2D Nonlinear Energy Channeling in the Locally Resonant Structure

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Outline

1. Background – locally resonant acoustic metamaterials

2. The new concept of 2D unidirectional wave entrapment in acoustic metamaterials

3. Dynamics of the 2D Unit-Cell-Model
   - Low Energy Excitations
   - High Energy Excitations

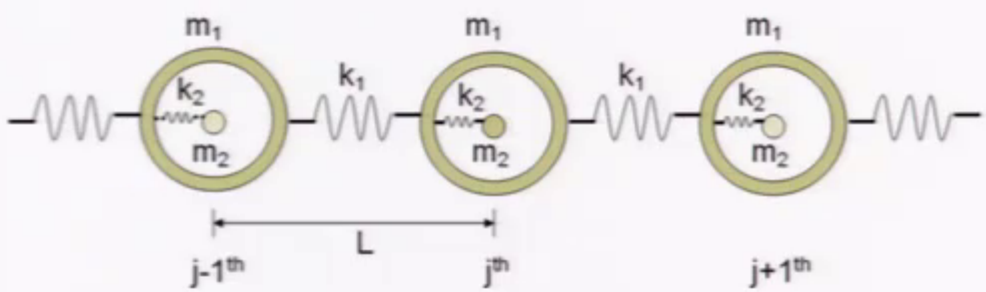
4. Conclusions and Future work

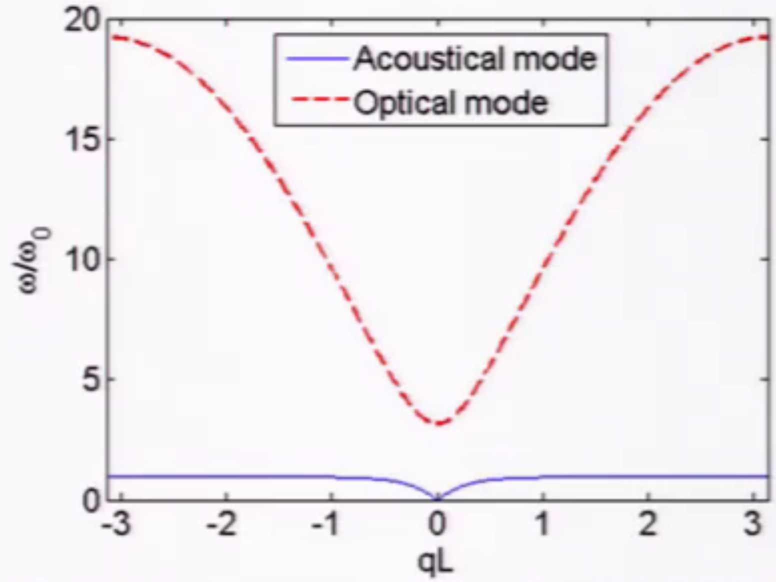
Background – Linear Absorptive Metamaterials

Mechanical metamaterials

Mechanical and acoustic metamaterials are engineered systems possessing selected material properties which are mainly affected by a unit-cell element structure rather than material composition

Locally resonant chain





H.H. Huang, C.T. Sun, G.L. Huang, *On the negative effective mass density in acoustic metamaterials*, International Journal of Engineering 47, 610-617, (2009)

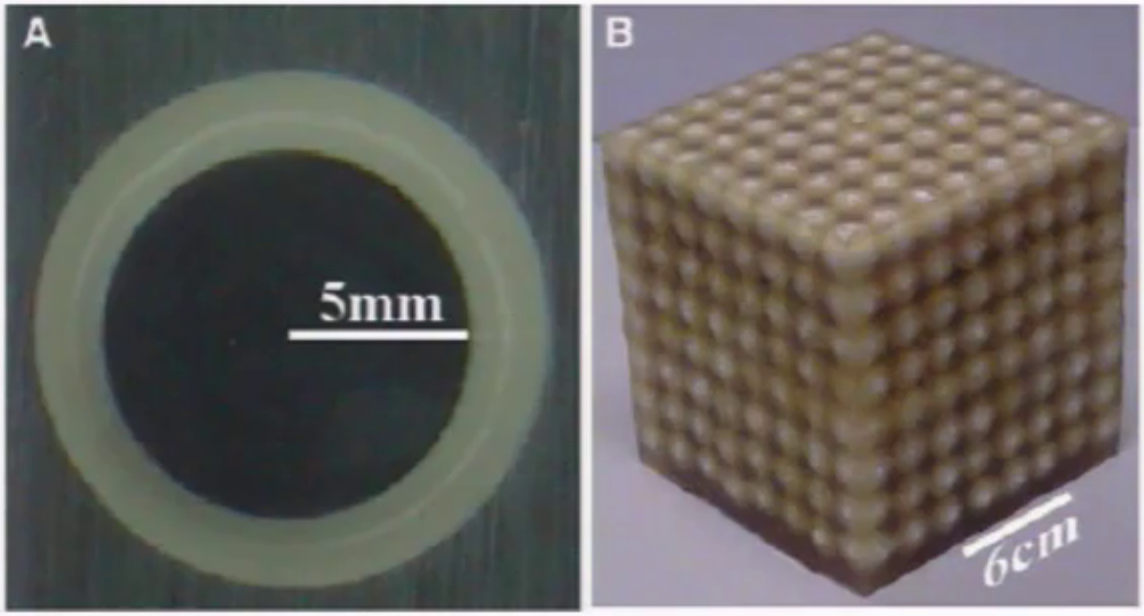
Liu Z., Zhang X., Mao Y., Zhu Y. Y., Yang Z., Chan C. T., Sheng P., Locally resonant sonic materials, Science 289, 1734 (2000)

Advantage: **low-frequency band gaps**

Properties: **negative effective mass, negative refractive index**

Lee S. et. al. , Acoustic metamaterial with negative density, Phys. Lett. A 373, 4464 (2009).
Liu X., et. Al., Wave propagation characterization and design of two-dimensional elastic chiral metacomposite, J. Sound Vib. 330, 2536 (2011)

Background – Linear Absorptive Metamaterials

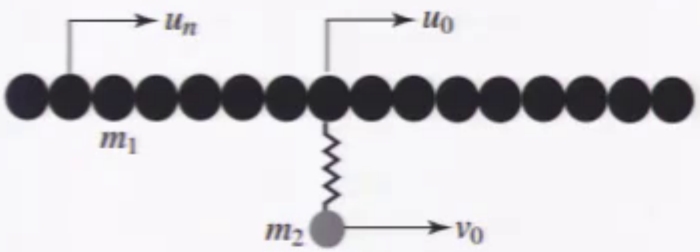


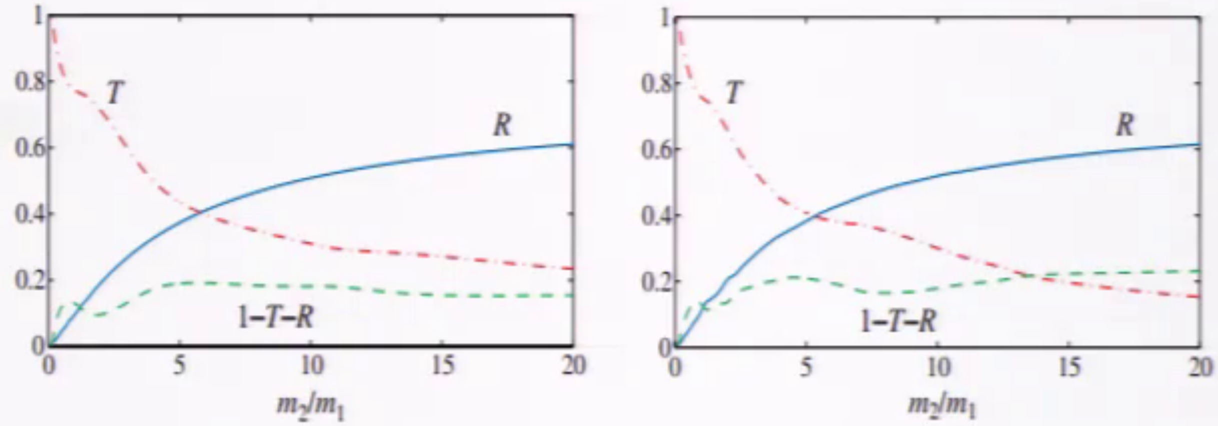
Liu Z., Zhang X., Mao Y., Zhu Y. Y., Yang Z., Chan C. T., Sheng P., Locally resonant sonic materials, Science 289, 1734 (2000)

The locally resonant sonic material is composed of inclusions of dense material shapes covered with a soft material, with the inclusions embedded in the matrix.

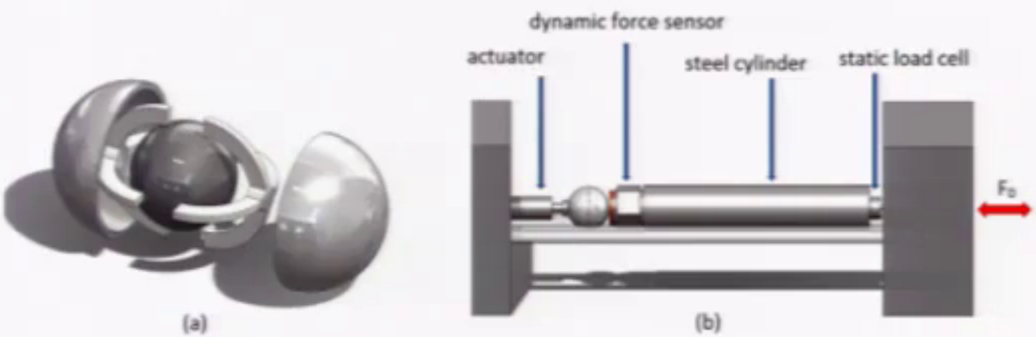
Background – Non Linear Absorptive Metamaterials

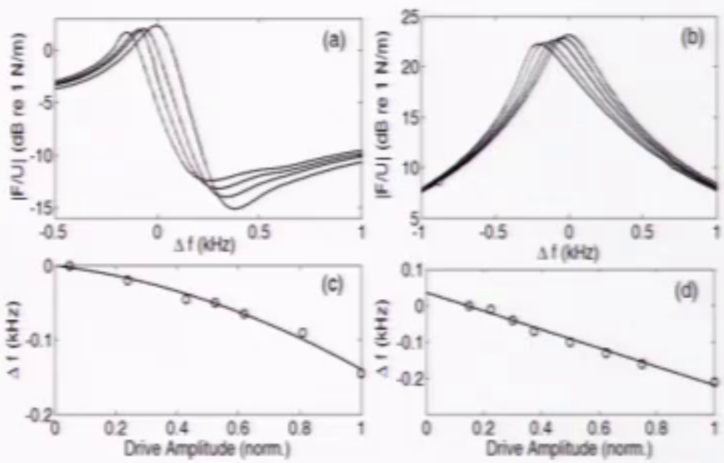
Dynamics of locally resonant, granular meta-materials





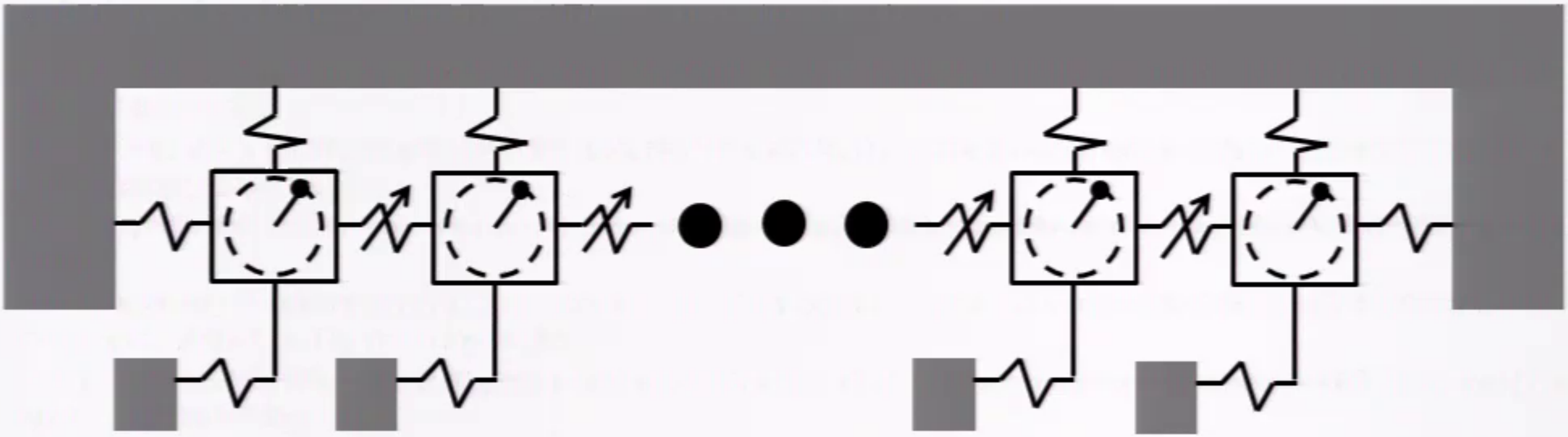
P. G. Kevrekidis, A. Vainchtein, M. Serra-Garcia, and C. Daraio. ***Interaction of traveling waves with mass-with-mass defects within a Hertzian chain*****. Phys. Rev. E 87, 042911, 2013**





L. Bonanomi, G. Theocharis, D. Ngo, and C. Daraio, ***Locally resonant granular crystals*****, arXiv:1403.1052**

New concept of the locally resonant structure



Governing Equations of Motion

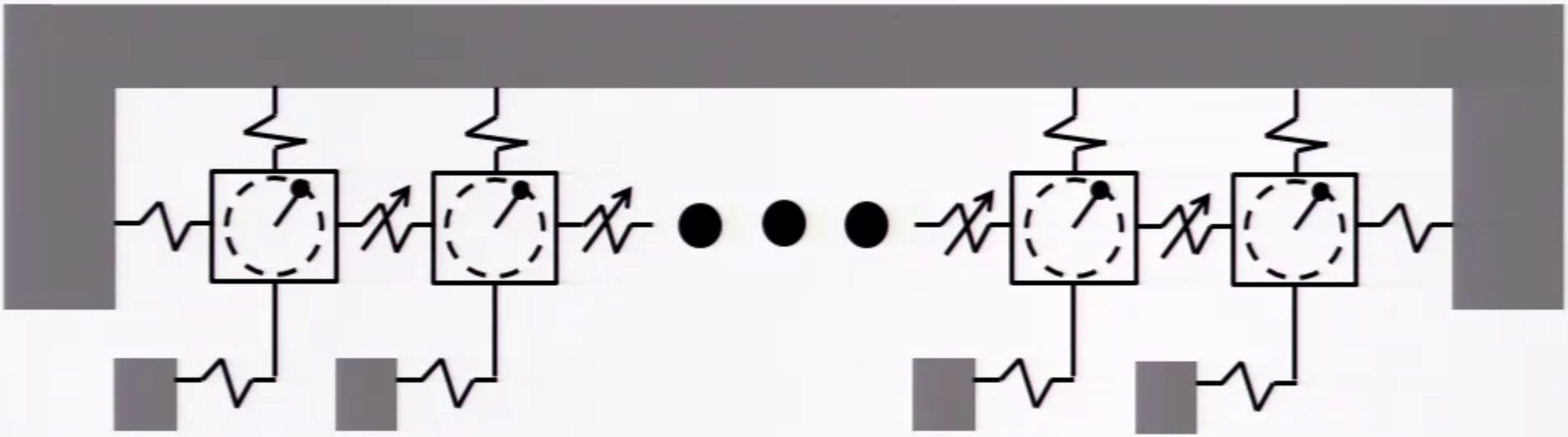
$$(M+m)\ddot{x}_i - mR\left(\ddot{\theta}_i \sin\theta_i + \dot{\theta}_i^2 \cos\theta_i\right) + kx_i - K\left[\left(x_{i-1}-x_i\right)^n - \left(x_i - x_{i+1}\right)^n\right] = 0$$

$$(M+m)\ddot{y}_i + mR\left(\ddot{\theta}_i \cos\theta_i - \dot{\theta}_i^2 \sin\theta_i\right) + ky_i = 0$$

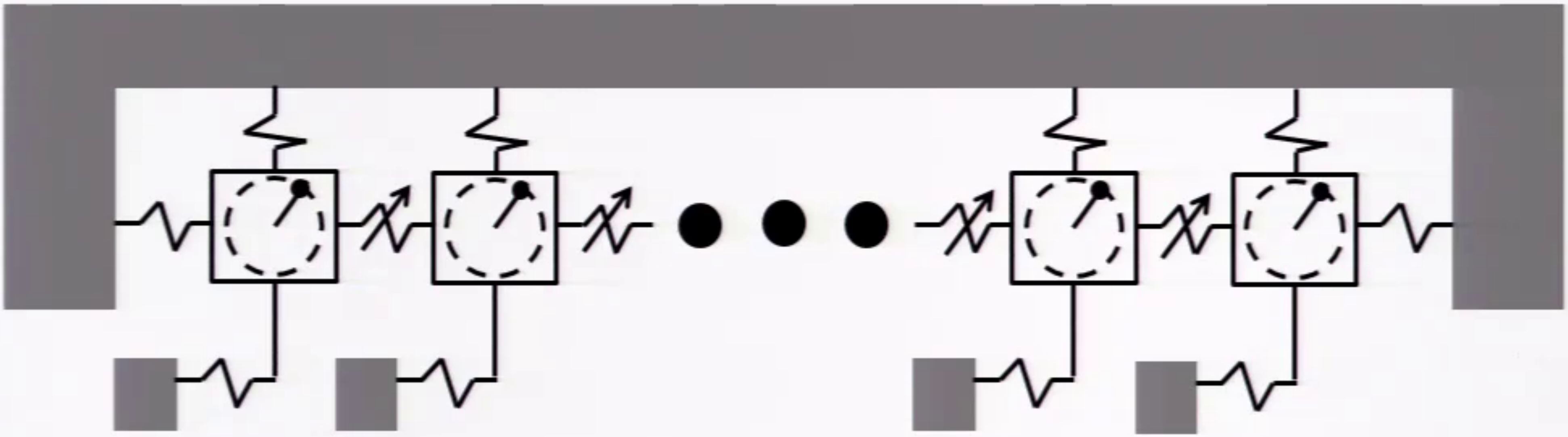
$$mR^2\ddot{\theta}_i - mR\ddot{x}_i \sin\theta_i + mR\ddot{y}_i \cos\theta_i = -c\dot{\theta}_i$$

x_i - is the displacement of the i^{th} element in the horizontal direction, y_i - is the displacement of the i^{th} element in the vertical direction, R – is the radius of the internal cavity of the rotator, M_i – is the mass of each element, θ_i - is the deviation angle from horizontal direction, k – is the constant of the linear springs, K – is the constant of the non-linear springs and c – is the linear dissipation term.

New concept of the locally resonant structure



New concept of the locally resonant structure



System nondimensionalization:

$$\tau = \omega_s t, \quad \omega_s^2 = \frac{k}{M}, \quad x_i = R\xi_i, \quad y_i = R\psi_i, \quad \varepsilon = \frac{m}{M}, \quad \kappa = \frac{K}{k}R^{n-1}, \quad \mu = \frac{c}{m\omega_s R^2}$$

Non-dimensional EOM

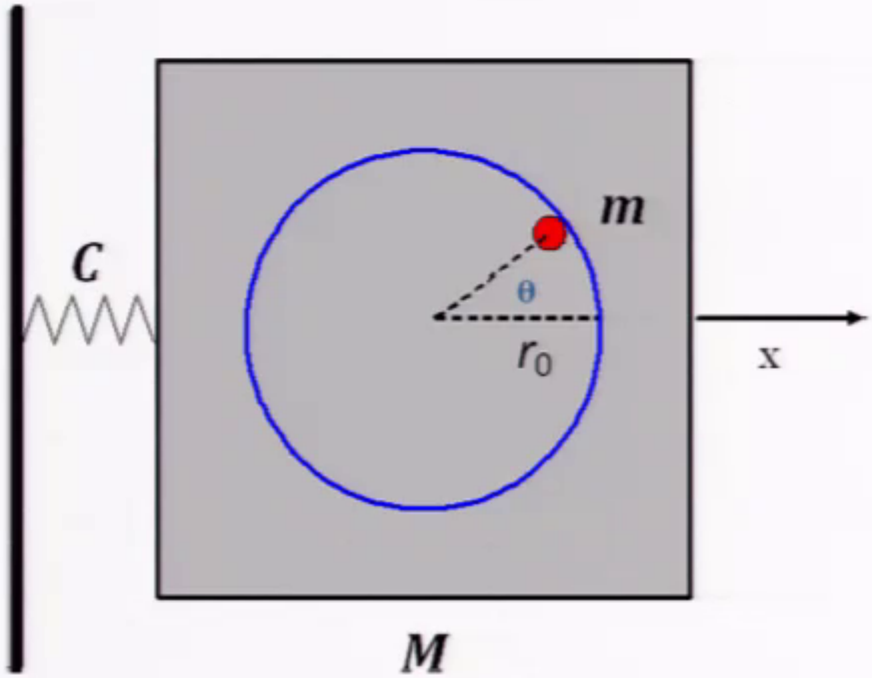
$$(1+\varepsilon)\xi_i'' - \varepsilon\left(\theta_i''\sin\theta_i + \theta_i'^2\cos\theta_i\right) + \xi_i - \kappa\left[\left(\xi_{i-1}-\xi_i\right)^n - \left(\xi_i - \xi_{i+1}\right)^n\right] = 0$$

$$(1+\varepsilon)\psi_i'' + \varepsilon\left(\theta_i''\cos\theta_i - \theta_i'^2\sin\theta_i\right) + \psi_i = 0$$

$$\theta_i'' - \xi_i''\sin\theta_i + \psi_i''\cos\theta_i = -\mu\theta_i'$$

Internal rotators as vibration absorbers (Nonlinear Energy Sinks)

Internal rotators as nonlinear vibration absorbers (NES)



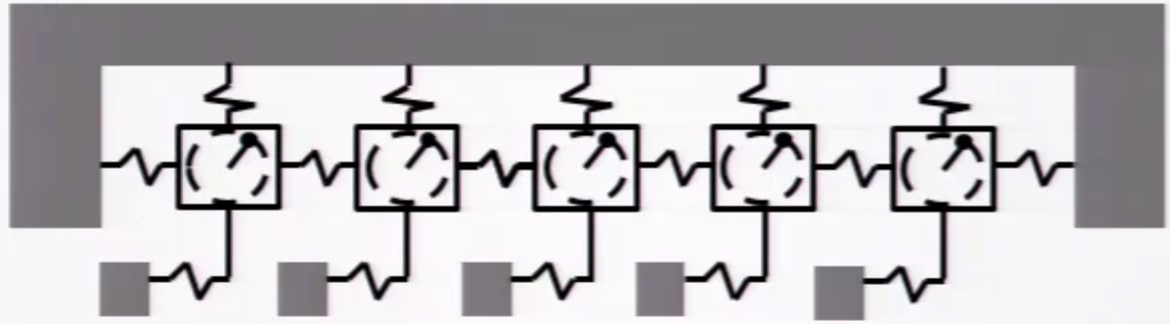
Gendelman O.V., Sigalov G., Manevitch L.I., Mane M., Vakakis A.F., Bergman L.A., Dynamics of an eccentric rotational nonlinear energy sink, J. Appl. Mech. 79(1), (2012), 011012.

Sigalov G., Gendelman O.V., AL-Shudeifat M.A., Manevitch L.I., Vakakis A.F., Bergman L.A., Resonance captures and targeted energy transfers in an inertially-coupled rotational nonlinear energy sink, Nonlinear Dynamics (2012) 69, 1693–1704.

Sigalov G., Gendelman O.V., AL-Shudeifat M.A., Manevitch L.I., Vakakis A.F., Bergman L.A., Alternation of regular and chaotic dynamics in a simple two-degree-of-freedom system with nonlinear inertial coupling, Chaos (2012) 22, 013118.

The concept of 2D energy channeling

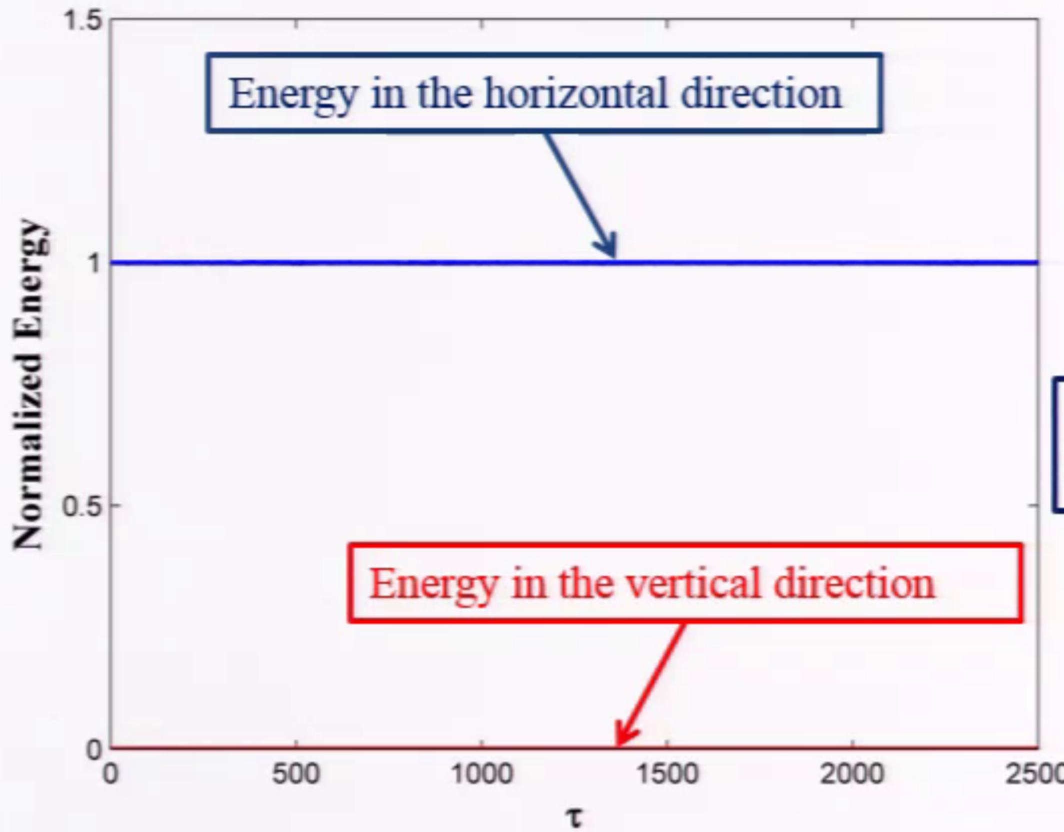
Numerical Evidences

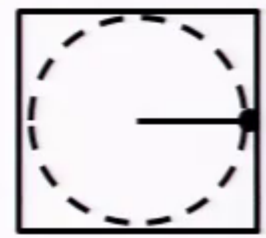


First scenario – Formation and unperturbed propagation of disturbance

Initial conditions:

$$\varepsilon = 0.05, \quad \mu = 0.0325, \quad n = 3, \quad \kappa = 1, \quad \xi_1'(0) = 0.08, \quad \theta_i(0) = 0, \quad \theta_i'(0) = 0, \quad 1 \le i \le 5$$

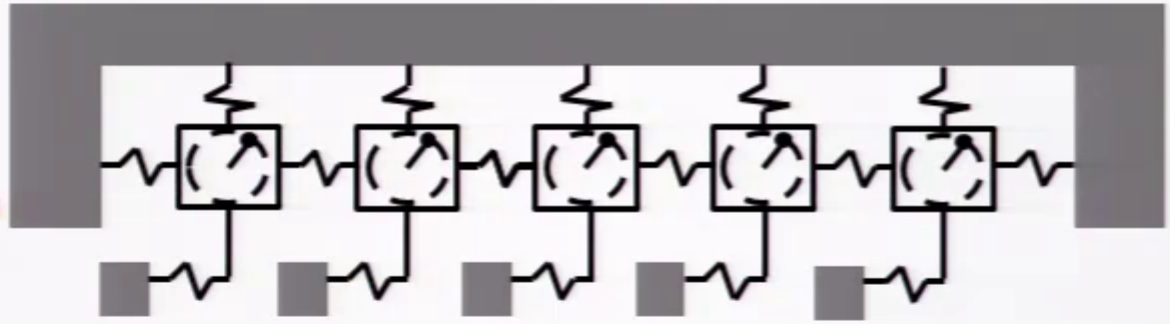




Conservation of Energy in the horizontal direction!

The concept of 2D energy channeling

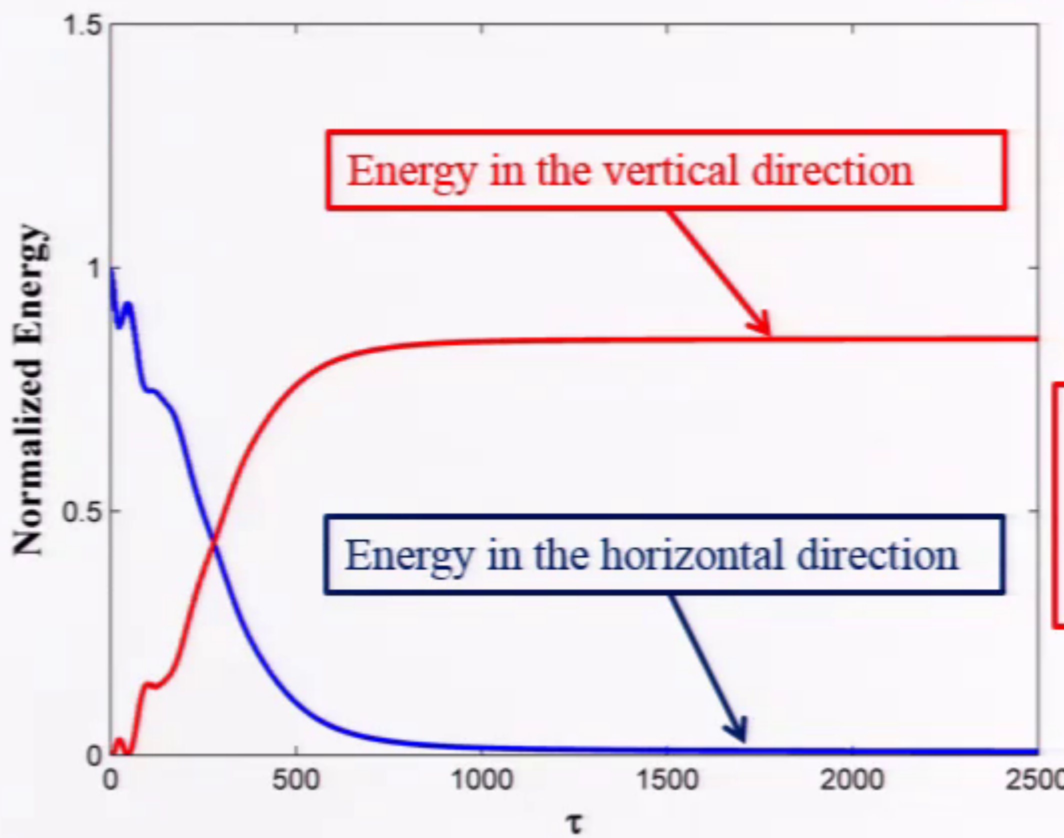
Numerical Evidences



Second scenario – **Nearly complete transport of the total system energy and its permanent entrapment in the vertical direction.**

Initial conditions:

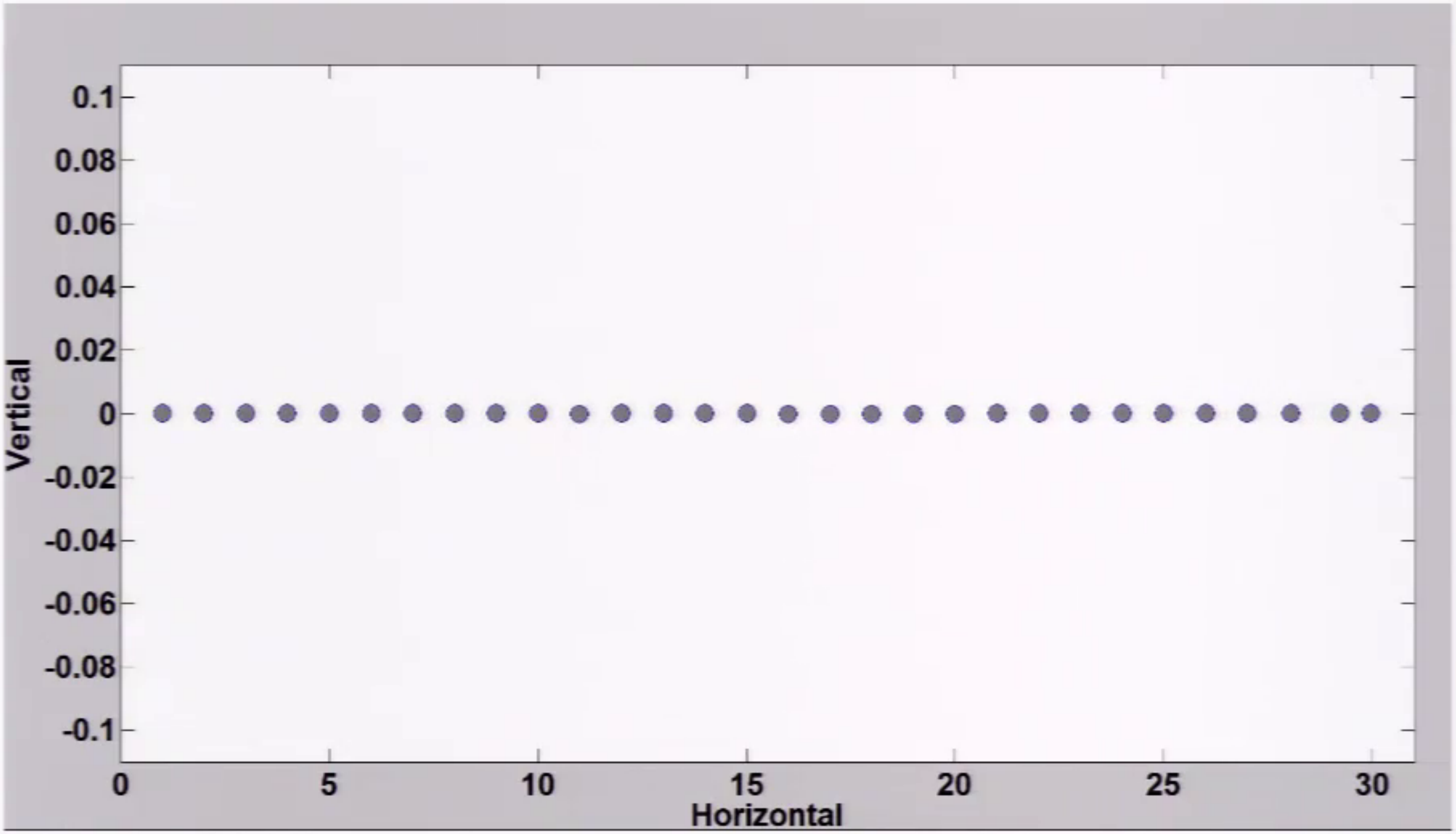
$$\varepsilon = 0.05, \quad \mu = 0.0325, \quad n = 3, \quad \kappa = 1, \quad \xi_1'(0) = 0.08, \quad \theta_i(0) = \pi/2, \quad \theta_i'(0) = 0, \quad 1 \le i \le 5$$



About 85% of Energy was transported from the horizontal direction to the vertical direction!

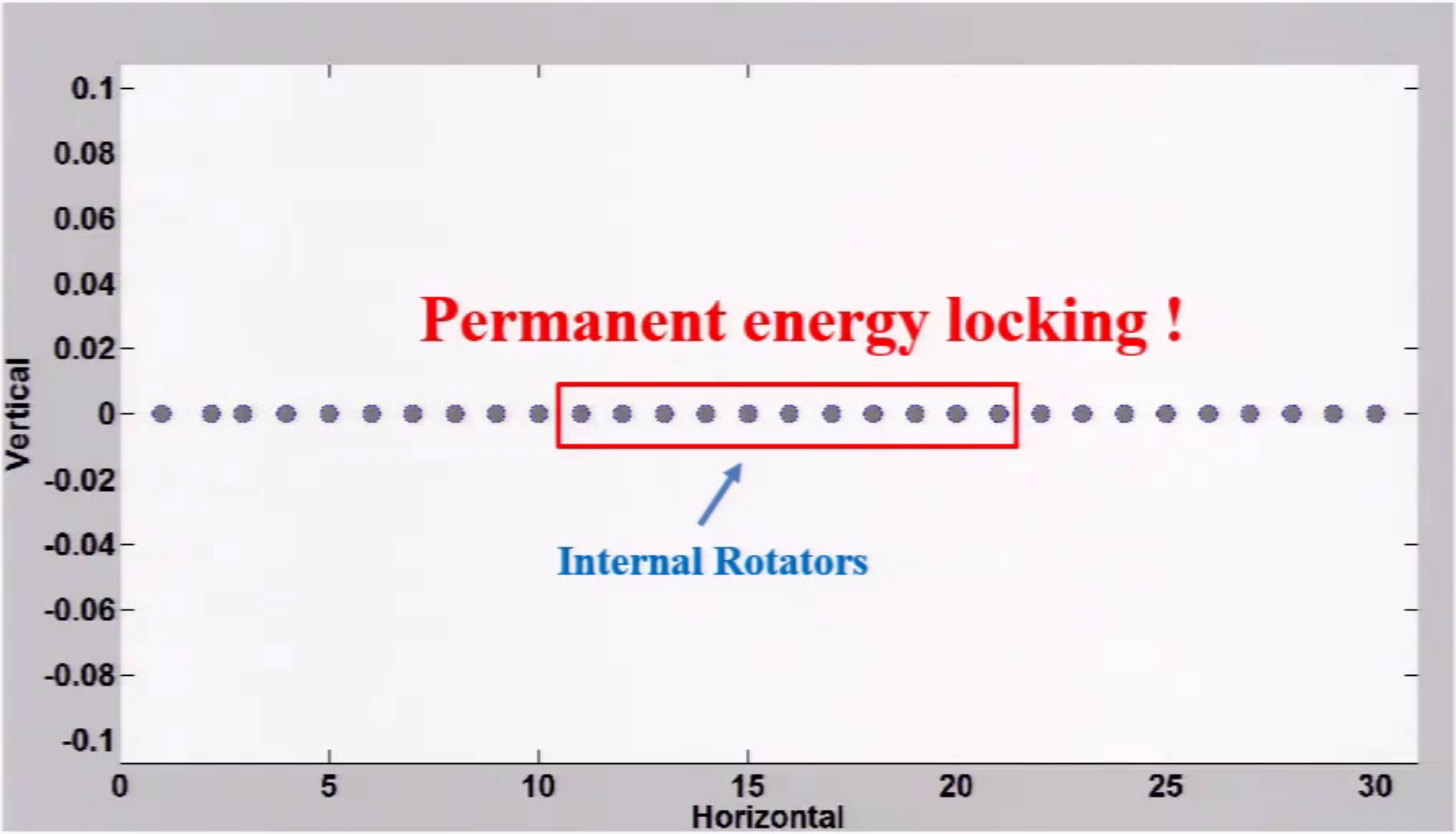
Animations of two different scenarios

First scenario – Formation and unperturbed propagation of moving breather.

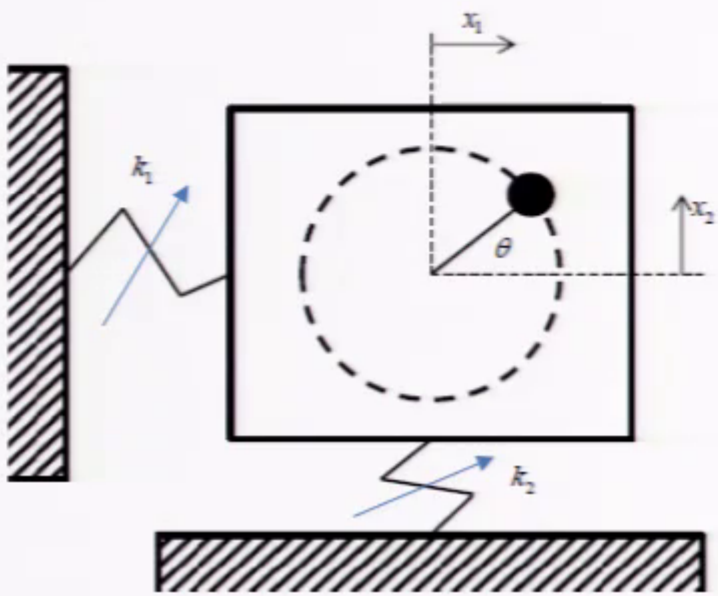


Animations of two different scenarios

Second scenario – **Nearly complete unidirectional energy channeling**



Analysis of energy channeling phenomena in the unit-cell model



Nondimensional Equations of Motion:

$$\tau = \omega_s t, \quad x_i = R\xi_i, \quad \omega_s^2 = \frac{k}{M+m}, \quad \varepsilon = \frac{m}{M+m}, \quad \varepsilon\mu = \frac{c}{\omega_s m R^2}$$

Non-dimensional Equations Of Motion

$$\xi_1'' - \varepsilon\left(\theta''\sin\theta + \theta'^2\cos\theta\right) + \xi_1 + \alpha\xi_1^{\,3} = 0$$

$$\xi_2'' + \varepsilon\left(\theta''\cos\theta - \theta'^2\sin\theta\right) + \xi_2 + \alpha\xi_2^{\,3} = 0$$

$$\xi_2''\cos\theta - \xi_1''\sin\theta + \theta'' = -\varepsilon\mu\theta'$$

Unit Cell Model – High Energy Pulsations

Model:

$$\xi''_1 + \xi_1 + \varepsilon\alpha\xi_1^3 = \varepsilon\left(\theta''\sin\theta + \theta'^2\cos\theta\right)$$
$$\xi''_1 + \xi_2 + \varepsilon\alpha\xi_2^3 = -\varepsilon\left(\theta''\cos\theta - \theta'^2\sin\theta\right)$$
$$\theta'' + \eta\theta' = \xi''_1\sin\theta - \xi''_2\cos\theta$$

Bidirectional, Recurrent, Energy Channeling – complete, recurrent energy transport from axial to lateral vibrations of the outer element (controlled by the motion of the internal rotator) being initially excited strictly in the axial (or lateral) direction.

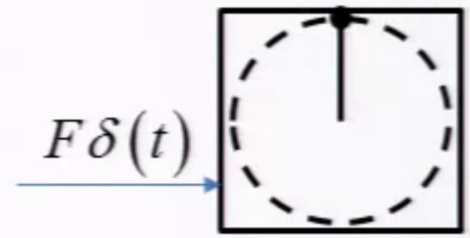
Unidirectional, Energy Locking – permanent, unidirectional energy localization in the outer element being initially excited strictly in the axial (or lateral) direction.

Unidirectional, Energy Channeling – complete energy transport from axial to lateral vibrations of the outer element (controlled by the motion of the internal rotator)

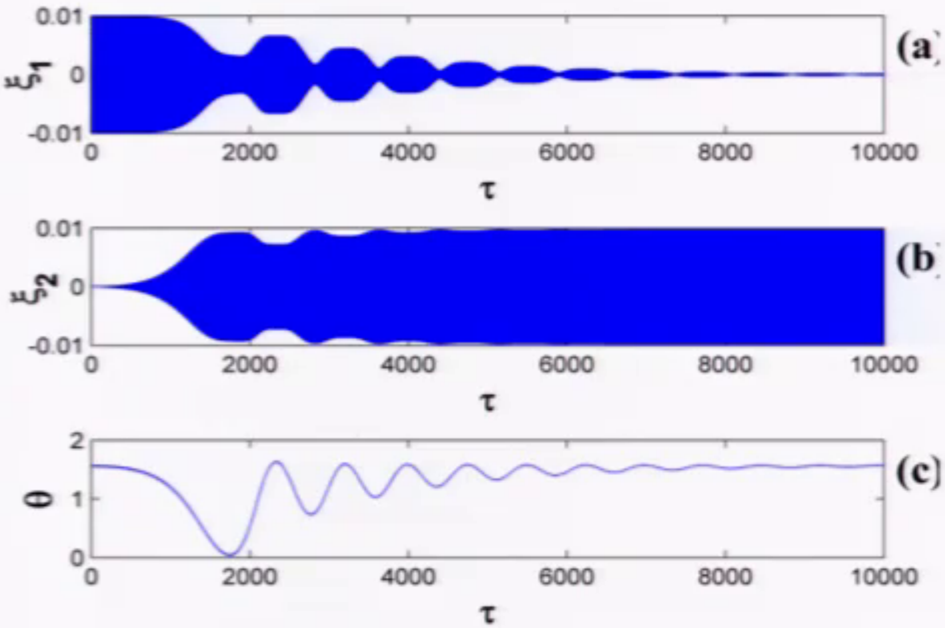
Unit Cell Model – Low Energy Excitations

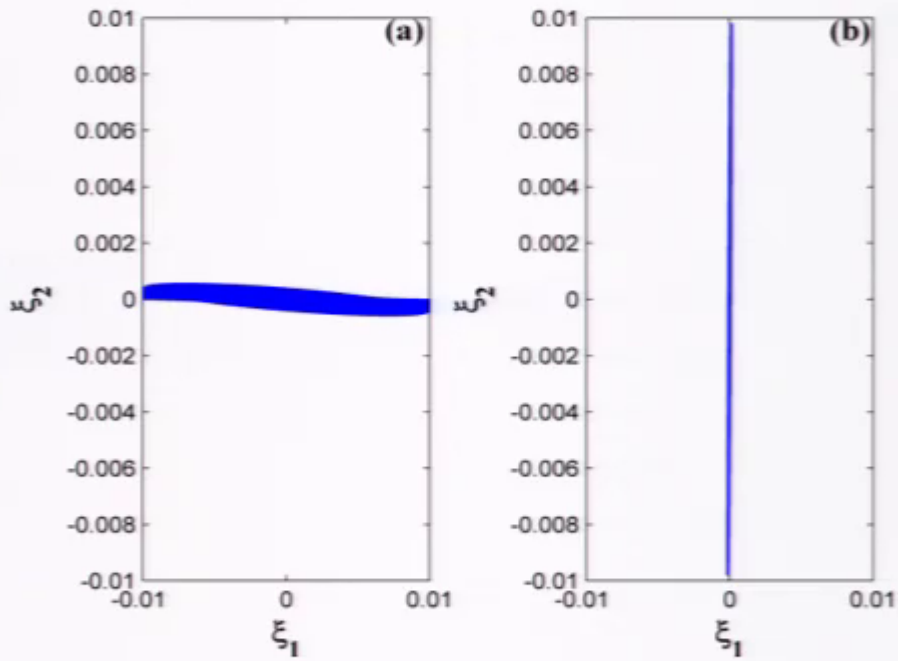
Model:

$$\xi''_1 + \xi_1 + \varepsilon\alpha\xi_1^3 = \varepsilon\left(\theta''\sin\theta + \theta'^2\cos\theta\right)$$
$$\xi''_1 + \xi_2 + \varepsilon\alpha\xi_2^3 = -\varepsilon\left(\theta''\cos\theta - \theta'^2\sin\theta\right)$$
$$\theta'' + \eta\theta' = \xi''_1\sin\theta - \xi''_2\cos\theta$$



Numerical Evidence:





Unit Cell Model – Low Energy Excitations

Regular Multi-Scale Analysis (low energy excitations):

Assuming the 1:1 resonant interaction between the axial and the lateral vibrations of the outer element we introduce complex variables in the following form (Manevitch et. al)

$$\psi_1 = \xi'_1 + i\xi_1$$
$$\psi_2 = \xi'_2 + i\xi_2$$

Equations of Motion (EOM) represented in the complex form:

$$\psi'_1 - i\psi_1 + i\frac{\varepsilon}{8}\alpha(\psi_1 - \psi_1^*)^3 = \varepsilon\left(\theta''\sin\theta + \theta'^2\cos\theta\right)$$
$$\psi'_2 - i\psi_2 + i\frac{\varepsilon}{8}\alpha(\psi_2 - \psi_2^*)^3 = -\varepsilon\left(\theta''\cos\theta - \theta'^2\sin\theta\right)$$
$$\left(\psi'_2 - i\frac{\psi_2 + \psi_2^*}{2}\right)\cos\theta - \left(\psi'_1 - i\frac{\psi_1 + \psi_1^*}{2}\right)\sin\theta + \theta'' = -\varepsilon\mu\theta'$$

To analyze the dynamics in the limit of low energy excitations we use the regular, multi-scale asymptotic expansion in the form (G. Sigalov, L. Manevitch):

$$\frac{d(\bullet)}{dt} = \frac{\partial(\bullet)}{\partial\tau_0} + \varepsilon\frac{\partial(\bullet)}{\partial\tau_1} + O(\varepsilon^2)$$
$$\psi_k(\tau) = \varepsilon\psi_{k0}(\tau_0,\tau_1) + \varepsilon^2\psi_{k1}(\tau_0,\tau_1) + O\left(\varepsilon^3\right),\ k = 1,2$$
$$\theta(\tau) = \theta_0(\tau_0,\tau_1) + \varepsilon\theta_1(\tau_0,\tau_1) + O(\varepsilon^2)$$

Unit Cell Model – Low Energy Excitations

Regular Multi-Scale Analysis (low energy excitations):

Zeroth order $O(\varepsilon^0)$

$$\frac{\partial^2\theta_0}{\partial\tau_0^2}=0 \Rightarrow \boxed{\theta_0(\tau_0,\tau_1)=A_1(\tau_1)\tau_0+A_2(\tau_1)}$$

First order $O(\varepsilon^1)$

$$\left.\begin{aligned}&\frac{\partial\psi_{k0}}{\partial\tau_0}-i\psi_{k0}=0,\ k=1,2\\&\frac{\partial^2\theta_1}{\partial\tau_0^2}=-\frac{i}{2}\left(\left(\psi_{20}-\psi_{20}^*\right)\cos\theta_0-\left(\psi_{10}-\psi_{10}^*\right)\sin\theta_0\right)\end{aligned}\right\}\boxed{\begin{aligned}&\psi_{k0}=\varphi_{k0}(\tau_1)e^{i\tau_0},k=1,2\\&\theta_1=-\frac{i}{2}\begin{pmatrix}\left(\varphi_{20}(\tau_1)e^{i\tau_0}-\varphi_{20}^*(\tau_1)e^{-i\tau_0}\right)\cos\theta_0\\-\left(\varphi_{10}(\tau_1)e^{i\tau_0}-\varphi_{10}^*(\tau_1)e^{-i\tau_0}\right)\sin\theta_0\end{pmatrix}\end{aligned}}$$

Unit Cell Model – Low Energy Excitations

Regular Multi-Scale Analysis (low energy excitations):

Next order $O(\varepsilon^2)$

$$\frac{\partial \psi_{11}}{\partial \tau_0} - i\psi_{11} = -\frac{\partial \psi_{10}}{\partial \tau_1} + \frac{i}{4}(1-\cos 2\theta_0)(\psi_{10} - \psi_{10}^*) + \frac{i}{4}(\psi_{20} - \psi_{20}^*)\sin 2\theta_0$$

$$\frac{\partial \psi_{21}}{\partial \tau_0} - i\psi_{21} = -\frac{\partial \psi_{20}}{\partial \tau_1} + \frac{i}{4}(1+\cos 2\theta_0)(\psi_{20} - \psi_{20}^*) + \frac{i}{4}(\psi_{10} - \psi_{10}^*)\sin 2\theta_0$$

$$\frac{\partial^2 \theta_0}{\partial \tau_1^2} = -\mu \frac{\partial \theta_0}{\partial \tau_1} - \frac{1}{4}\begin{pmatrix} \left(|\varphi_{10}(\tau_1)|^2 - |\varphi_{20}(\tau_1)|^2\right)\sin 2\theta_0 \\ -\left(\varphi_{20}^*(\tau_1)\varphi_{10}(\tau_1) + \varphi_{20}(\tau_1)\varphi_{10}^*(\tau_1)\right)\cos 2\theta_0 \end{pmatrix}$$

Slow Flow Model

$$\varphi_{10}'(\tau_1) = \frac{i}{4}\varphi_{10}(\tau_1)(1-\cos 2\theta_0) - \frac{i}{4}\varphi_{20}(\tau_1)\sin 2\theta_0$$

$$\varphi_{20}'(\tau_1) = \frac{i}{4}\varphi_{20}(\tau_1)(1+\cos 2\theta_0) - \frac{i}{4}\varphi_{10}(\tau_1)\sin 2\theta_0$$

$$\frac{\partial^2 \theta_0}{\partial \tau_1^2} = -\mu \frac{\partial \theta_0}{\partial \tau_1} - \frac{1}{4}\begin{pmatrix} \left(|\varphi_{10}(\tau_1)|^2 - |\varphi_{20}(\tau_1)|^2\right)\sin 2\theta_0 \\ -\left(\varphi_{20}^*(\tau_1)\varphi_{10}(\tau_1) + \varphi_{20}(\tau_1)\varphi_{10}^*(\tau_1)\right)\cos 2\theta_0 \end{pmatrix}$$

Unit Cell Model – Low Energy Excitations

Regular Multi-Scale Analysis (low energy excitations):

Integral of motion: $$\left|\varphi_{10}\left(\tau_1\right)\right|^2+\left|\varphi_{20}\left(\tau_1\right)\right|^2=N^2$$

Angular coordinates:

$$\varphi_{10}\left(\tau_1\right)=N\cos\Theta\left(\tau_1\right)e^{i\delta_1(\tau_1)}$$

$$\varphi_{20}\left(\tau_1\right)=N\sin\Theta\left(\tau_1\right)e^{i\delta_2(\tau_1)}$$

Angular representation:

$$\frac{\partial\Theta}{\partial\tau_1}=\frac{1}{4}\sin\Delta\sin 2\theta_0$$

$$\frac{\partial\Delta}{\partial\tau_1}=-\frac{1}{2}\left[\cos 2\theta_0-\cos\Delta\sin 2\theta_0\cot 2\Theta\right]$$

$$\frac{\partial^2\theta_0}{\partial\tau_1^2}=-\mu\frac{\partial\theta_0}{\partial\tau_1}-\frac{N^2}{4}\left(\cos 2\Theta\sin 2\theta_0-\cos\Delta\sin 2\Theta\cos 2\theta_0\right)$$

where $\Delta=\delta_1-\delta_2$

Unit Cell Model – Low Energy Excitations

Dynamics of the underlying Hamiltonian System

$$\frac{\partial \Theta}{\partial \tau_1} = \frac{1}{4} \sin \Delta \sin 2\theta_0$$

$$\frac{\partial \Delta}{\partial \tau_1} = -\frac{1}{2}\left[\cos 2\theta_0 - \cos \Delta \sin 2\theta_0 \cot 2\Theta\right]$$

$$\frac{\partial^2 \theta_0}{\partial \tau_1^2} = -\frac{N^2}{4}\left(\cos 2\Theta \sin 2\theta_0 - \cos \Delta \sin 2\Theta \cos 2\theta_0\right)$$

Unit Cell Model – Low Energy Excitations

Dynamics of the underlying Hamiltonian System

$$\frac{\partial\Theta}{\partial\tau_1} = \frac{1}{4}\sin\Delta\sin 2\theta_0$$

$$\frac{\partial\Delta}{\partial\tau_1} = -\frac{1}{2}\left[\cos 2\theta_0 - \cos\Delta\sin 2\theta_0\cot 2\Theta\right]$$

$$\frac{\partial^2\theta_0}{\partial\tau_1^2} = -\frac{N^2}{4}\left(\cos 2\Theta\sin 2\theta_0 - \cos\Delta\sin 2\Theta\cos 2\theta_0\right)$$

Integrability

$$\boxed{\begin{gathered} L = \frac{1}{2}N^2\sin\Delta\sin 2\Theta + \frac{\partial\theta_0}{\partial\tau_1} \\ H = -\cos 2\Theta\cos 2\theta_0 - \cos\Delta\sin 2\Theta\sin 2\theta_0 + 4N^{-2}\theta_0'^2 \end{gathered}}$$

Reduction

$$\boxed{\begin{gathered} \frac{\partial\Theta}{\partial\tau_1} = \frac{1}{4}\sin\Delta\sin 2\theta_0 \\ \frac{\partial\Delta}{\partial\tau_1} = -\frac{1}{2}\left[\cos 2\theta_0 - \cos\Delta\sin 2\theta_0\cot 2\Theta\right] \\ \frac{\partial\theta_0}{\partial\tau_1} = L - \frac{1}{2}N^2\sin\Delta\sin 2\Theta \end{gathered}}$$

Unit Cell Model – Low Energy Excitations

Dynamics of the underlying Hamiltonian System

Stationary points: $\partial\Theta/\partial\tau_1 = \partial\Delta/\partial\tau_1 = \partial\theta_0/\partial\tau_1 = 0$

$$\sin\Delta\sin 2\theta_0 = 0$$
$$\Rightarrow \cos 2\theta_0 - \cos\Delta\sin 2\theta_0 \cot 2\Theta = 0$$
$$2L - N^2 \sin\Delta\sin 2\Theta = 0$$

Two sets:

$$\Delta = \pi n,\quad \theta_0 = (-1)^n\,\Theta + \frac{\pi m}{2},\quad n,m\in Z\,,m-odd$$

$$\boxed{\Delta = \pi n,\quad \theta_0 = (-1)^n\,\Theta + \frac{\pi m}{2},\quad n,m\in Z\,,m-even}$$

Next order verification:

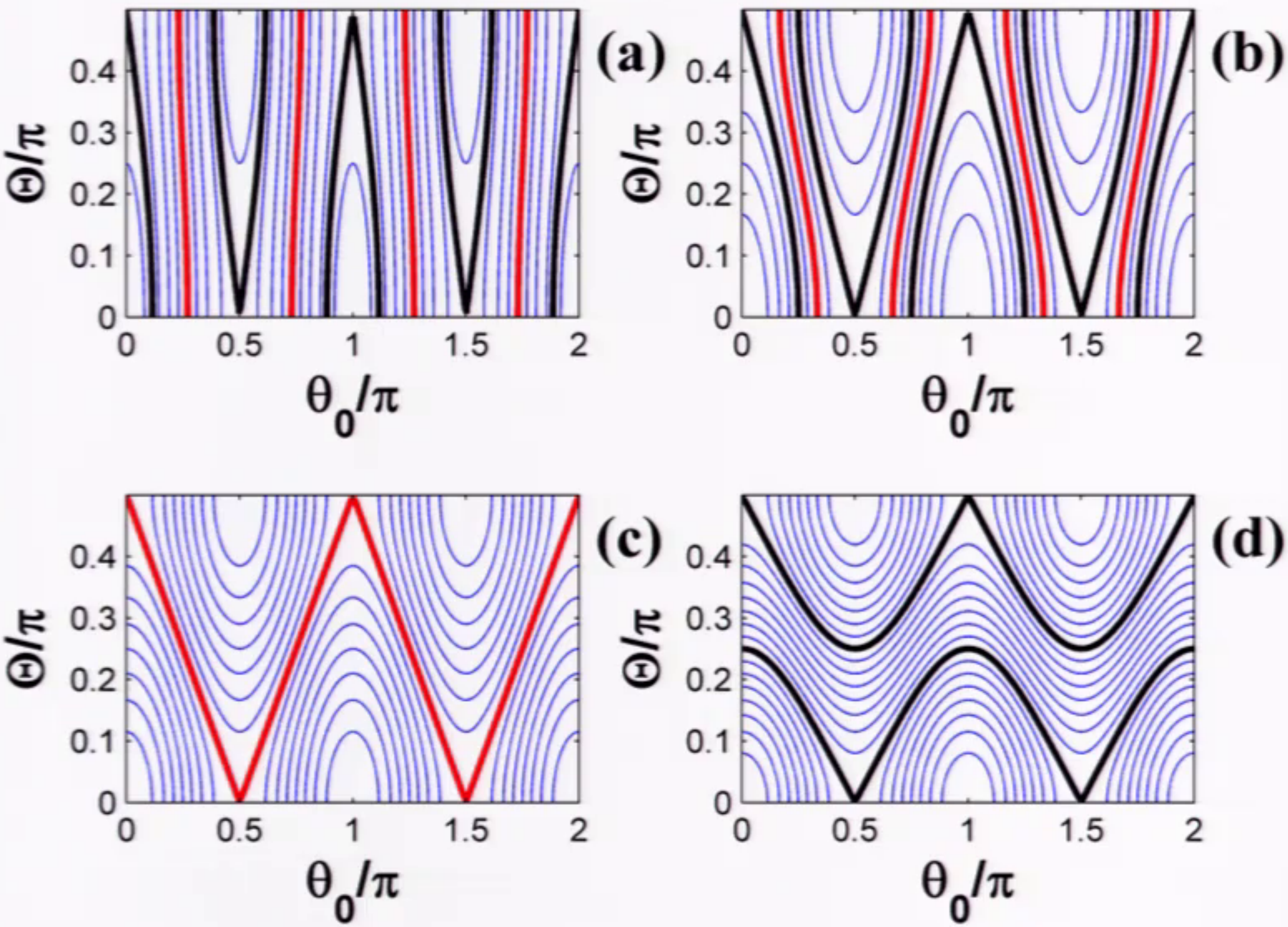
$$\theta(\tau) = \theta_0(\tau_0,\tau_1) + \boxed{\varepsilon\theta_1(\tau_0,\tau_1)} + O(\varepsilon^2)$$

$\theta_1(\tau_0,\tau_1) = -N(-1)^k \sin\left(\delta_1(\tau_1)+\tau_0\right)$ (First set)

$\theta_1(\tau_0,\tau_1) = 0$ (Second set) $\longrightarrow$ **Stable NNM**

Unit Cell Model – Low Energy Excitations

Special Case of Bi-Directional Energy Channeling $(L = 0)$



N : (a) $N = 0.25$, (b) $N = 0.5$, (c) $N = 2^{-1/2}$ and (d) $N = 1$.

Unit Cell Model – Low Energy Excitations

Special Case of Bi-Directional Energy Channeling $(L = 0)$

$$\frac{\partial \Theta}{\partial \tau_1} = \frac{1}{4} \sin\Delta \sin 2\theta_0$$

$$\frac{\partial \Delta}{\partial \tau_1} = -\frac{1}{2}\left[\cos 2\theta_0 - \cos\Delta \sin 2\theta_0 \cot 2\Theta\right]$$

$$\frac{\partial \theta_0}{\partial \tau_1} = -\frac{1}{2} N^2 \sin\Delta \sin 2\Theta$$

Internal rotator is at rest at the initial state !

Conserved quantity: $C = 2N^2 \cos 2\Theta + \cos 2\theta_0$

Unit Cell Model – Low Energy Excitations

Dynamics of the underlying Hamiltonian System

Stationary points: $\partial\Theta/\partial\tau_1 = \partial\Delta/\partial\tau_1 = \partial\theta_0/\partial\tau_1 = 0$

$$\sin\Delta\sin 2\theta_0 = 0$$
$$\Rightarrow \cos 2\theta_0 - \cos\Delta\sin 2\theta_0 \cot 2\Theta = 0$$
$$2L - N^2 \sin\Delta\sin 2\Theta = 0$$

Two sets:

$$\Delta = \pi n,\quad \theta_0 = (-1)^n\,\Theta + \frac{\pi m}{2},\quad n,m \in Z\,, m - odd$$

$$\boxed{\Delta = \pi n,\quad \theta_0 = (-1)^n\,\Theta + \frac{\pi m}{2},\quad n,m \in Z\,, m - even}$$

Next order verification:

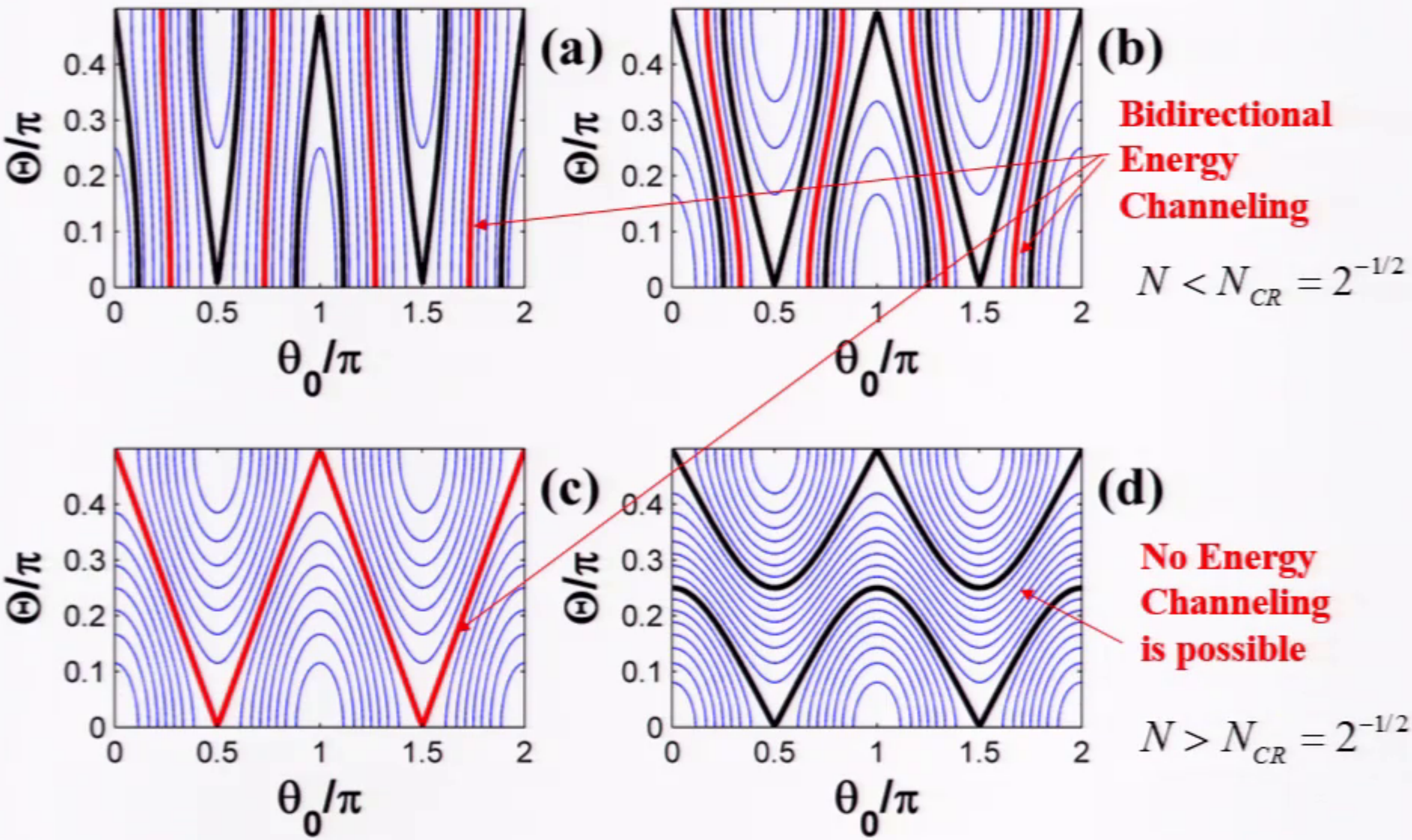
$$\theta(\tau) = \theta_0(\tau_0,\tau_1) + \boxed{\varepsilon\theta_1(\tau_0,\tau_1)} + O(\varepsilon^2)$$

$\theta_1(\tau_0,\tau_1) = -N(-1)^k \sin\left(\delta_1(\tau_1) + \tau_0\right)$ (First set)

$\theta_1(\tau_0,\tau_1) = 0$ (Second set) $\longrightarrow$ **Stable NNM**

Unit Cell Model – Low Energy Excitations

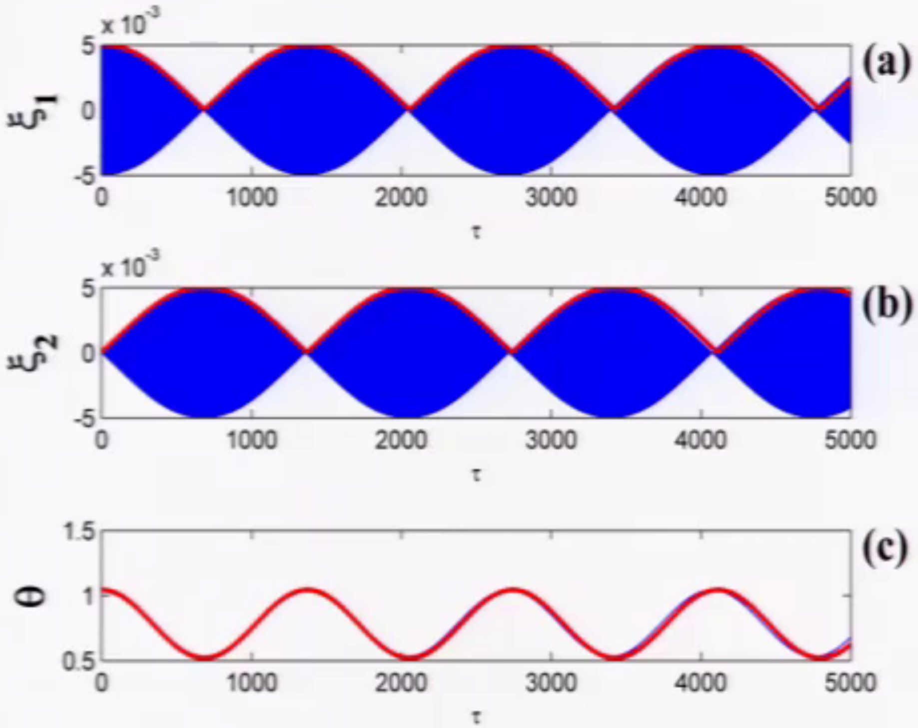
Special Case of Bi-Directional Energy Channeling $(L=0)$



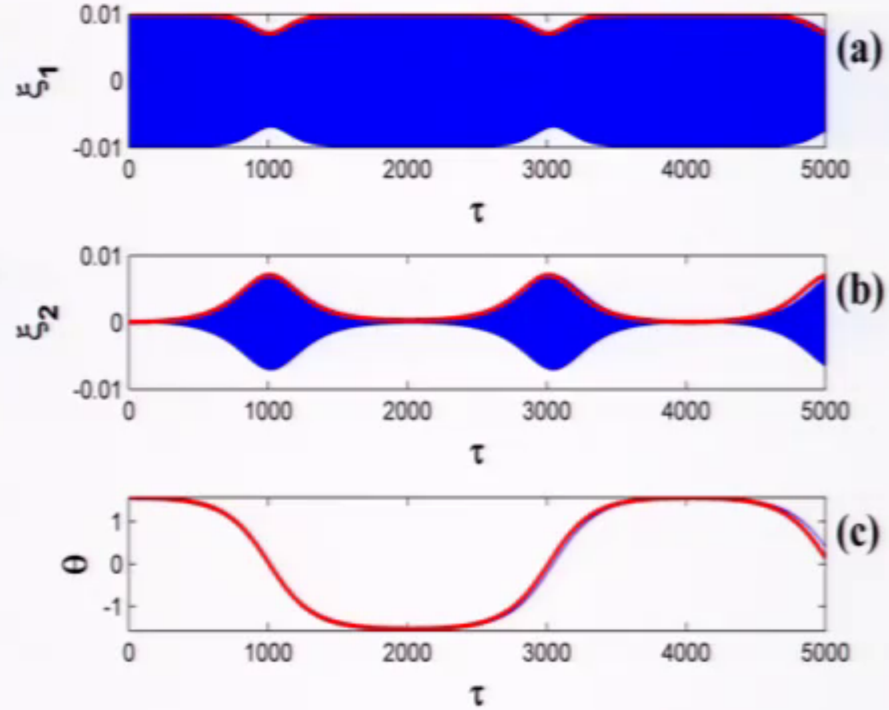
N : (a) $N = 0.25$, (b) $N = 0.5$, (c) $N = 2^{-1/2}$ and (d) $N = 1$.

Unit Cell Model – Low Energy Excitations

Special Case of Bi-Directional Energy Channeling $(L = 0)$



Bi-Directional Energy Channeling



Unidirectional Localization

Unit Cell Model – Low Energy Excitations

Unidirectional Energy Channeling (Dissipative Case)

$$L(\tau_1) = \frac{\partial\theta_0}{\partial\tau_1} + \frac{1}{2}N^2 \sin\Delta \sin 2\Theta$$

Differentiating w.r.t slow time scale

$$\frac{dL}{d\tau_1} = \frac{1}{2}N^2\left(\frac{\partial\Delta}{\partial\tau_1}\cos\Delta\sin 2\Theta + 2\frac{\partial\Theta}{\partial\tau_1}\sin\Delta\cos 2\Theta\right) + \frac{\partial^2\theta_0}{\partial\tau_1^2}$$

Performing additional algebraic manipulations

$$\frac{dL}{d\tau_1} = -\mu\frac{\partial\theta_0}{\partial\tau_1} \;\Rightarrow L = C - \mu\theta_0$$

Dissipative flow can be further reduced

$$\frac{\partial\Theta}{\partial\tau_1} = \frac{1}{4}\sin\Delta\sin 2\theta_0$$

$$\frac{\partial\Delta}{\partial\tau_1} = -\frac{1}{2}\left[\cos 2\theta_0 - \cos\Delta\sin 2\theta_0 \cot 2\Theta\right]$$

$$\frac{\partial\theta_0}{\partial\tau_1} = C - \mu\theta_0 - \frac{1}{2}N^2\sin\Delta\sin 2\Theta$$

Unit Cell Model – Low Energy Excitations

Unidirectional Energy Channeling (Dissipative Case)

Analysis of fixed points: $\partial\Theta/\partial\tau_1 = \partial\Delta/\partial\tau_1 = \partial\theta_0/\partial\tau_1 = 0$

$$\sin\Delta\sin 2\theta_0 = 0$$

$$\cos 2\theta_0 - \cos\Delta\sin 2\theta_0\cot 2\Theta = 0$$

$$\boxed{C - \mu\theta_0} - \frac{1}{2}N^2\sin\Delta\sin 2\Theta = 0$$

Fixed points: $\boxed{\Delta = \pi n, \quad \theta_0 = \dfrac{C}{\mu}, \quad \Theta = (-1)^n\theta_0 + \dfrac{\pi m}{2}}$

Map from the initial state to the final state:

$$\boxed{\Delta^F = \pi n, \quad \theta_0^F = \frac{L(0) + \mu\theta_0^I}{\mu}, \quad \Theta^F = (-1)^n\frac{L(0) + \mu\theta_0^I}{\mu} + \pi k}$$

Unit Cell Model – Low Energy Excitations

Unidirectional Energy Channeling (Dissipative Case)

Initial Tuning:

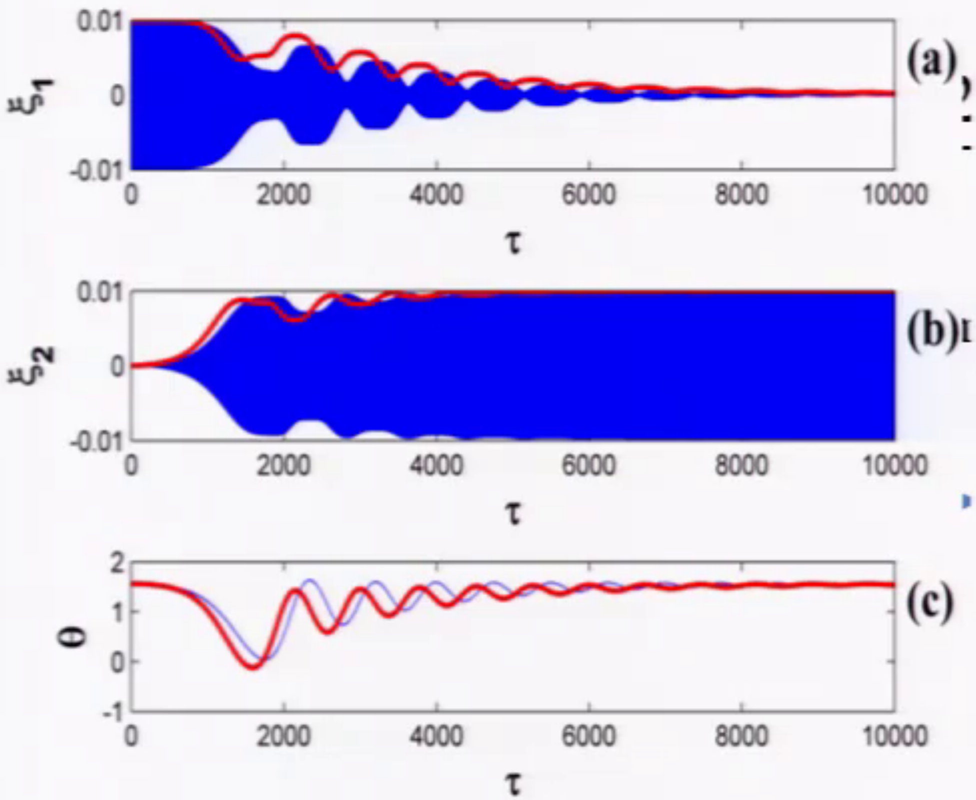
$$\Delta^{I} = \pi n, \theta_0^{\,I} = \frac{\pi\mu - 2l}{2\mu}, \Theta^{I} = 0, L(0) = l$$

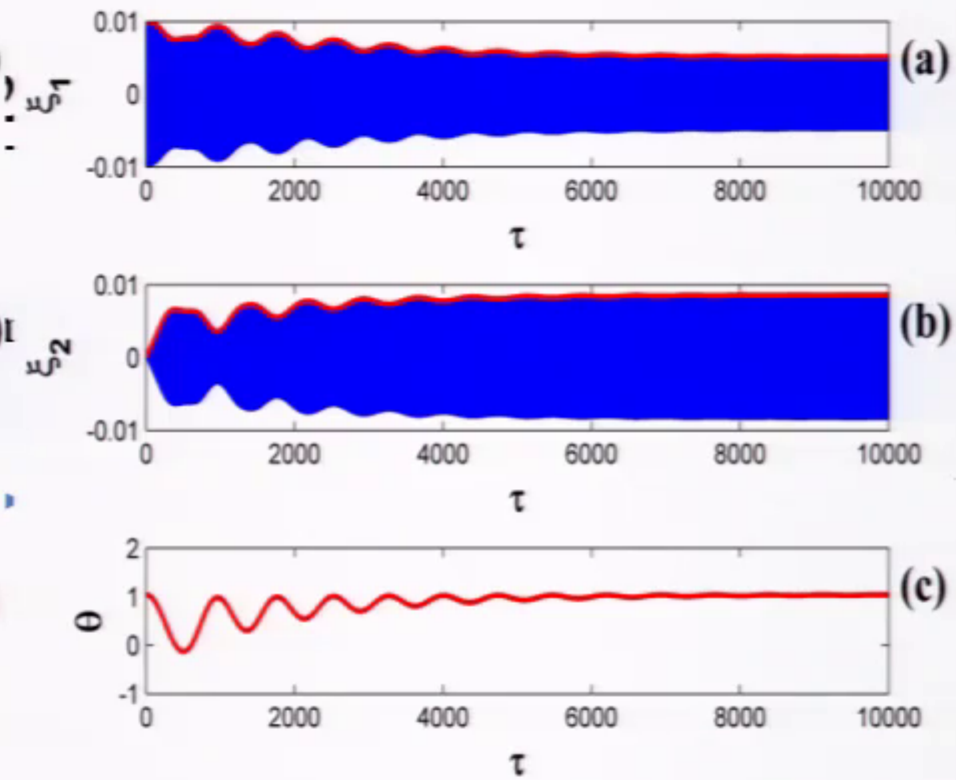
To achieve the complete energy transport from axial to lateral vibrations for the given initial state, one should require that

Final state: $\Theta^{F} = \pi / 2 \longrightarrow$ **Initial state:** $\theta_0^{\,I} = \dfrac{\pi\mu / 2 - l}{\mu} + \pi k$

Unit Cell Model – Low Energy Excitations

Unidirectional Energy Channeling (Dissipative Case)





$$\theta_0^{\,I} = \frac{\pi\mu/2 - l}{\mu} + \pi k$$

Complete Energy Channeling !!!

Partial Energy Channeling

Unit Cell Model – High Energy Pulsations

Multi-Scale Analysis in the vicinity of 1:1:1 resonance manifold

Complex coordinates: $\psi_k = \xi'_k + i\xi_k, \quad k = 1,2$

$$\psi'_1 - i\psi_1 + i\frac{\varepsilon\alpha}{8}(\psi_1 - \psi_1^*)^3 = \varepsilon\left(\theta''\sin\theta + \theta'^2\cos\theta\right)$$

$$\psi'_2 - i\psi_2 + i\frac{\varepsilon\alpha}{8}(\psi_2 - \psi_2^*)^3 = -\varepsilon\left(\theta''\cos\theta - \theta'^2\sin\theta\right)$$

$$\theta'' + \eta\theta' = \left(\psi'_1 - \frac{i}{2}(\psi_1 + \psi_1^*)\right)\sin\theta - \left(\psi'_2 - \frac{i}{2}(\psi_2 + \psi_2^*)\right)\cos\theta$$

1:1:1 Resonance - Assumption

$\theta = t + \beta(t)$ – Resonant rotations of the rotator

$\psi_k = \varphi_k(t)\exp(it), k = 1,2$ - Resonant motion of the outer element

Unit Cell Model – High Energy Pulsations

Multi-Scale Analysis in the vicinity of 1:1:1 resonance manifold

Complexification Averaging:

Following the C-A procedure we average out the fast components with respect to the dominant resonant frequency

$$\varphi'_k = \varepsilon\left[i\frac{3\alpha}{8}|\varphi_k|^2\varphi_k + \frac{(-i)^k}{2}(\beta'' + i(1+\beta')^2)\exp\big(i\beta(\tau)\big)\right], \quad k = 1,2$$

$$\beta'' + \eta\beta' = -\frac{1}{4}\begin{bmatrix}\Big(\varphi_1^*\exp\big(i\beta(\tau)\big) + \varphi_1\exp\big(-i\beta(\tau)\big)\Big) - \\ -i\Big(\varphi_2^*\exp\big(i\beta(\tau)\big) - \varphi_2\exp\big(-i\beta(\tau)\big)\Big)\end{bmatrix} - \eta$$

Multi-scale analysis of the averaged flow:

$$\frac{d(\bullet)}{dt} = \frac{\partial(\bullet)}{\partial\tau_1} + \varepsilon\frac{\partial(\bullet)}{\partial\tau_2}, \quad \varphi_k(\tau) = \varphi_{k0}(\tau_1,\tau_2) + O(\varepsilon), \quad \beta(\tau) = \beta_0(\tau_1,\tau_2) + O(\varepsilon)$$

Unit Cell Model – High Energy Pulsations

Multi-Scale Analysis in the vicinity of 1:1:1 resonance manifold

Leading Order: $O(\varepsilon^0)$

$$\partial\varphi_{k0}/\partial\tau_1 = 0, \quad k = 1{,}2 \Rightarrow \varphi_{k0} = \varphi_{k0}(\tau_2)$$

$$\frac{\partial^2\beta_0}{\partial\tau_1^2} + \eta\frac{\partial\beta_0}{\partial\tau_1} = -\frac{1}{4}\begin{bmatrix}\left(\varphi_{10}^*(\tau_2)\exp(i\beta_0) + \varphi_{10}(\tau_2)\exp(-i\beta_0)\right) - \\ -i\left(\varphi_{20}^*(\tau_2)\exp(i\beta_0) - \varphi_{20}(\tau_2)\exp(-i\beta_0)\right)\end{bmatrix} - \eta$$

First Order: $O(\varepsilon^1)$

$$\frac{\partial\varphi_{k0}}{\partial\tau_2} = i\frac{3\alpha}{8}|\varphi_{k0}|^2\varphi_{k0} + \frac{(-i)^k}{2}\left[\frac{\partial^2\beta_0}{\partial\tau_1^2} + i\left(1 + \frac{\partial\beta_0}{\partial\tau_1}\right)^2\right]\exp(i\beta_0), \quad k = 1{,}2$$

Evolution on the Slow Invariant Manifold (SIM) with respect to the super slow time scale (Gendelman et. al.) τ_2

$$\frac{\partial\beta_0(\tau_1,\tau_2)}{\partial\tau_1} = \frac{\partial^2\beta_0(\tau_1,\tau_2)}{\partial\tau_1^2} = 0$$

$$\Rightarrow \beta_0 = B(\tau_2)$$

$$\frac{\partial\varphi_{k0}}{\partial\tau_2} - i\frac{3\alpha}{8}\varphi_{k0}|\varphi_{k0}|^2 = \frac{(-i)^{k-1}}{2}\exp(iB), \quad k = 1{,}2$$

$$4\eta + \begin{bmatrix}\left(\varphi_{10}^*(\tau_2)\exp\left(iB(\tau_2)\right) + \varphi_{10}(\tau_2)\exp\left(-iB(\tau_2)\right)\right) - \\ -i\left(\varphi_{20}^*(\tau_2)\exp\left(iB(\tau_2)\right) - \varphi_{20}(\tau_2)\exp\left(-iB(\tau_2)\right)\right)\end{bmatrix} = 0$$

Unit Cell Model – High Energy Pulsations

Multi-Scale Analysis in the vicinity of 1:1:1 resonance manifold

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$$\Rightarrow \beta_0 = B(\tau_2)$$

$$\boxed{\begin{gathered}\frac{\partial\varphi_{k0}}{\partial\tau_2} - i\frac{3\alpha}{8}\varphi_{k0}|\varphi_{k0}|^2 = \frac{(-i)^{k-1}}{2}\exp(iB), \quad k = 1,2 \\ 4\eta + \begin{bmatrix}\left(\varphi_{10}^*(\tau_2)\exp(iB(\tau_2)) + \varphi_{10}(\tau_2)\exp(-iB(\tau_2))\right) - \\ -i\left(\varphi_{20}^*(\tau_2)\exp(iB(\tau_2)) - \varphi_{20}(\tau_2)\exp(-iB(\tau_2))\right)\end{bmatrix} = 0\end{gathered}}$$

Unit Cell Model – High Energy Pulsations

Intrinsic Dynamics on Slow Invariant Manifold (SIM)

Nonlinear Elastic Foundation (No dissipation):

$$\frac{\partial \varphi_{k0}}{\partial \tau_2} - i\frac{3\alpha}{8}\varphi_{k0}\left|\varphi_{k0}\right|^2 = \frac{(-i)^{k-1}}{2}\exp(iB), \quad k = 1, 2$$

$$\begin{bmatrix} i\left(\varphi_{20}(\tau_2)\exp(-iB) - \varphi_{20}^*(\tau_2)\exp(iB)\right) + \\ +\left(\varphi_{10}(\tau_2)\exp(-iB) + \varphi_{10}^*(\tau_2)\exp(iB)\right) \end{bmatrix} = 0$$

Integrals of motion: $|\varphi_{10}|^2 + |\varphi_{20}|^2 = N^2$

Angular coordinates:

$$\varphi_{10}(\tau_2) = N\cos\Theta(\tau_2)\exp\left(i\delta_1(\tau_2)\right)$$

$$\varphi_{20}(\tau_2) = N\sin\Theta(\tau_2)\exp\left(i\delta_2(\tau_2)\right)$$

Dynamics of the outer element can be reduced to the plane:

$$\Theta' = (-1)^m \frac{\cos\Delta}{2N\sqrt{1+\sin\Delta\sin2\Theta}}$$

$$\Delta' = 2\sigma N^2\cos2\Theta - \frac{(-1)^m}{N}\left(\frac{\cos2\Theta\sin\Delta}{\sin2\Theta\sqrt{1+\sin\Delta\sin2\Theta}}\right)$$

Unit Cell Model – High Energy Pulsations

Intrinsic Dynamics on Slow Invariant Manifold (SIM)

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Unit Cell Model – High Energy Pulsations

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Additional Integral of Motion:

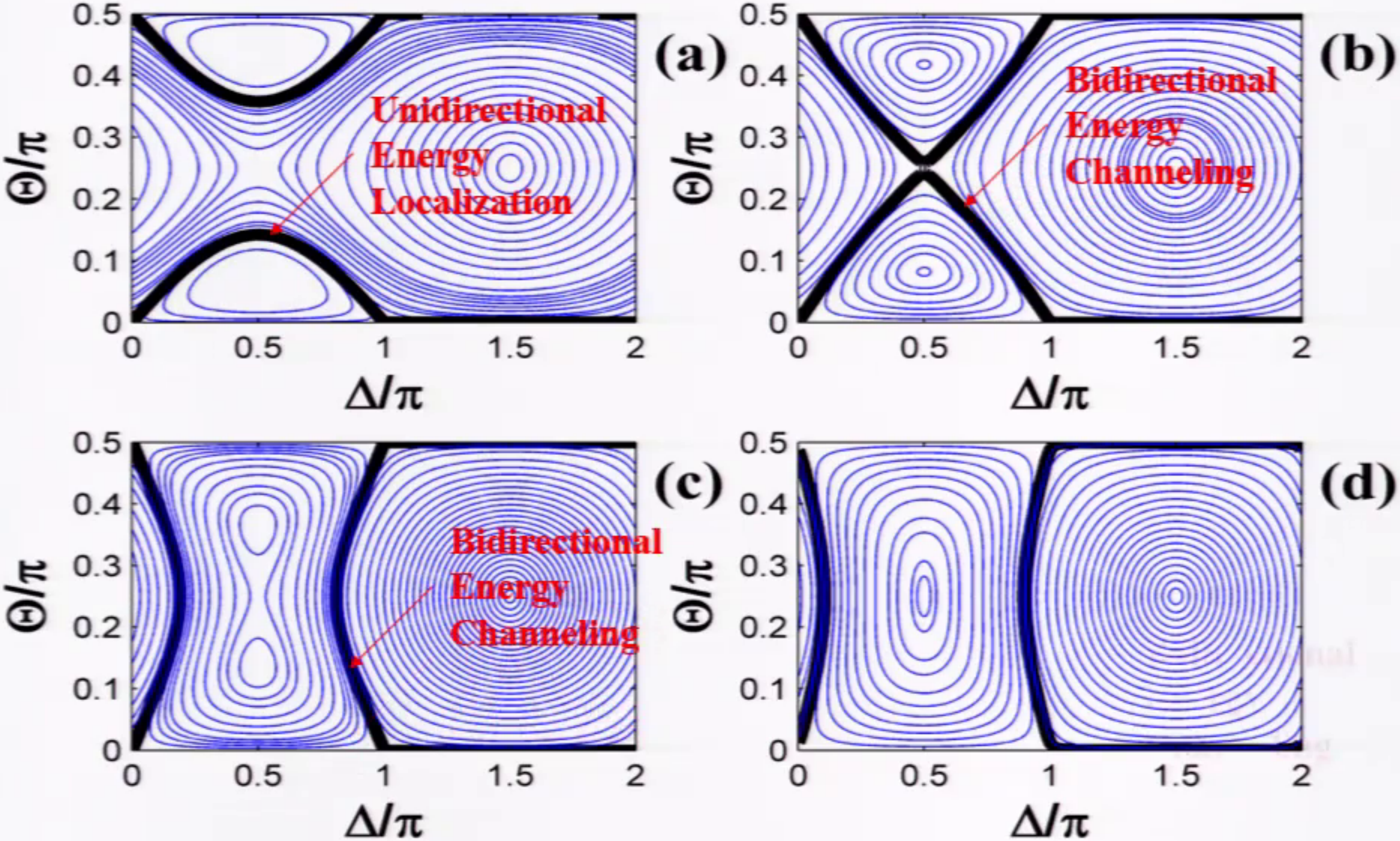
$$H = \sigma\left(\cos^4\Theta + \sin^4\Theta\right) + (-1)^m\sqrt{1+\sin 2\Theta\sin\Delta}$$

Planar system defines the two independent planar flows for the two distinct values of m (i.e. m – even, odd)

- **Stable branch of SIM (m - even)**
- **Unstable branch of SIM (m - odd)**

Unit Cell Model – High Energy Pulsations

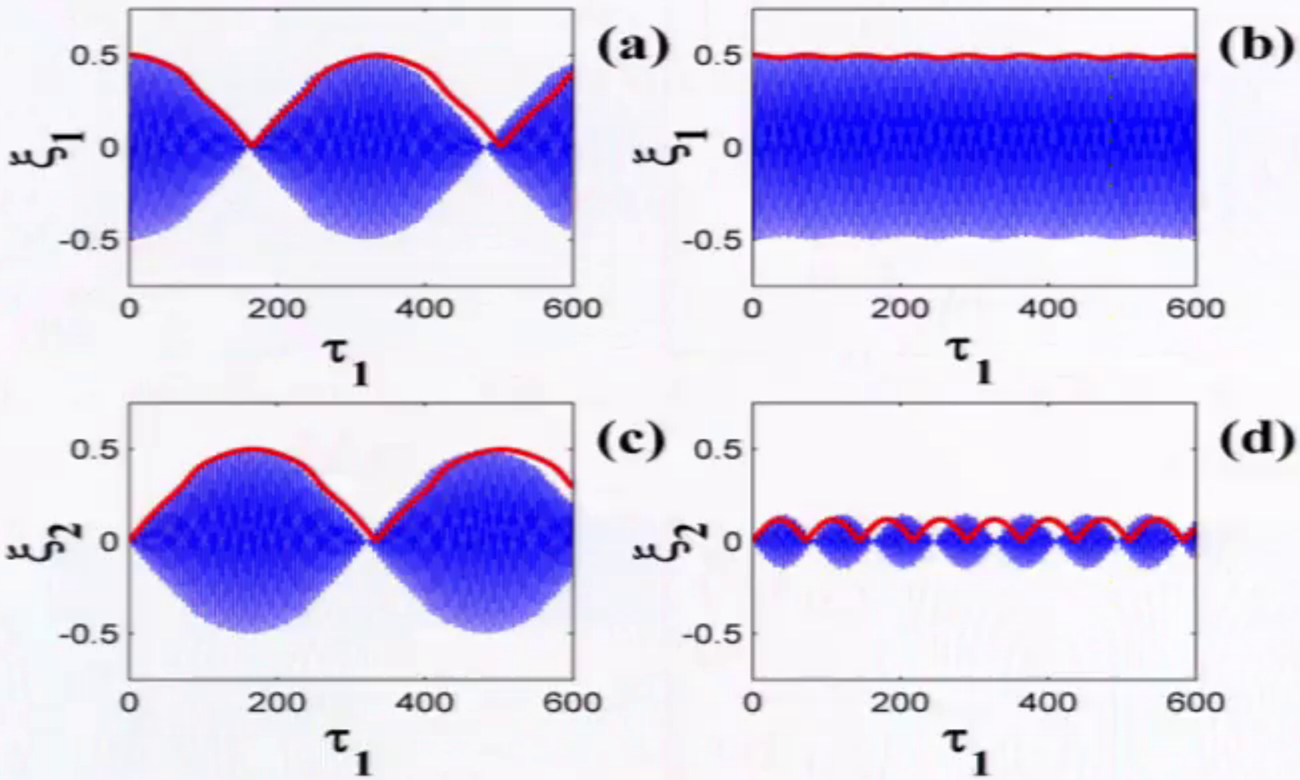
Intrinsic Dynamics on Slow Invariant Manifold (SIM)

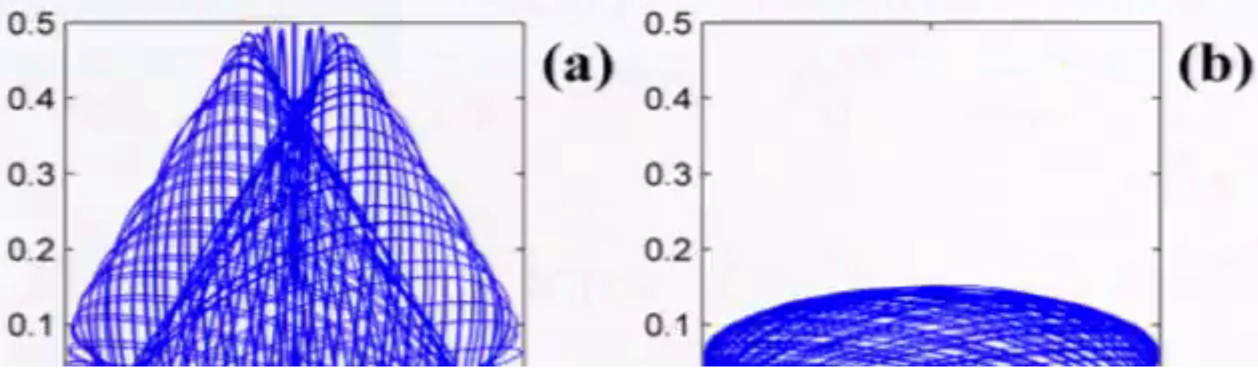


(a) $\mu = 0.91$ (b) $\mu = 1.21$ (c) $\mu = 2$ and (d) $\mu = 3.33$ $\mu = 0.55$

Unit Cell Model – Numerical Evidence

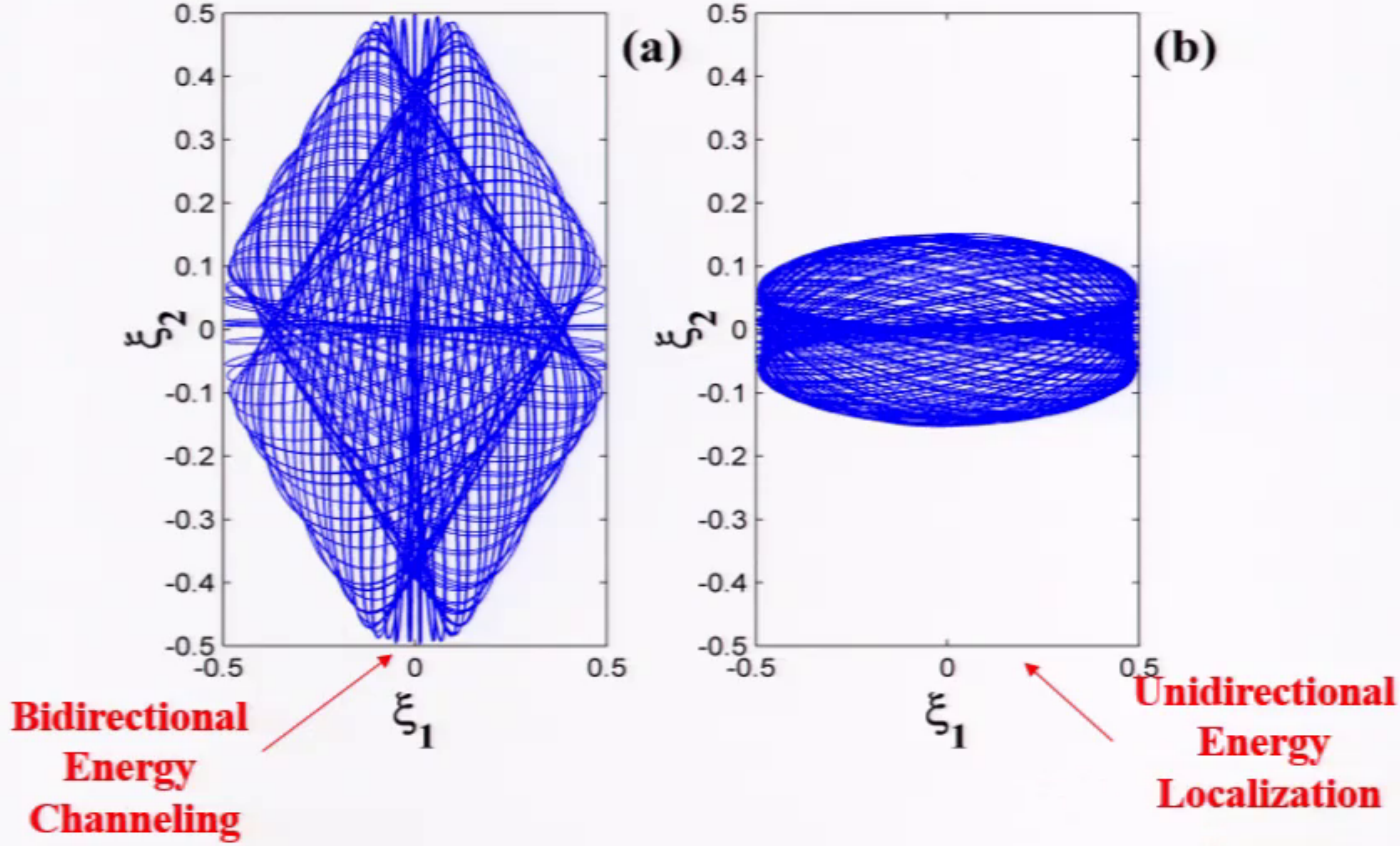
Unidirection energy localization and energy channeling





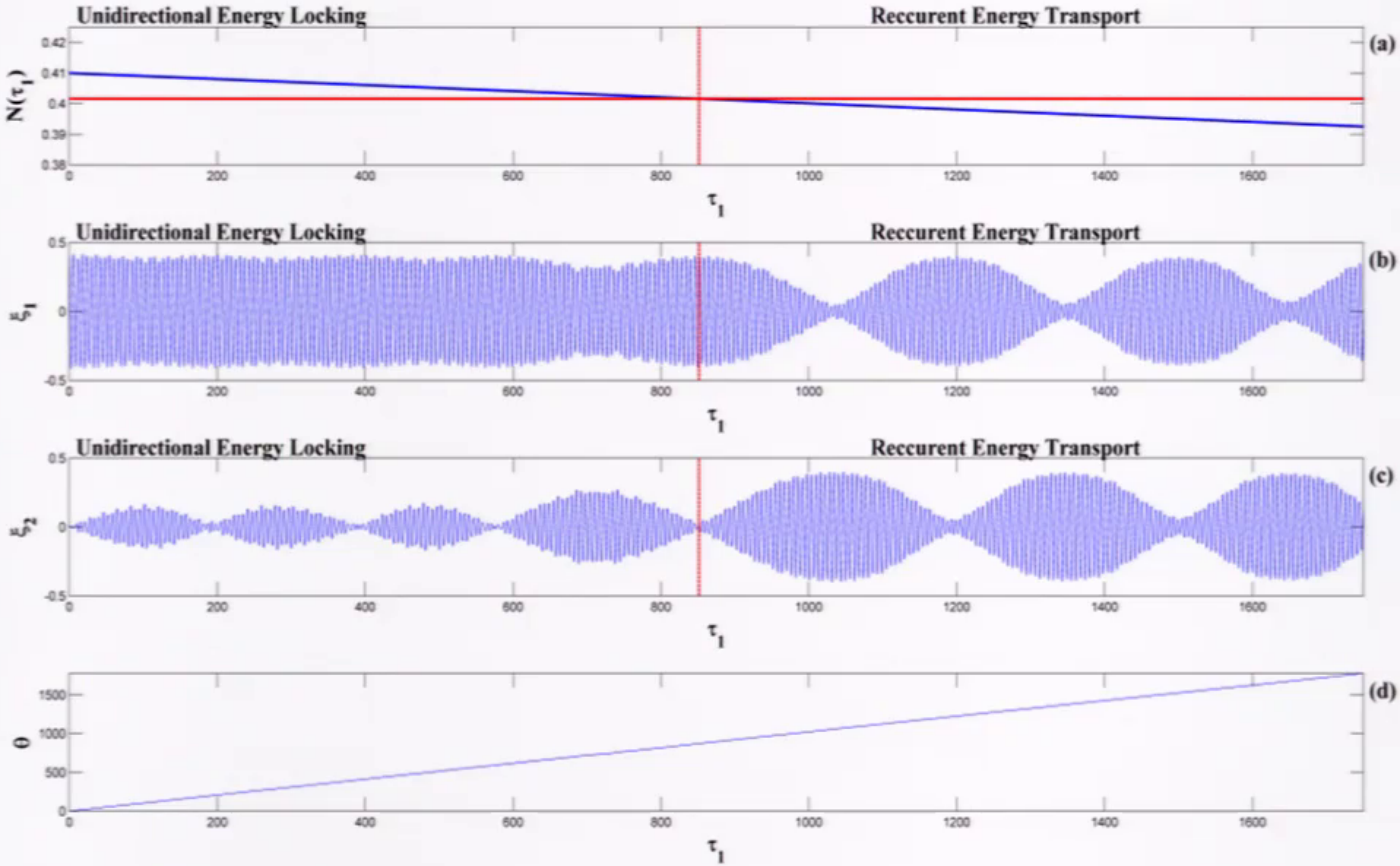
Unit Cell Model – Numerical Evidence

Unidirection energy localization and energy channeling



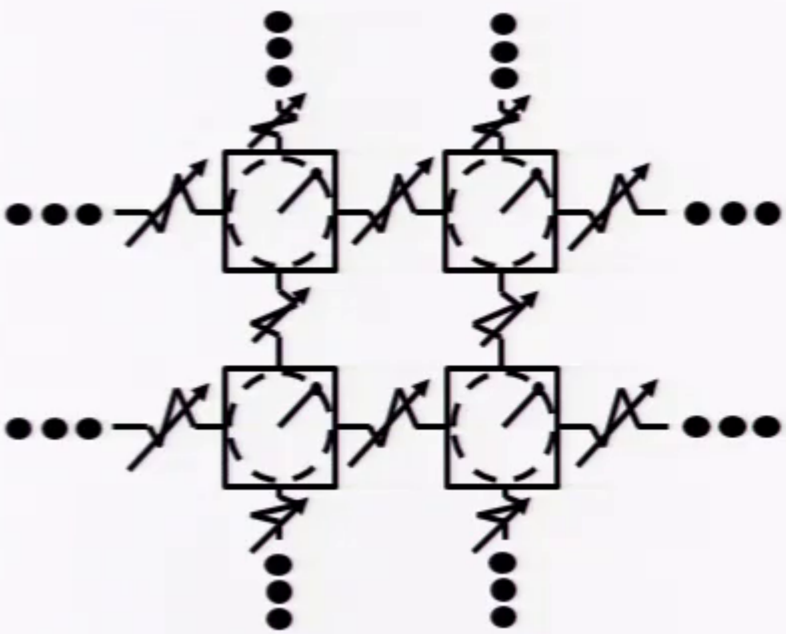
Unit Cell Model – Numerical Evidence

Damped Transition



Conclusions and Future Work

1. Dynamics of the quasi 1D, inertially coupled lattice shows unidirectional energy channeling phenomena
2. Channeling phenomena is analyzed for the two asymptotic limits, i.e. low amplitude excitations and high energy pulsations
3. Uni - and bi-directional energy channeling can be fully predicted for both asymptotic limits
4. Analysis of wave entrapment and redirection in quasi – 1D chain and 2D lattice



Thank you!