# Acknowledgements

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- Chiara Daraio

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*References*

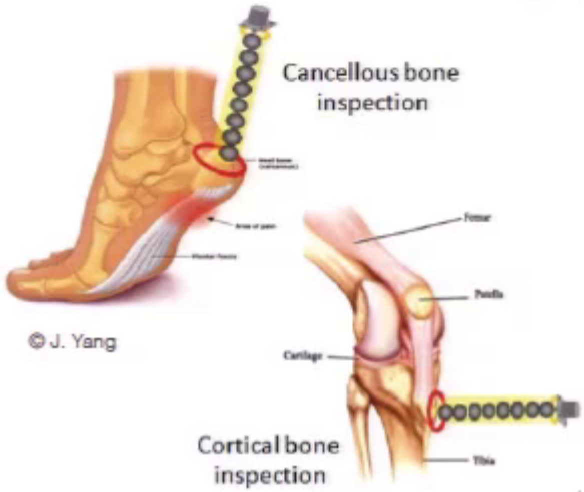
**Dark Breathers in Granular Crystals**
C. Chong, P.G. Kevrekidis, G. Theocharis, and C. Daraio
*PRE* **87** (2013) 042202

**Damped-Driven Granular Crystals: An Ideal Playground for Dark Breathers and Multibreathers**
C. Chong, F. Li, J. Yang, M.O. Williams, I.G. Kevrekidis, P.G. Kevrekidis, and C. Daraio
*PRE* **89** (2014) 032924

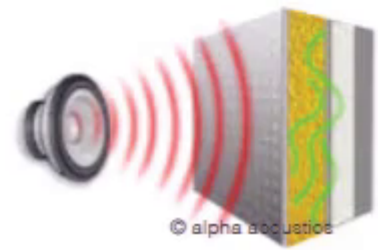
# Outline

- Hamiltonian System: Breathers and the NLS
- Vibration Energy Harvesting
- Damped-Driven Dynamics
- Revisiting NLS Prediction

# Applications




Non-destructive Evaluation



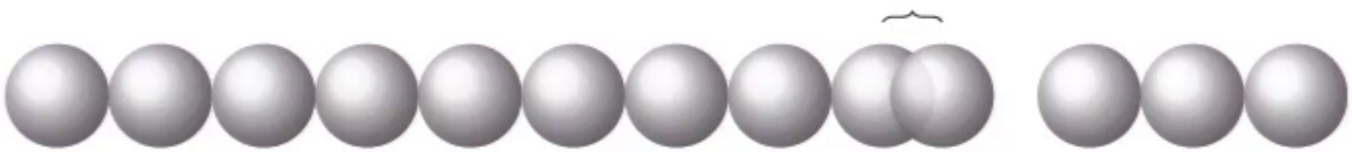

Sound Proofing

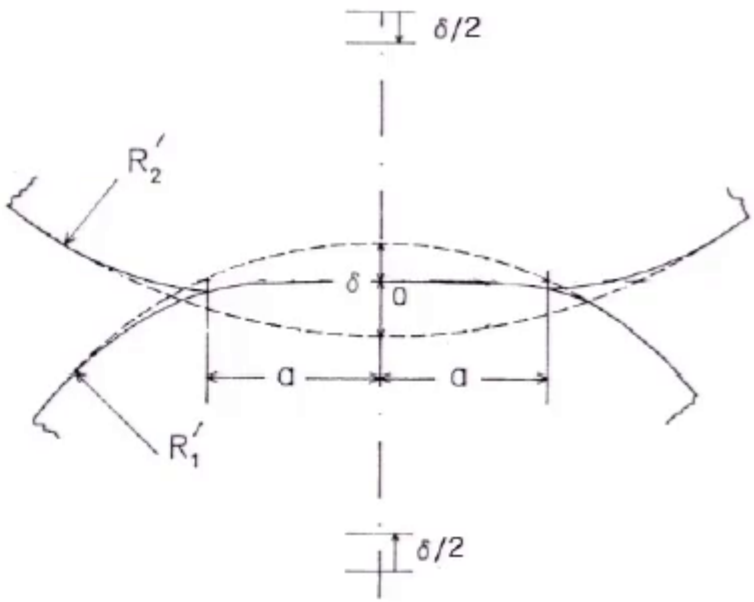
Vibration Energy Harvesting


Impact Absorption

# Modeling the Granular Chain

$$\delta_n = u_n - u_{n+1}$$






$$F(\delta) = \gamma\,[\,\delta\,]_+^{3/2}$$
$$[s]_+ = \max(0, s)$$

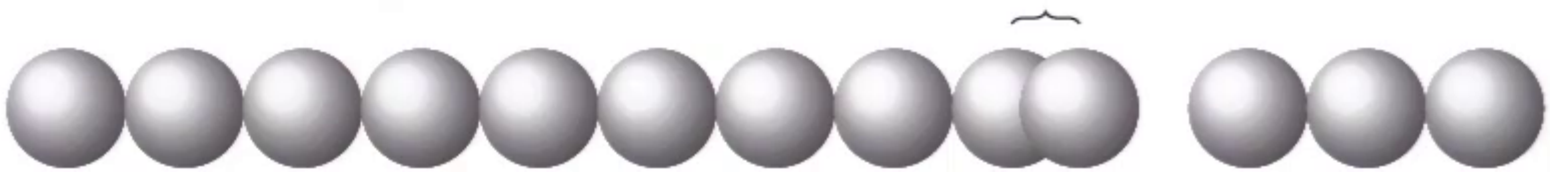
Source: Johnson, K. L. (1985) *Contact Mechanics* (Cambridge University Press, Cambridge)

# Modeling the Granular Chain

$$F(\delta) = \gamma \, [\, \delta \,]_+^{3/2}$$

$$[s]_+ = \max(0, s)$$

$$\delta_n = u_n - u_{n+1}$$



Assuming a static load induced by the force $F_0 = \gamma \delta_0^{3/2}$ we have the following model

$$M \partial_t^2 u_n = \gamma[\delta_0 + u_{n-1} - u_n]_+^{3/2} - \gamma[\delta_0 + u_n - u_{n+1}]_+^{3/2}$$

# Strain Variable

We will work in terms of the strain variable $y_n = u_n - u_{n+1}$.

$$\ddot{y}_n = F(y_{n+1}) - 2F(y_n) + F(y_{n-1}).$$

A displacement variable solution can be recovered by using the relation,

$$u_{n+1} = u_1 - \sum_{i=1}^{n} y_i$$

where an arbitrary choice for the first node is made.

# Weakly Nonlinear Regime

Considering small amplitude solutions $\frac{|y_n|}{\delta_0} \ll 1$, implies we can Taylor expand $F(x)$ :

$$F(x) \approx \gamma \delta_0^{3/2} + \frac{3}{2}\gamma \delta_0^{1/2} x + \gamma \frac{3}{8} \delta_0^{-1/2} x^2 - \gamma \frac{3}{48} \delta_0^{-3/2} x^3$$

such that the linear problem becomes:

$$M \ddot{y}_n = \frac{3}{2}\gamma \delta_0^{1/2} (y_{n-1} - 2y_n + y_{n+1})$$

# Spectral Situation

The linear problem is solved by,

$$y_n(t) = e^{i(kn+\omega t)}$$

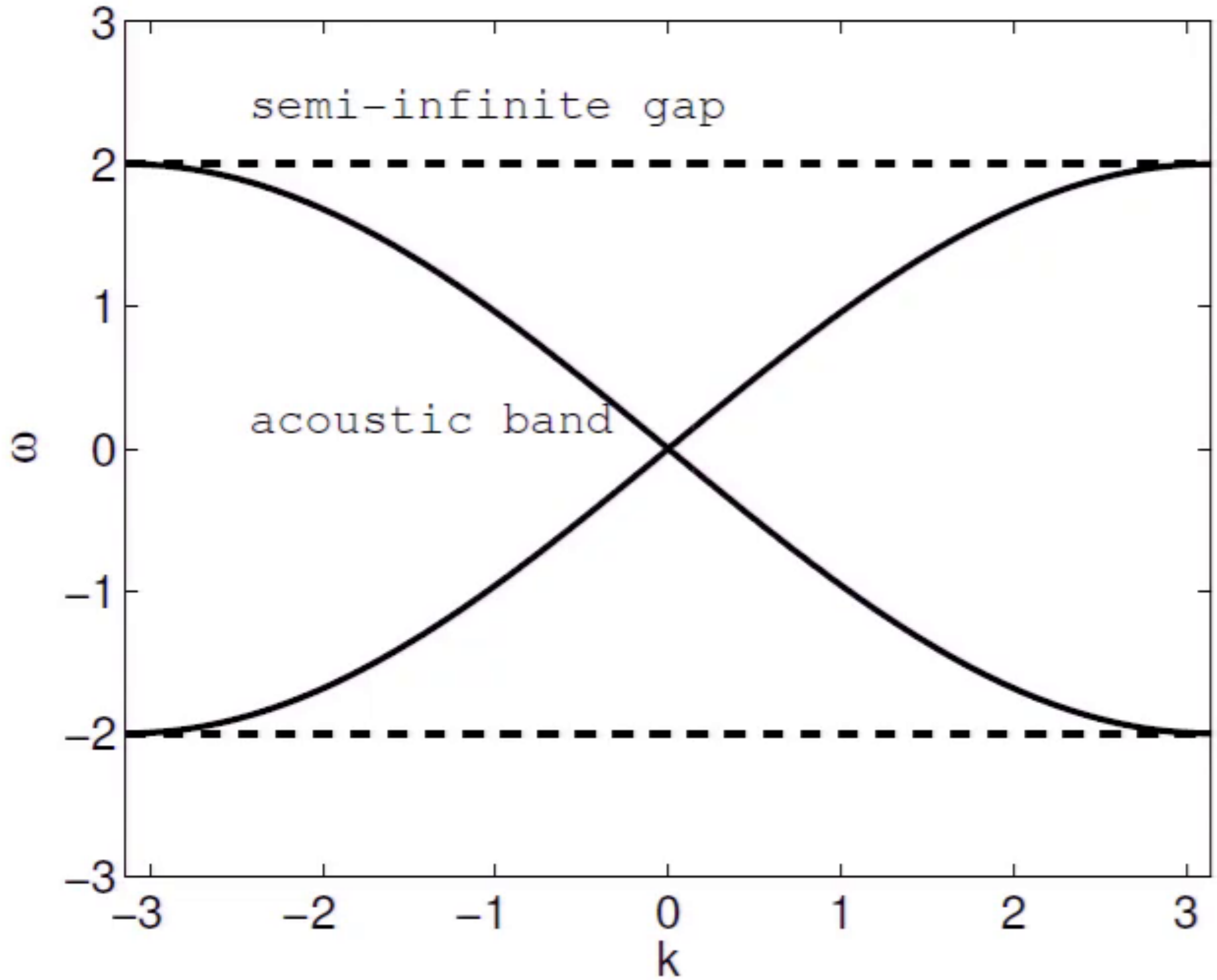
for all $k \in \mathbb{R}$ where $\omega$ and $k$ are related through the dispersion relation,

$$\omega(k)^2 = 2K_2/M(1 - \cos k)$$

such that linear spectrum is given by

$$\sigma = \left[-2\sqrt{K_2/M}, 2\sqrt{K_2/M}\right]$$

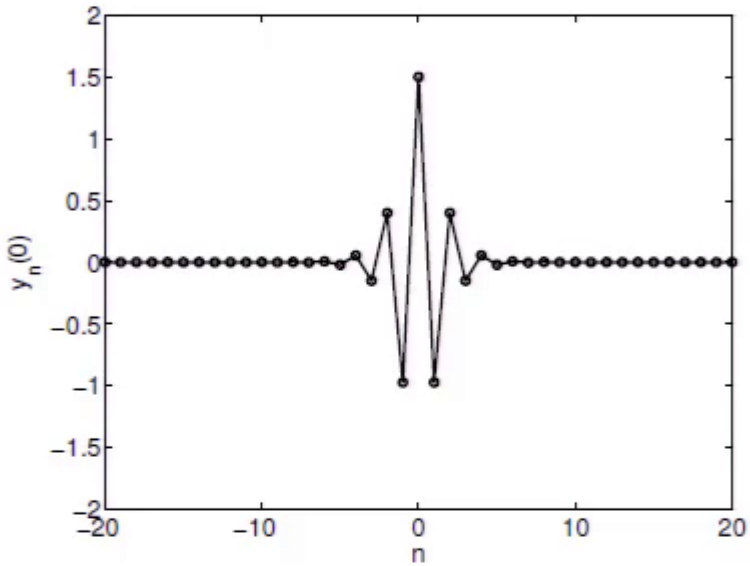
# Spectral Situation



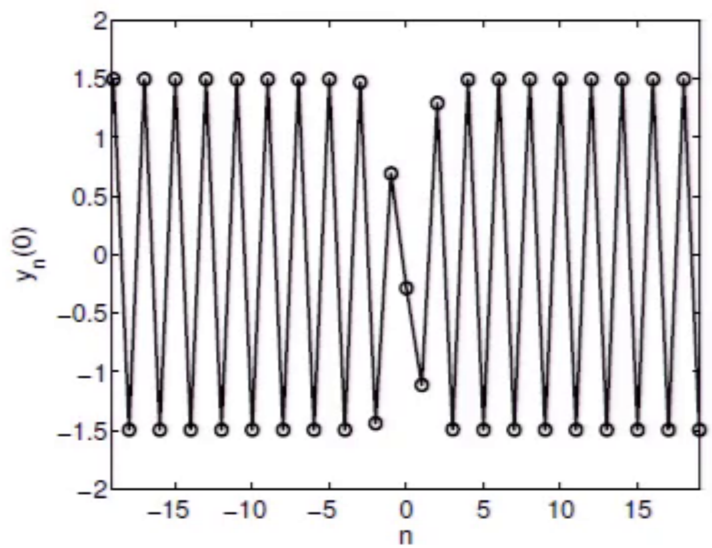
# Discrete Breathers

Breathers are time period solutions that are localized in space, e.g.:

$$y_n = (-1)^n a_n \cos(\omega t) + h.o.t.$$




Bright: $a_n = \alpha \text{sech}(\beta n)$




Dark: $a_n = \alpha \tanh(\beta n)$

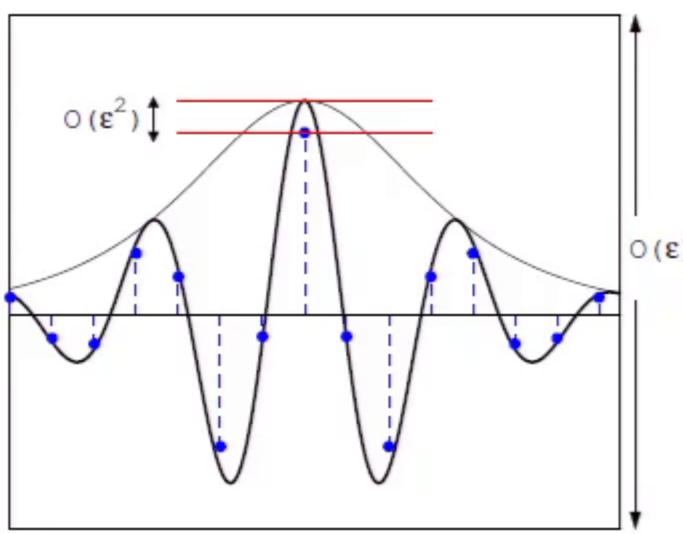
# Derivation of NLS Equation

The multiple scale ansatz

$$y_n(t) \approx \psi_n(t) := \epsilon A(X,T)e^{i(k_0 n+\omega_0 t)}+\text{c.c.}\,, \quad X = \epsilon(n+ct), \quad T = \epsilon^2 t$$

where $\epsilon \ll 1$ is a small parameter, is the canonical ansatz to derive the NLS equation from wave equations e.g. in optics and water waves.

** $A(X,T)$ is *continuous* in both its arguments **



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where $\epsilon \ll 1$ is a small parameter, is the canonical ansatz to derive the NLS equation from wave equations e.g. in optics and water waves.

$\mathcal{O}(\varepsilon^{-1}\varepsilon)$ : the dispersion relation $\omega_0 = \omega(k_0)$

$\mathcal{O}(\varepsilon^{-1}\varepsilon^2)$:the group velocity relation $c = \omega'(k_0)$

$\mathcal{O}(\varepsilon^{-1}\varepsilon^3)$:and the nonlinear Schrödinger equation (NLS),

$$i\partial_T A(X,T) + \nu_2 \partial_X^2 A(X,T) + \nu_3 A(X,T)|A(X,T)|^2 = 0$$

# The NLS equation

The NLS equation,

$$i\partial_T A(X,T) + \nu_2 \partial_X^2 A(X,T) + \nu_3 A(X,T)|A(X,T)|^2 = 0$$

has steady-states which have the form,

$$A(X,T) = \tilde{A}(X)e^{i\kappa T}$$

where $\kappa \in \mathbb{R}$ and $\tilde{A}(X)$ satisfies the Duffing equation,

$$\partial_X^2 \tilde{A} = \frac{1}{\nu_2}\left(\kappa\tilde{A} - \nu_3\tilde{A}^3\right)$$

Which has the exact solutions,

$$\tilde{A}(X) = c_1 \tanh(c_2 X) \qquad \text{if } \nu_3 < 0, \kappa < 0$$

$$\tilde{A}(X) = c_3 \operatorname{sech}(c_4 X) \qquad \text{if } \nu_3 > 0, \kappa > 0$$

# Non-Existence of (bright) breathers

Since we seek standing wave solutions, we chose the wavenumber to be at the edge of the phonon band $k_0 = \pi$, such that the group velocity vanishes, $\omega_0 = 2\sqrt{K_2/M}$, and

$$\nu_3|_{k_0=\pi} = 3K_2K_4 - 4K_3^2 = B < 0.$$

**In the case of the granular chain, NLS is defocusing**

Thus, the only possible (NLS) steady-state is,

$$\tilde{A}(X) = c_1 \tanh(c_2 X) \qquad \text{if } \nu_3 < 0, \kappa < 0$$

- Bright breathers only if $B > 0$ (modulation instability) [Flach 1997]
- Bright breathers only if $B > 0$ (center manifold) [James 2003]

# NLS approximation

Returning to our ansatz we have the following approximation,

$$\begin{aligned} y_n(t) &\approx \epsilon \tilde{A} e^{i\kappa T} e^{i(k_0 n + \omega_0 t)} + \text{c.c.} \\ &= 2\epsilon(-1)^n \sqrt{\frac{\kappa}{\nu_3}} \tanh\left(\sqrt{\frac{-\kappa}{2\nu_2}} \epsilon n\right) \cos(\omega_b t) \qquad\qquad (\text{for } k_0 = \pi) \end{aligned}$$

where $\omega_b = \omega_0 + \kappa\epsilon^2$ is the frequency of the breather, where $\kappa < 0$ is a fixed but arbitrary parameter.

# Non-Existence of (bright) breathers

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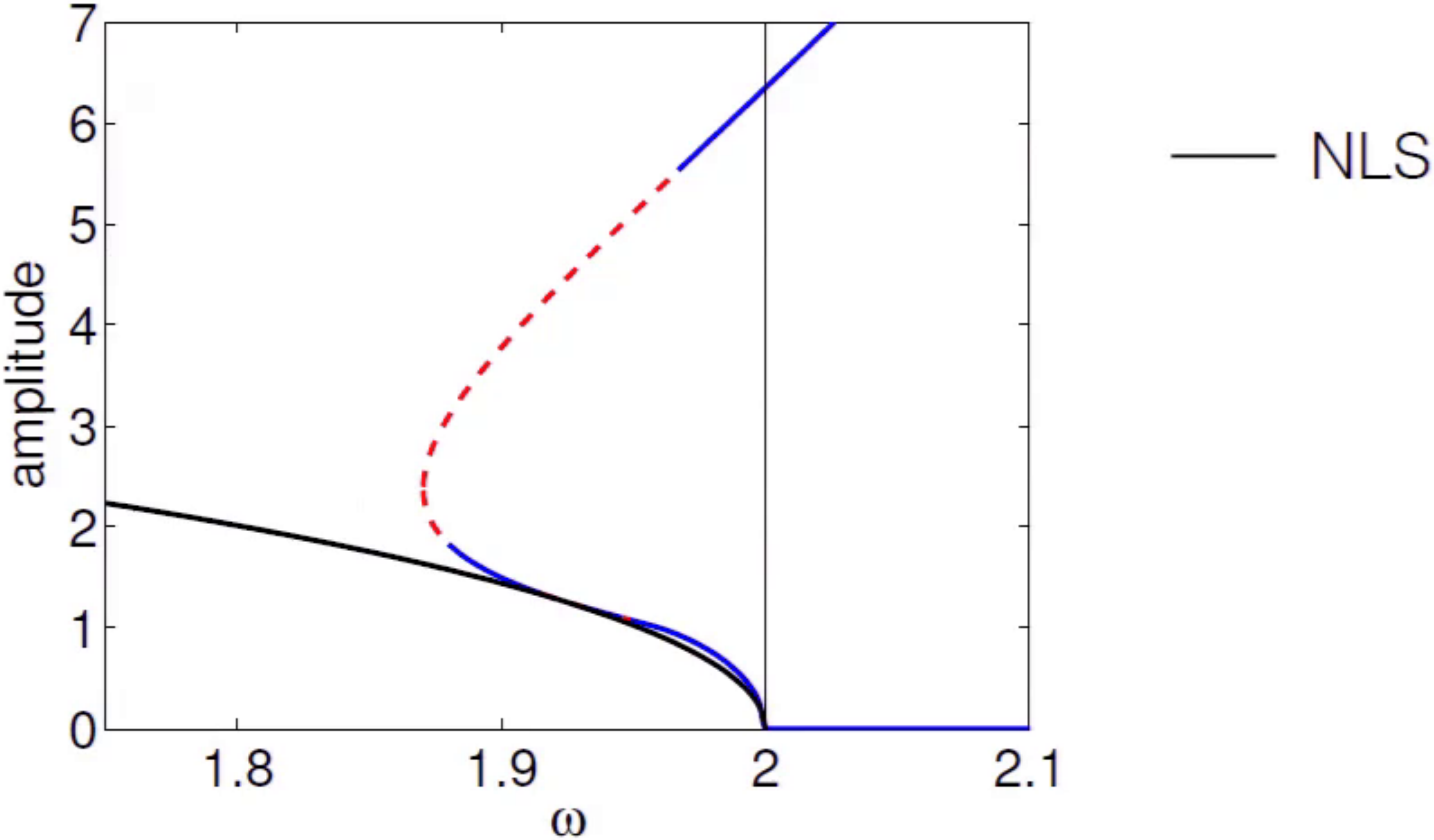
# Time-period solutions and their spectral stability and bifurcations

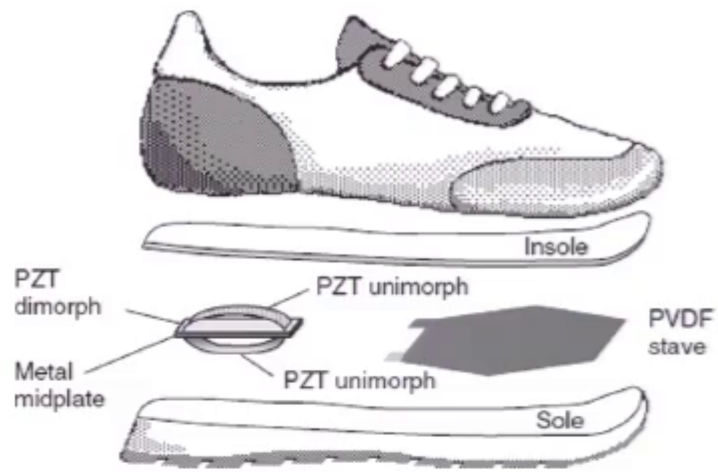
- We numerically compute time-periodic solutions by finding roots of the map $F(x) = x(0) - x(T)$ via Newton iterations.
- Solutions are continued via pseudo-arclength continuation
- We linearize about these time periodic solutions and compute the Floquet multipliers of the resulting Hills equation

# Comparison of NLS Prediction to Numerics



# Vibration Energy Harvesting

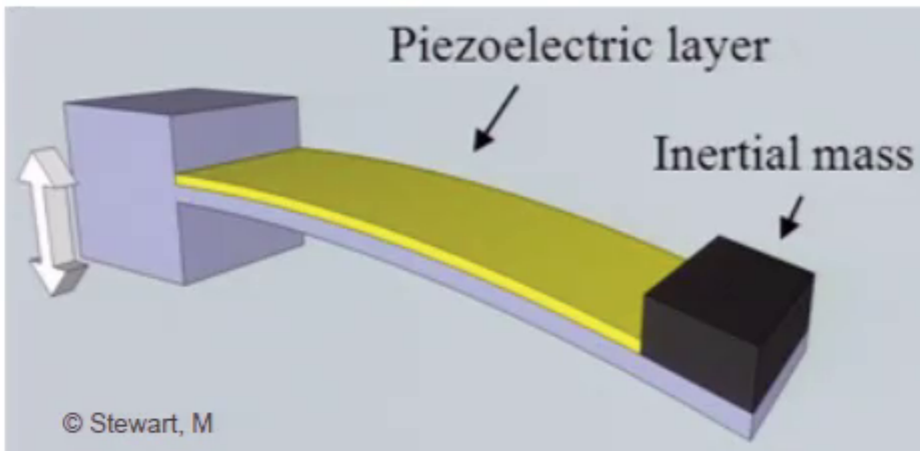
- Goal: Enable self-powered electronic devices by scavenging ambient energy (vibrations)
- Energy Conversion: Electromagnetic Induction, Electrostatic Transduction, **Piezoelectric Transduction**
- Ambient vibration:
  white noise <—> **harmonic**


Energy scavenging with shoe mounted piezoelectrics
N. A. Shenck, J. A. Paradiso
*IEEE Micro, vol. 21, p. 30–42, 2001-05/06*

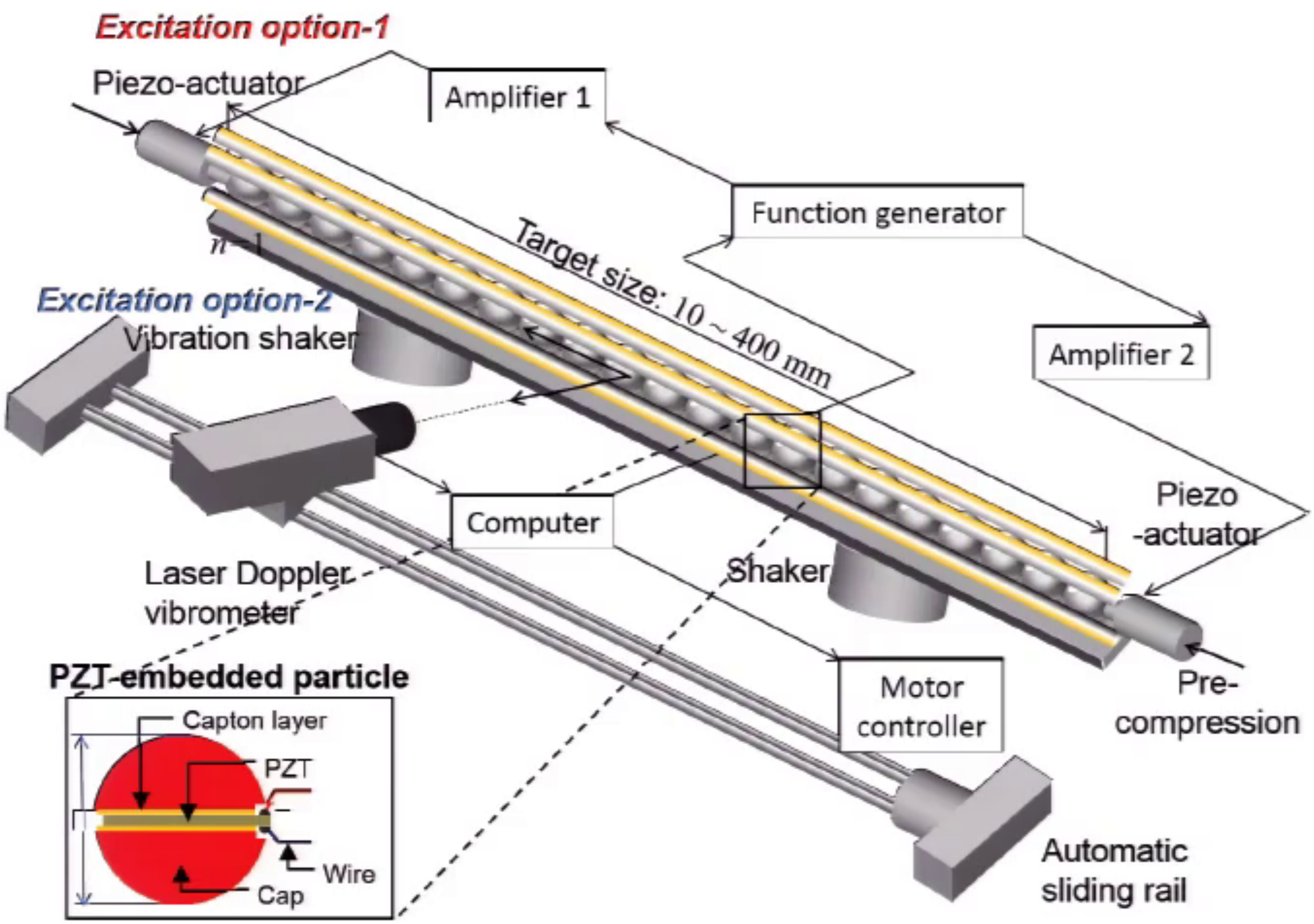
# Linear Harvester

- Conventional devices are linear and well studied (Erturk 2013)
- Inefficient if input frequency varies or is not known a priori

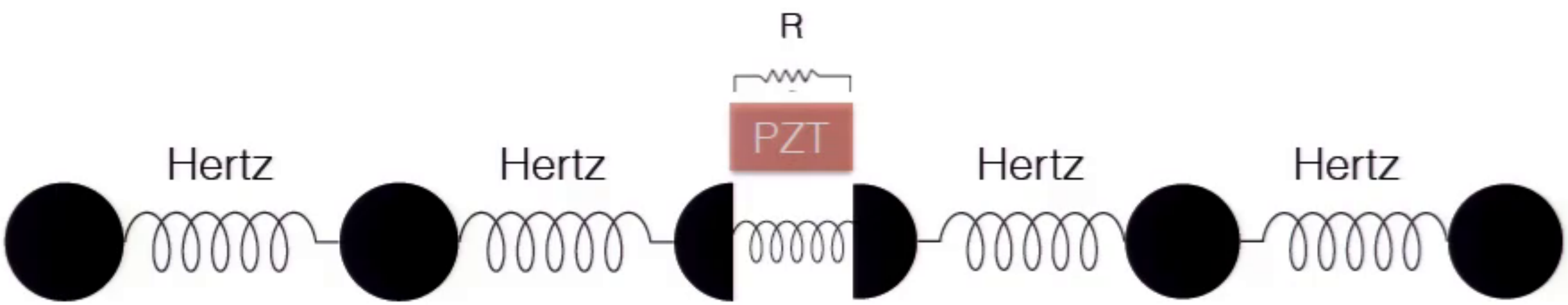


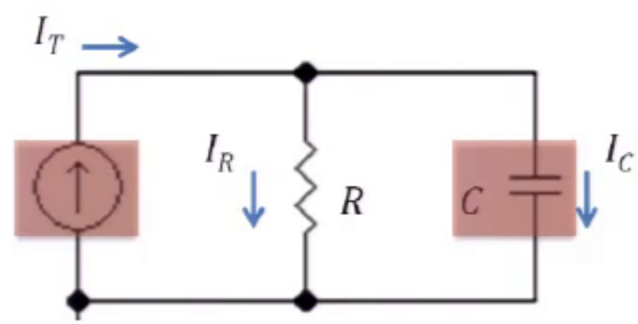

- We consider a spatially extended and nonlinear system and test its efficiency as an energy harvester

# Experimental Set-up for Energy Harvesting



# Electromechanical Model



# Electromechanical Model

**Constitutive relation**

$$\begin{aligned} D &= \epsilon^T E + d_{33} T \\ S &= d_{33} E + s^E T \end{aligned}$$

+

**Granular Model**

$$M \partial_t^2 u_n = \gamma[\delta_0 + u_{n-1} - u_n]_+^{3/2} - \gamma[\delta_0 + u_n - u_{n+1}]_+^{3/2}$$

$$\begin{aligned} M \ddot{u}_n &= \gamma[\delta_0 + u_{n-1} - u_n]_+^{3/2} - \gamma[\delta_0 + u_n - u_{n+1}]_+^{3/2} - \frac{M}{\tau}\dot{u}_n, \quad n \notin \{m, m+1\} \\ \frac{M}{2}\ddot{u}_m &= \gamma[\delta_0 + u_{m-1} - u_m]_+^{3/2} - K_a(\delta_1 + u_m - u_{m+1}) - d_{33} K_a V - \frac{M}{2\tau}\dot{u}_m, \\ \frac{M}{2}\ddot{u}_{m+1} &= K_a(\delta_1 + u_m - u_{m+1}) - \gamma[\delta_0 + u_{m+1} - u_{m+2}]_+^{3/2} + d_{33} K_a V - \frac{M}{2\tau}\dot{u}_{m+1}, \\ RC(1-k^2)\dot{V} &= d_{33} K_a R(\dot{u}_m - \dot{u}_{m+1}) - V \end{aligned}$$

$$k^2 = \frac{d_{33}^2}{S^E \epsilon^T}, \qquad K_a = \frac{A}{S^E d} \qquad C = \frac{\epsilon^T A}{d}$$

**Boundary Actuation**

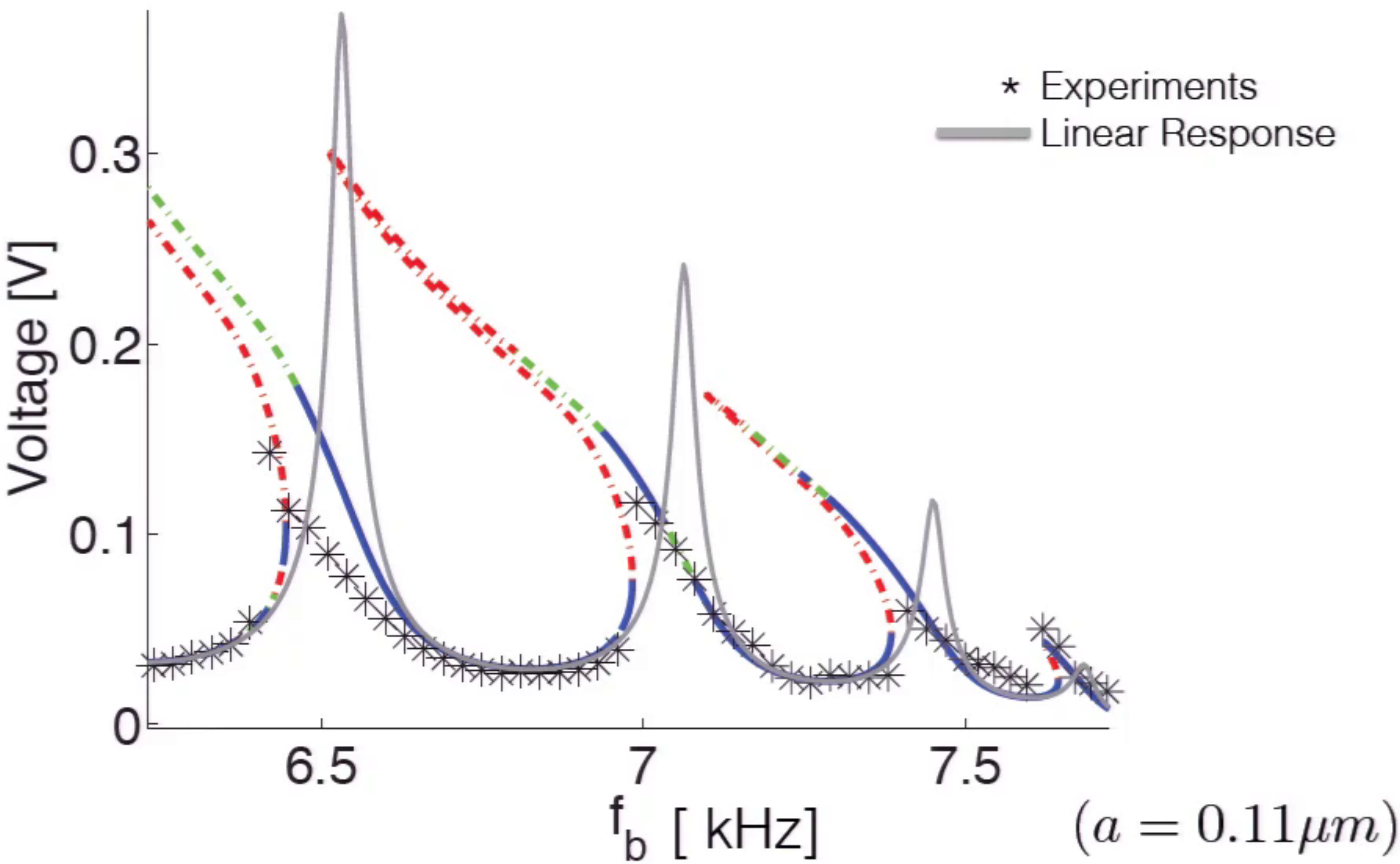
$$\begin{aligned} u_0 &= a\cos(2\pi f_b t) \\ u_{N+1} &= b\cos(2\pi f_b t) \end{aligned}$$

# Experimental Values and Set-up

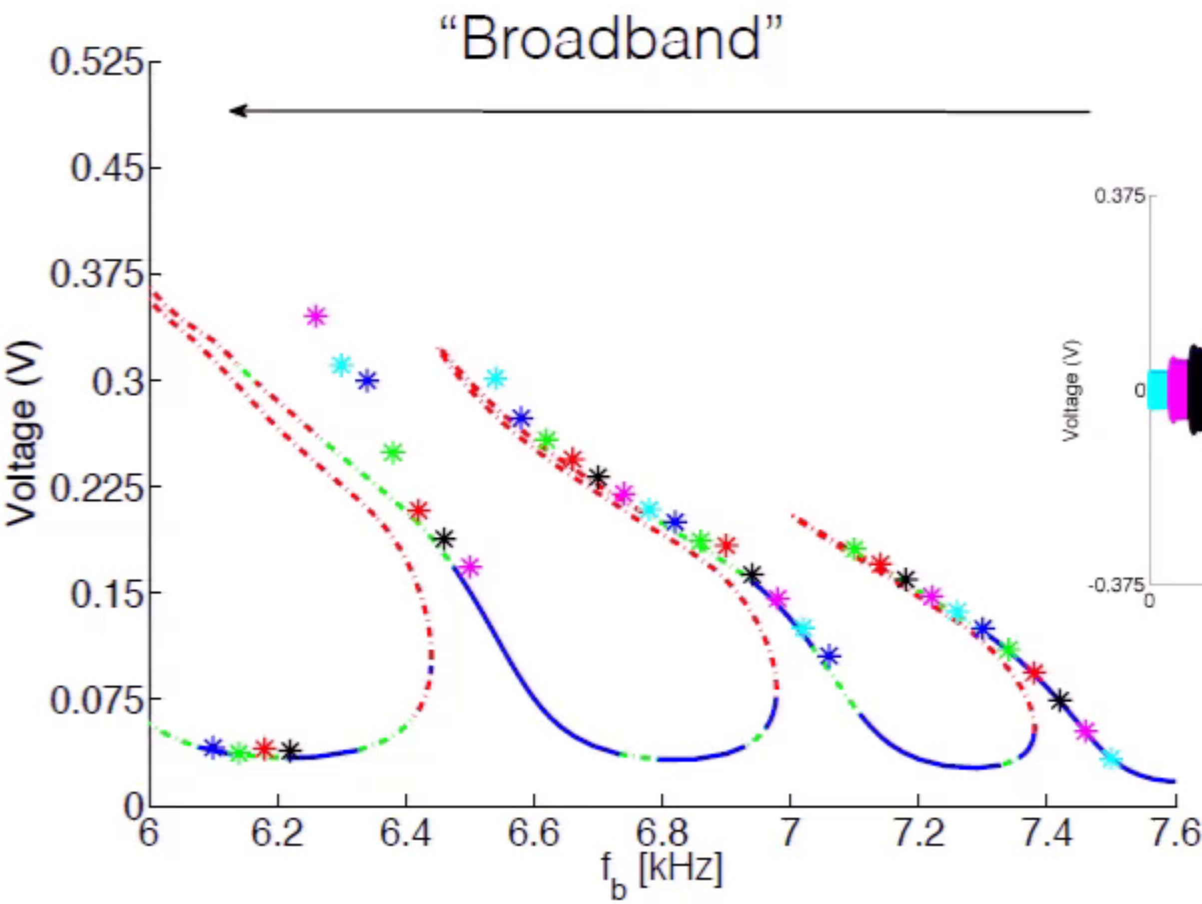
| Mechanical Parameters | | | Electrical Parameters | | |
|---|---|---|---|---|---|
| Bead Mass | $M$ | 28.2 [g] | Piezo Constant | $d_{33}$ | $360 \cdot 10^{-12}$ m/V |
| Bead Young's Modulus | $E$ | 200 [Gpa] | Piezo Permittivity | $\epsilon^T$ | $14166 \cdot 10^{-12}$ F/m |
| Bead Radius | $r$ | 9.53 [mm] | Piezo Compliance | $S^E$ | 72 Gpa$^{-1}$ |
| Bead Poisson Ration | $\nu$ | 0.3 | Piezo Disc Area | $A$ | 28.353 mm |
| Damping Coefficient | $\tau$ | 5 [ms] | Piezo Thickness | $d$ | 0.3 mm |
| | | | Resistance | $R$ | $3\,K\Omega$ |

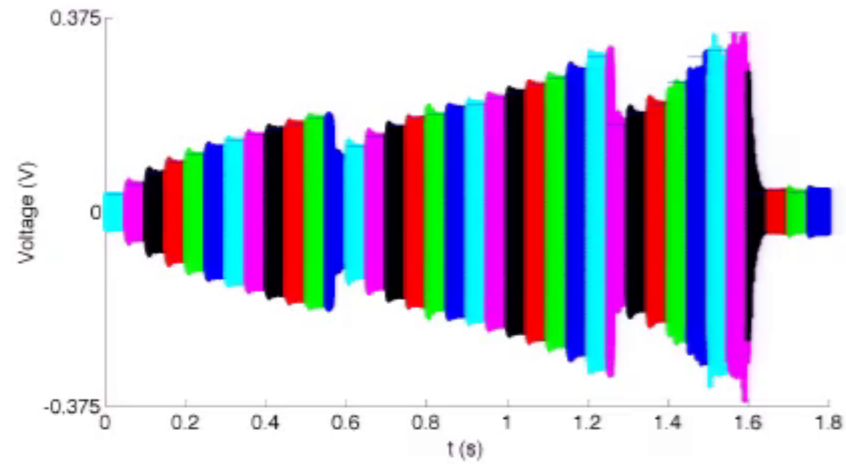
TABLE I: Theoretical values of the electromechanical parameters.

# Theory vs Experiments



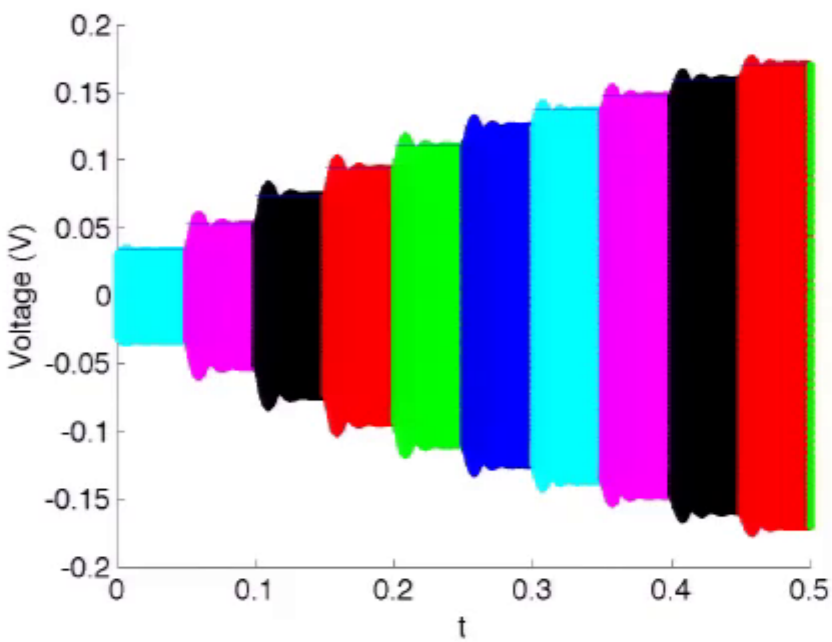
# High to Low Frequency Sweep



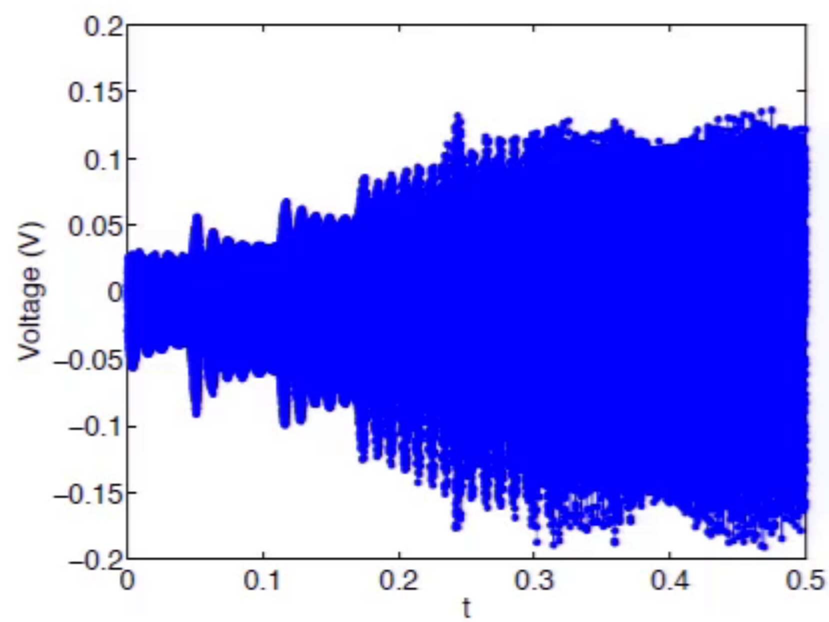




decrease frequency 40 Hz every .05 seconds

# Numerics vs Experiment: High to Low Frequency Sweep

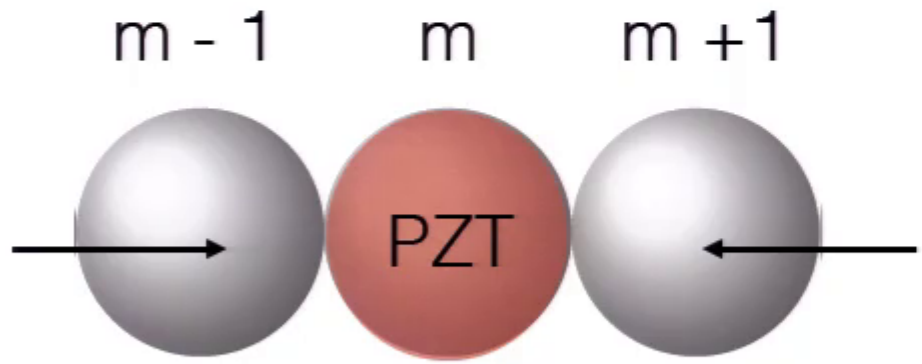



Theory




Experiment

# Mechanical System




The amount the PZT bead is squeezed is related to voltage production:

$$s_m(t) = u_{m-1} - u_{m+1}$$

In terms of the strain $y_m = u_{m-1} - u_m$ we have,

$$s_m(t) = y_m + y_{m+1}$$

# NLS Prediction

Recall, the NLS approximation:

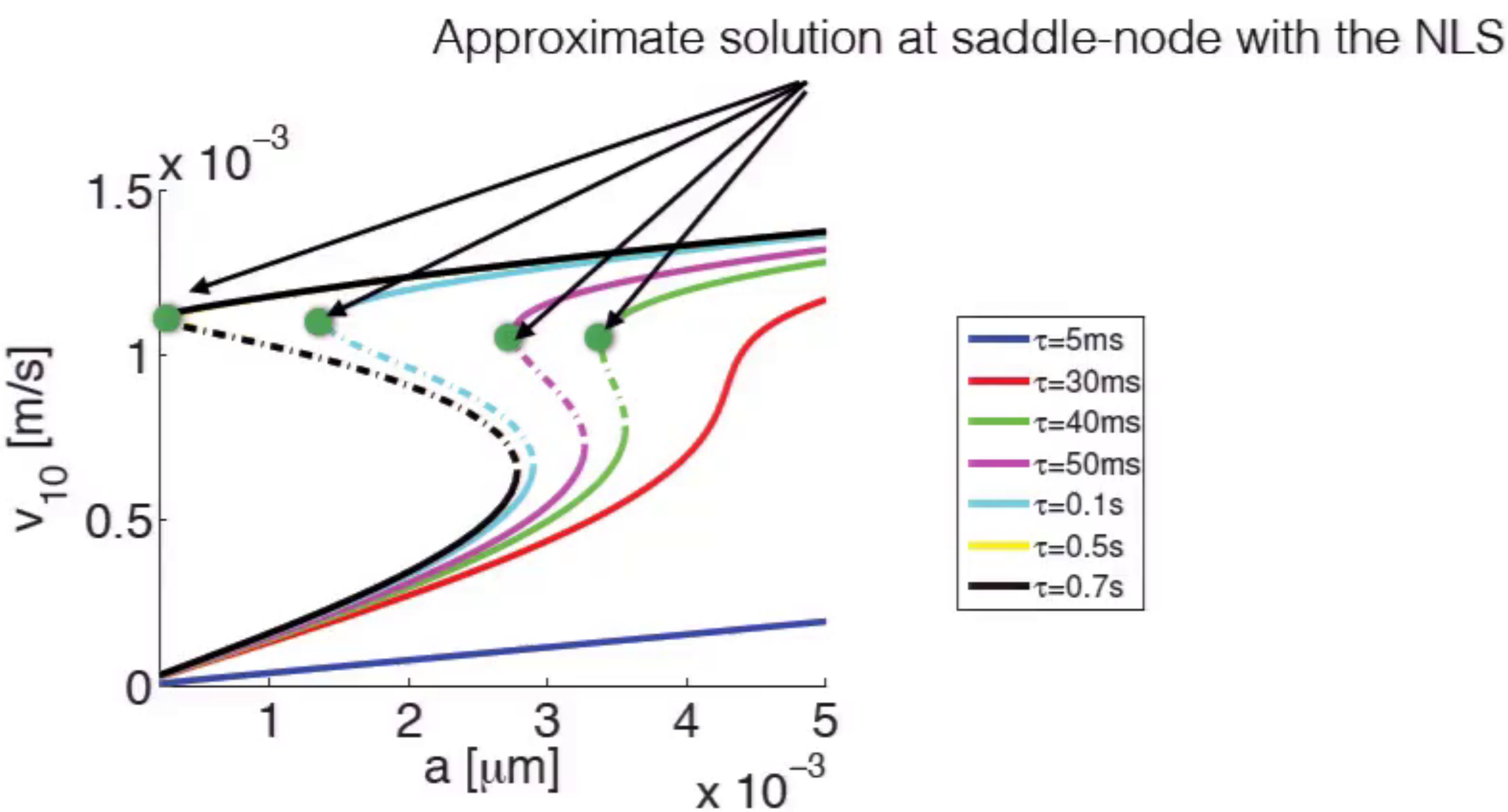
$$y_n(t) = 2\varepsilon(-1)^n\sqrt{\frac{\kappa}{\nu_3}}\tanh\left(\sqrt{\frac{-\kappa}{2\nu_2}}\varepsilon(n-x_0)\right)\cos(\omega_b t)$$

Where $\omega_b = \omega_0 + \kappa\epsilon^2$ is the frequency and $\kappa < 0$ is a fixed but arbitrary parameter. In terms of the "squeeze", we have

$$s_n(t) = 2\epsilon(-1)^{n+1}\sqrt{\tfrac{\kappa}{\nu_3}}\left(\tanh\left(\sqrt{\tfrac{-\kappa}{2\nu_2}}\epsilon(n+1)\right) - \tanh\left(\sqrt{\tfrac{-\kappa}{2\nu_2}}\epsilon n\right)\right)\cos(\omega_b t)$$

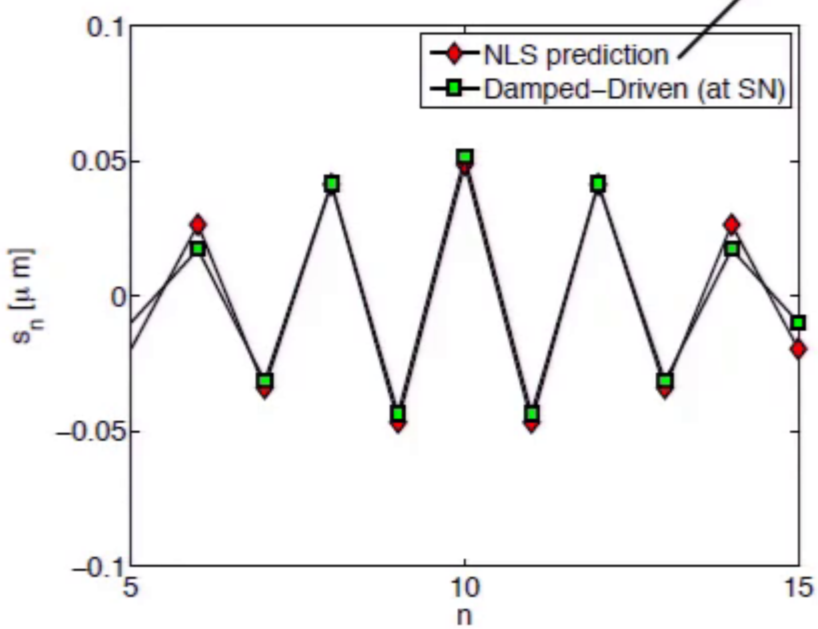
$$\approx 2\epsilon^2(-1)^{n+1}\sqrt{\tfrac{\kappa}{\nu_3}}\left(1 - \tanh^2\left(\sqrt{\tfrac{-\kappa}{2\nu_2}}\epsilon n\right)\right)\cos(\omega_b t)$$

# Continuation in Amplitude and the NLS approximation



# NLS Prediction

$$s_n(t) = 2\epsilon(-1)^{n+1}\sqrt{\frac{\kappa}{\nu_3}}\left(\tanh\left(\sqrt{\frac{-\kappa}{2\nu_2}}\epsilon(n+1.5)\right) - \tanh\left(\sqrt{\frac{-\kappa}{2\nu_2}}\epsilon(n+.5)\right)\right)\cos(\omega_b t)$$




We can back calculate the voltage: