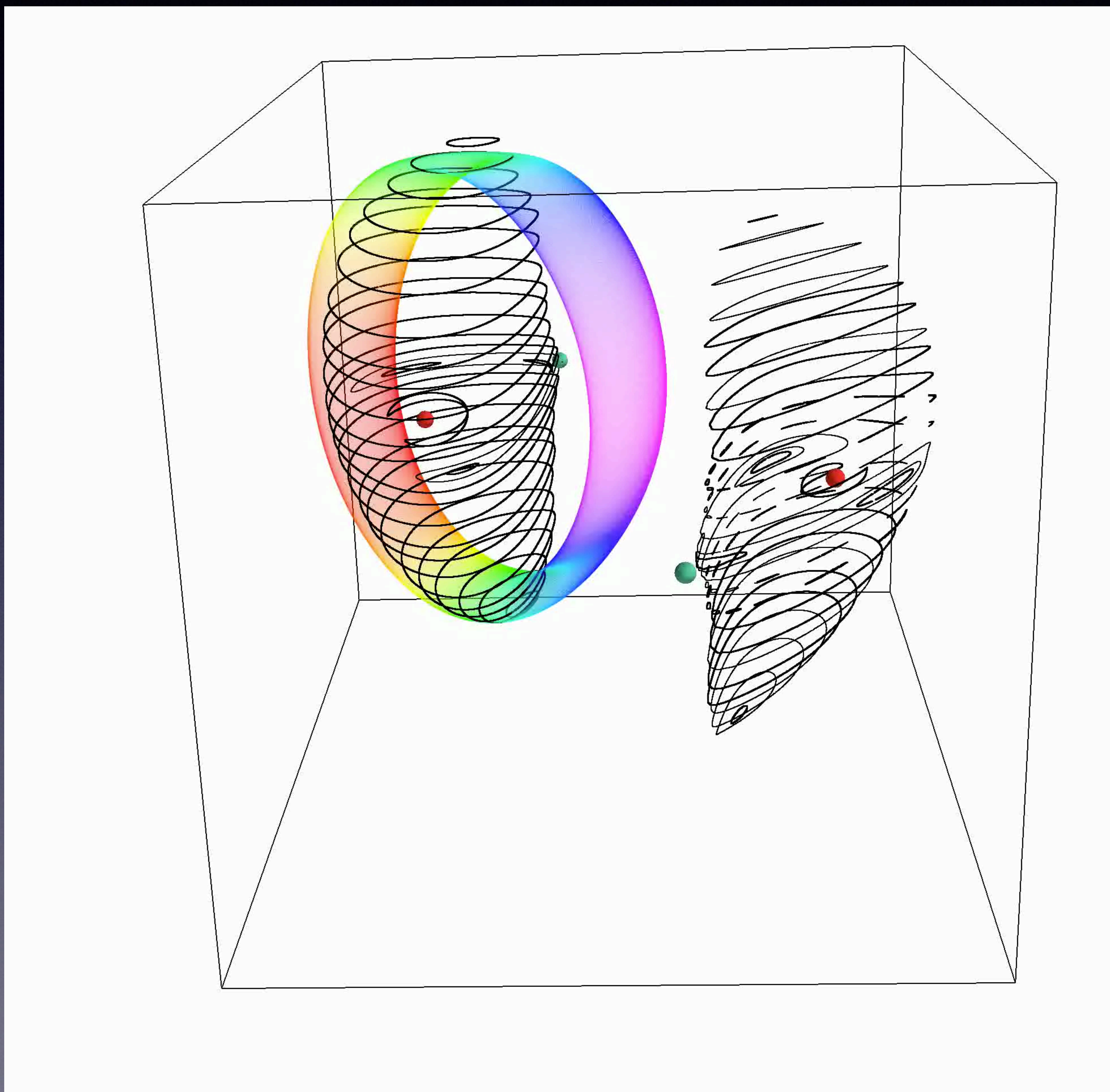
$(a, b, c) = -0.075(1, 0.25, 0.5)$

$(\alpha, \beta, \mu) = (1, 1, 0.5) \varepsilon_1 = \varepsilon_2 = 1$

Slices + Projection

- 3D Slices of Phase space ($\varepsilon_1 = 1$)

EE + EE + EH + EH



$(a, b, c) = (-0.015, -0.005, -0.001)$

$$(\alpha, \mu, \delta) = (1, 0.1, 0.5)$$
$$\varepsilon_1 = \varepsilon_2 = 1$$
$$(a, b, c) = (-0.0015, -0.005, -0.001)$$

Rotated coordinates
to 3-plane containing fixed
points

For More Information

- Dullin, H. R. and J. D. Meiss (2009). "Quadratic Volume-Preserving Maps: Invariant Circles and Bifurcations." SIAM J. Appl. Dyn. Sys. 8(1): 76-128.
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