2015 SIAM Conference on Dynamical Systems

Snowbird, Utah, USA - May 17-21, 2015

Dynamics of Sweeping Processes
with
Jumps in the Driving Term

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Notations

$\mathcal{H}$ is a real Hilbert space,

$$y_0 \in \mathcal{H},$$

$$\mathscr{C}_{\mathcal{H}} := \left\{ \mathcal{C} \subseteq \mathcal{H} \ : \ \mathcal{C} \text{ nonempty, closed, convex} \right\}$$

$$d_{\mathscr{H}}(\mathcal{A}, \mathcal{B}) := \max \left\{ \sup_{a \in \mathcal{A}} d(a, \mathcal{B}), \sup_{b \in \mathcal{B}} d(b, \mathcal{A}) \right\}, \qquad \mathcal{A}, \mathcal{B} \in \mathscr{C}_{\mathcal{H}}.$$

Lipschitz sweeping processes

Theorem (J.J. Moreau, *Proceeding CIME*, 1973)**.**

$$\forall \mathcal{C} \in Lip_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}}) \quad \exists! y \in Lip_{\mathrm{loc}}([0,\infty[\,;\mathcal{H}) \ :$$

$$\begin{cases} y(t) \in \mathcal{C}(t) & \forall t \geq 0 \\ -y'(t) \in N_{\mathcal{C}(t)}(y(t)) & \textit{for } \mathcal{L}^1\textit{-a.e. } t \geq 0 \\ y(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0) \end{cases}$$

Proof: Moreau-Yosida regularization

$$N_{\mathcal{C}(t)}(y(t)) \ \rightsquigarrow \ (N_{\mathcal{C}(t)}(y(t)))_\lambda := \frac{1}{\lambda}\Big(y(t) - \mathrm{Proj}_{\mathcal{C}(t)}(y(t))\Big)$$

Solve

$$\begin{cases} y_\lambda'(t) + \frac{1}{\lambda}\Big(y_\lambda(t) - \mathrm{Proj}_{\mathcal{C}(t)}(y_\lambda(t))\Big) = 0 \quad \forall t \geq 0 \\ y_\lambda(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0) \end{cases}$$

Then

$$y(t) := \lim_{\lambda \searrow 0} y_\lambda(t) \text{ solves the sweeping process}$$

Operator solution of sweeping processes

The solution operator of the sweeping processes

$$\begin{array}{rccc} \mathsf{S} \ : & Lip_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}}) & \longrightarrow & Lip_{\mathrm{loc}}([0,\infty[\,;\mathcal{H}) \\ & \mathcal{C} & \longmapsto & y \end{array}$$

is rate independent:

$$\mathsf{S}(\mathcal{C} \circ \phi) = \mathsf{S}(\mathcal{C}) \circ \phi$$

if $\phi \in Lip_{\mathrm{loc}}([0,\infty[\,;\mathbb{R})$ is increasing, $\phi(0) = 0$.

$\mathcal{C}(t)$ constant shape: the play operator

If

$$\mathcal{Z} \in \mathscr{C}_{\mathcal{H}}, \quad z_0 \in \mathcal{Z}, \quad u \in Lip_{\mathrm{loc}}([0,\infty[\,;\mathcal{H}),$$
$$\mathcal{C}(t) := u(t) - \mathcal{Z},$$

then the sweeping process reads

$$\begin{cases} u(t) - y(t) \in \mathcal{Z} & \forall t \geq 0 \\ y'(t) \in N_{\mathcal{Z}}(y(t) - u(t)) & \text{for } \mathcal{L}^1\text{-a.e. } t \geq 0 \\ y(0) = u(0) - z_0 \end{cases}$$

$$\begin{array}{rcl} \mathsf{P} \ : \ Lip_{\mathrm{loc}}([0,\infty[\,;\mathcal{H}) & \longrightarrow & Lip_{\mathrm{loc}}([0,\infty[\,;\mathcal{H}) \\ u & \longmapsto & y \end{array} \quad \textit{play operator.}$$

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More general data $\mathcal{C}(t)$

If $\mathcal{C} \in BV_{\mathrm{loc}}([0, \infty[\,; \mathscr{C}_{\mathcal{H}})$ then

$$-y'(t) \in N_{\mathcal{C}(t)}(y(t))$$

is not the proper formulation:

another notion of solution is needed.

$\boldsymbol{BV}$-solutions

Theorem (J.J. Moreau, *JDE*, 1977)**.**

$$\forall \mathcal{C} \in BV^r_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}}) \quad \exists! y \in BV^r_{\mathrm{loc}}([0,\infty[\,;\mathcal{H}) \ :$$

$$\begin{cases} y(t) \in \mathcal{C}(t) & \forall t \geq 0 \\ \mathrm{D}y = w\mu & \\ -w(t) \in N_{\mathcal{C}(t)}(y(t)) & \textit{for } \mu\textit{-a.e. } t \geq 0 \\ y(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0) & \end{cases}$$

Moreau's proof: catching up algorithm

For every $n \in \mathbb{N}$ discretize the time interval

$$0 = t_0^n < t_1^n < \cdots < t_j^n \to \infty \qquad \text{as } j \to \infty.$$

Define the step function $y_n : [0, \infty[\longrightarrow \mathcal{H}$ by

$$y_0^n := y_0, \qquad y_j^n := \mathrm{Proj}_{\mathcal{C}(t_j^n)}(y_{j-1}^n)$$
$$y_n(t) := y_j^n \quad \text{if } t \in [t_{j-1}^n, t_j^n[$$

Then there exists $y \in BV_{\mathrm{loc}}^r([0, \infty[\,; \mathcal{H})$ such that

$$y_n \to y \qquad \text{locally uniformly on } [0, \infty[,$$

and y solves the BV-sweeping process.

A "rate independent" method in $\boldsymbol{BV \cap C}$

Theorem (V. Recupero, *JDE*, 2011).
If

$$\mathcal{C} \in BV_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}}) \cap C([0,\infty[\,;\mathscr{C}_{\mathcal{H}}),$$

$$\ell_{\mathcal{C}} : [0,\infty[\longrightarrow [0,\infty[\,, \quad \ell_{\mathcal{C}}(t) := \mathrm{V}(\mathcal{C},[0,t]),$$

then $\exists! \widetilde{\mathcal{C}} \in Lip([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$ *such that* $\mathcal{C} = \widetilde{\mathcal{C}} \circ \ell_{\mathcal{C}}$ *and*

$$\overline{\mathsf{S}}(\mathcal{C}) := \mathsf{S}(\widetilde{\mathcal{C}}) \circ \ell_{\mathcal{C}}$$

"easily" solves the $(BV \cap C)$*-sweeping process.*

“Easily” means: simple measure theory proof

$$\text{Measure theory:}\begin{cases} \mathrm{D}\overline{\mathsf{S}}(\mathcal{C}) = (\mathsf{S}(\widetilde{\mathcal{C}})' \circ \ell_{\mathcal{C}})\,\mathrm{D}\ell_{\mathcal{C}}, \\ \mathrm{D}\ell_{\mathcal{C}}(\ell_{\mathcal{C}}^{-1}(B)) = \mathcal{L}^1(B) \quad \forall B \in \mathscr{B}(\ell_{\mathcal{C}}([0,\infty[)). \end{cases}$$

If $\widehat{y} := \mathsf{S}(\widetilde{\mathcal{C}}) \in Lip([0,\infty[\,;\mathcal{H})$ then

$$Z := \{t \ : \ -\widehat{y}'(t) \notin N_{\widetilde{\mathcal{C}}(t)}(\widehat{y}(t))\} \Longrightarrow \ \mathcal{L}^1(Z) = 0,$$

hence

$$\begin{aligned} &\mathrm{D}\ell_{\mathcal{C}}(\{t \ : \ -\widehat{y}'(\ell_{\mathcal{C}}(t)) \notin N_{\widetilde{\mathcal{C}}(\ell_{\mathcal{C}}(t))}(\widehat{y}(\ell_{\mathcal{C}}(t)))\}) \\ =\,&\mathrm{D}\ell_{\mathcal{C}}(\{t \ : \ \ell_{\mathcal{C}}(t) \in Z\}) = \mathrm{D}\ell_{\mathcal{C}}(\ell_{\mathcal{C}}^{-1}(Z)) = \mathcal{L}^1(Z) = 0, \end{aligned}$$

i.e.

$$-\widehat{y}'(\ell_{\mathcal{C}}(t)) \in N_{\widetilde{\mathcal{C}}(\ell_{\mathcal{C}}(t))}(\widehat{y}(\ell_{\mathcal{C}}(t))) \quad \text{for } \mathrm{D}\ell_{\mathcal{C}}\text{-a.e. } t \geq 0. \quad \blacksquare$$

The discontinuous BV case

If

$$\mathcal{C} \in BV^{r}_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$$

$$\ell_{\mathcal{C}} : [0,\infty[\longrightarrow [0,\infty[\,, \quad \ell_{\mathcal{C}}(t) := \mathrm{V}(\mathcal{C},[0,t]),$$

then $\exists! \widetilde{\mathcal{C}} \in Lip(\ell_{\mathcal{C}}([0,\infty[);\mathscr{C}_{\mathcal{H}})$ such that $\mathcal{C} = \widetilde{\mathcal{C}} \circ \ell_{\mathcal{C}}$.

We need to extend $\widetilde{\mathcal{C}}$ to $[0,\infty[$,

i.e.

we need to define $\widetilde{\mathcal{C}}$ on $[\ell_{\mathcal{C}}(t-),\ell_{\mathcal{C}}(t+)]$ for $t \in \mathrm{Discont}(\mathcal{C})$

i.e.

we need to fill in the jumps of $\mathcal{C}$ at every $t \in \mathrm{Discont}(\mathcal{C})$.

Jump

t_0 jump point

The particular case $\mathsf{S} = \mathsf{P}$

A natural choice:

fill in the jumps with segments.

If $u \in BV^{\mathrm{r}}_{\mathrm{loc}}([0,\infty[\,;\mathcal{H})$ and $\ell_u(t) := \mathrm{V}(u,[0,t])$
then $\exists! \widetilde{u} \in Lip([0,\infty[\,;\mathcal{H})$ such that

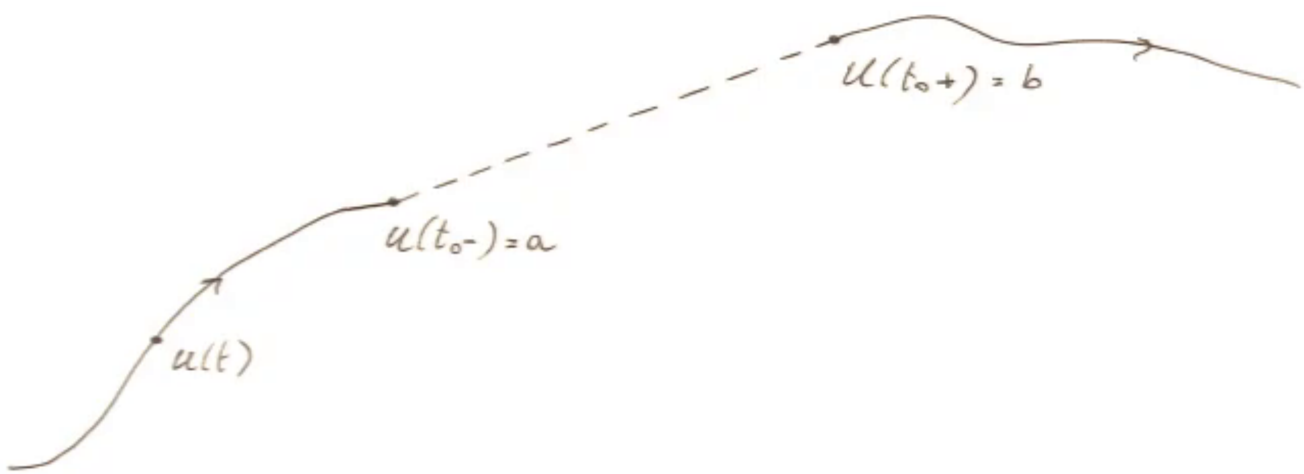
$$u = \widetilde{u} \circ \ell_u$$

$$\widetilde{u} \quad \text{is a geodesic segment } on\, [\ell_u(t-), \ell_u(t+)]\,.$$

Is $\overline{\mathsf{P}}(u) := \mathsf{P}(\widetilde{u}) \circ \ell_u$ the BV-solution?

Geodesic segment in $\mathcal{H}$

$s(t) = (1-t)a + tb$



No!

Theorem (V. Recupero, *Ann. SNS Pisa*, 2011).

$$\overline{\mathsf{P}}(u) := \mathsf{P}(\widetilde{u}) \circ \ell_u, \qquad u \in BV^r_{\mathrm{loc}}([0,\infty[\,;\mathcal{H}),$$

is the unique continuous extension of
$\mathsf{P} : Lip_{\mathrm{loc}}([0,\infty[\,;\mathcal{H}) \longrightarrow Lip_{\mathrm{loc}}([0,\infty[\,;\mathcal{H})$ *when*

the domain has the BV*-strict topology,*

the codomain has the L^1*-topology.*

But $\overline{\mathsf{P}}(u)$ *is* not *the* BV*-solution.*

Theorem (P. Krejčí, V. Recupero, *JCA*, 2014).
If $dim(\mathcal{H}) < \infty$,

$$\overline{\mathsf{P}}(u) \textit{ is the } BV\textit{-solution} \iff \mathcal{Z} \textit{ is a non-obtuse polyhedron}$$

Convex-valued "segments"

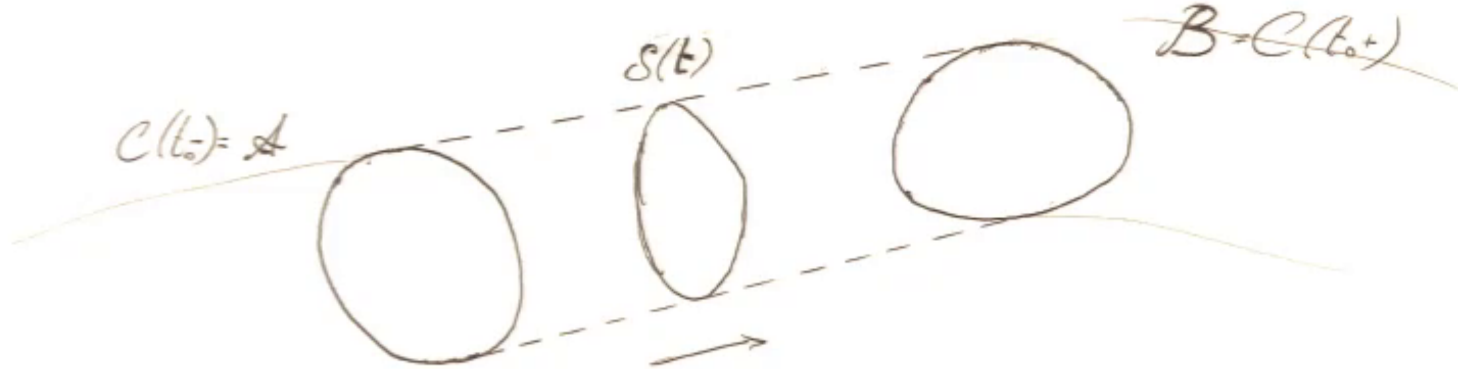
Segments correspond to convex-valued geodesics of the form

$$\mathcal{S}(t) := (1-t)\mathcal{A} + t\mathcal{B}, \qquad t \in [0,1]\,.$$

connecting $\mathcal{A}$ to $\mathcal{B}$. Therefore $\mathcal{S}$ is not a good choice.

Minkowski sum-type geodesic

$S(t) = (1-t)A + tB$ No!



Another convex-valued curve is needed

We look for a curve $\mathcal{G}(t)$ connecting $\mathcal{A}$ and $\mathcal{B}$ such that

any point $a \in \mathcal{A}$ *is swept by* $\mathcal{G}(t)$ *to its projection on* $\mathcal{B}$,

i.e. for *any* initial datum $a \in \mathcal{A}$, we want $\mathrm{Proj}_{\mathcal{B}}(a)$ to be the final point of trajectory of the solution of the sweeping process driven by $\mathcal{G}(t)$.

This is consistent with the catching-up algorithm.

A convex-valued geodesic which works

If $\rho := d_{\mathcal{H}}(\mathcal{A}, \mathcal{B})$

$$\mathcal{G}_{\mathcal{A},\mathcal{B}}(t) := (\mathcal{A} + B_{t\rho}(0)) \cap (\mathcal{B} + B_{(1-t)\rho}(0)), \quad t \in [0,1]\,,$$

is a geodesic connecting $\mathcal{A}$ to $\mathcal{B}$.

For any $a \in \mathcal{A}$, the solution $y \in Lip([0,1]\,;\mathcal{H})$ of

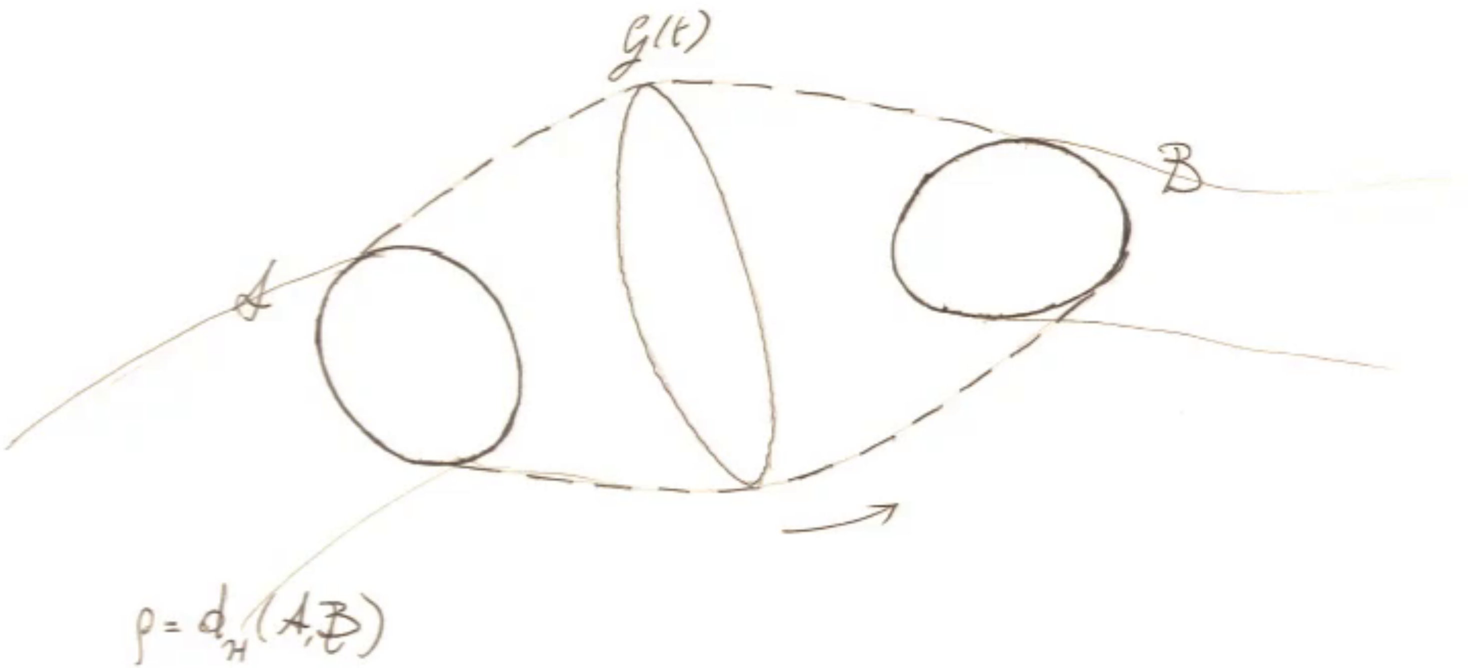
$$\begin{cases} y(t) \in \mathcal{G}(t) & \forall t \in [0,1]\,, \\ -y'(t) \in N_{\mathcal{G}(t)}(y(t)) & \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,1], \\ y(0) = a \end{cases}$$

satisfies

$$y(1) = \mathrm{Proj}_{\mathcal{B}}(a).$$

The geodesic $\mathcal{G}_{\mathcal{A},\mathcal{B}}$

$$\mathcal{G}(t) = (\mathcal{A} + B_{t\rho}(0)) \cap (\mathcal{B} + B_{(1-t)\rho}(0))$$ YES



Reduction from *BV* to *Lip*

Theorem (V. Recupero, 2015)**.**
If

$$\mathcal{C} \in BV^{r}_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}}), \quad \ell_{\mathcal{C}}(t) := \mathrm{V}(\mathcal{C},[0,t]),$$

then $\exists! \widetilde{\mathcal{C}} \in Lip([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$ *such that*

$$\mathcal{C} = \widetilde{\mathcal{C}} \circ \ell_{\mathcal{C}}$$

$$\widetilde{\mathcal{C}} \ \text{ is the geodesic } \mathcal{G}_{\mathcal{C}(t-),\mathcal{C}(t+)} \ \text{ on } [\ell_{\mathcal{C}}(t-), \ell_{\mathcal{C}}(t+)]\,.$$

and

$\overline{\mathsf{S}}(\mathcal{C}) := \mathsf{S}(\widetilde{\mathcal{C}}) \circ \ell_{\mathcal{C}}$ *"easily" solves the BV-sweeping process:*

existence, continuous dependence, convergence
of the catching-up algorithm are deduced from the Lip-case.

Basic ("new") vector measure theory tools

If

$$f \in Lip_{\mathrm{loc}}([0,\infty[\,;\mathcal{H}), \quad h : [0,\infty[\longrightarrow [0,\infty[\text{ increasing },$$

then

$$\begin{cases} \mathrm{D}h(h^{-1}(B)) = \mathscr{L}^1(B) & \forall B \in \mathscr{B}(h(\mathrm{Cont}(h))). \\ \mathrm{D}(f \circ h) = g\,\mathrm{D}h \end{cases}$$

where

$$g(t) := \begin{cases} f'(h(t)) & \text{if } t \in \mathrm{Cont}(h) \\ \\ \dfrac{f(h(t+)) - f(h(t-))}{h(t+) - h(t-)} & \text{if } t \in \mathrm{Discont}(h) \end{cases}$$