Global Challenge: Critical Factors for Tipping Points

"The ultimate objective is stabilisation of greenhouse gas concentrations at a level that would prevent dangerous interference with the climate system

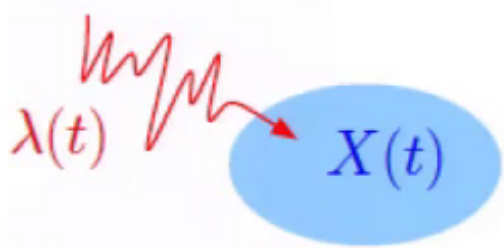
... such a level should be achieved within a time frame sufficient to allow ecosystems to adapt naturally to climate change"

dangerous levels and dangerous rates

[UNFCCC, 2007]

Mathematics Needed to Address the Global Challenge

Non-autonomous Dynamical Systems

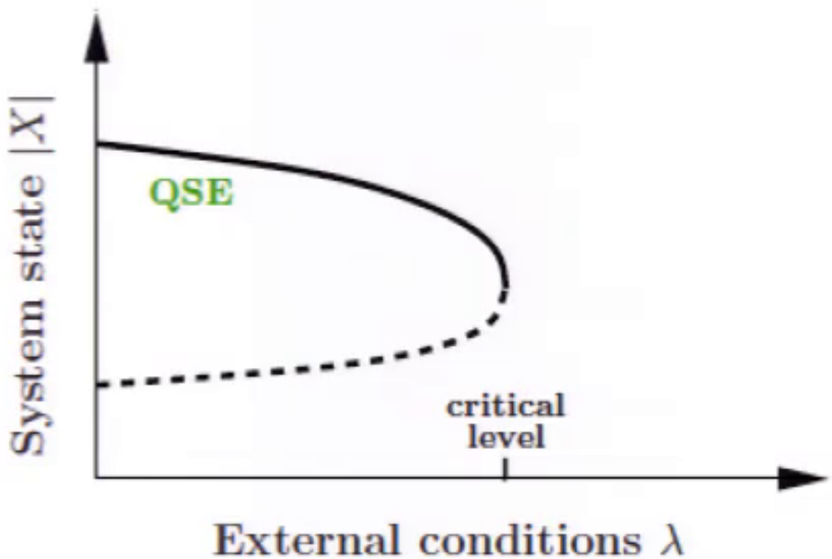


$$\frac{d}{dt}X = f(X, \lambda(t))$$

$X(t)$ - state of an open system at time t
$\lambda(t)$ - time-varying external conditions (external forcing)

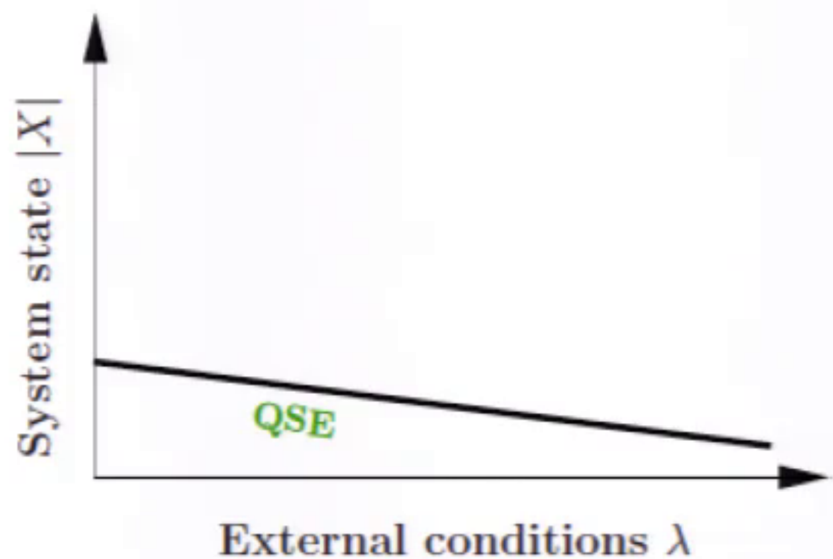
Two Bifurcation Types:

Dynamic Bifurcation



critical level

Rate-induced Bifurcation

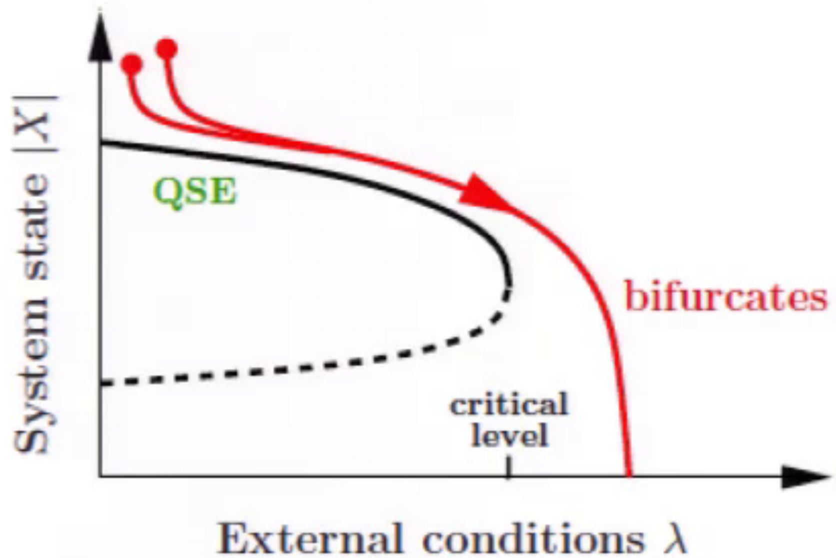


no critical level

Quasi-Static Equilibrium QSE: $\dfrac{d\lambda}{dt} = 0$ and $\dfrac{dx}{dt} = f(x, \lambda) = 0$

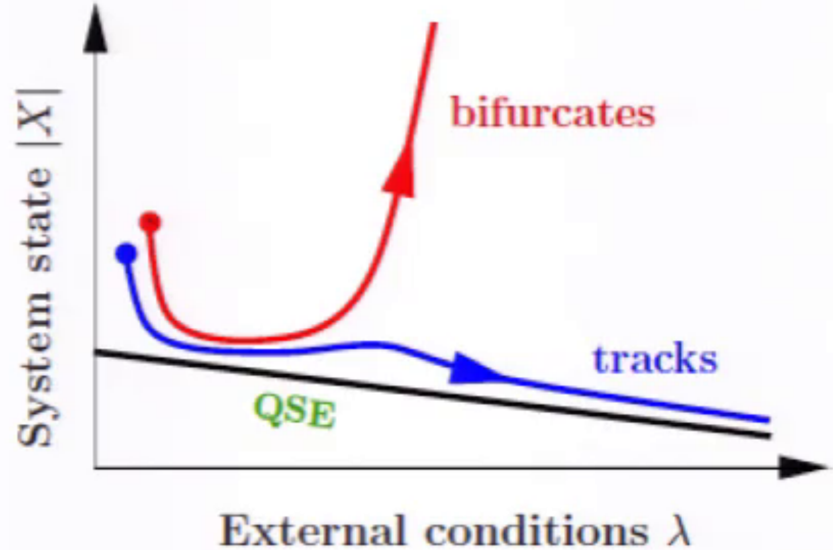
Two Bifurcation Types:

Dynamic Bifurcation



bifurcates for any rate of change

Rate-induced Bifurcation



tracks the QSE below critical rate

may track or bifurcate above critical rate

Rate-Induced Bifurcations in Carbon Cycle Models

Carbon Facts:

Peatland soils contain 400–1000 billion tones of carbon.

Spontaneous combustion is the most common cause of compost fires!

Question:

How will peatlands respond to hot weather anomalies?

Photo: Peatland fires in Russia in Summer 2010

Example: Soil-Carbon & Temperature Model

$$\frac{d}{dt}C = \Pi - C\, r_0\, e^{\alpha T},$$

$$\epsilon\ \frac{d}{dt}T = -\frac{\lambda}{A}(T - T_a) + C\, r_0 e^{\alpha T}, \qquad \epsilon = \frac{\mu}{A} = 0.064$$

$$\frac{d}{dt}T_a = r.$$

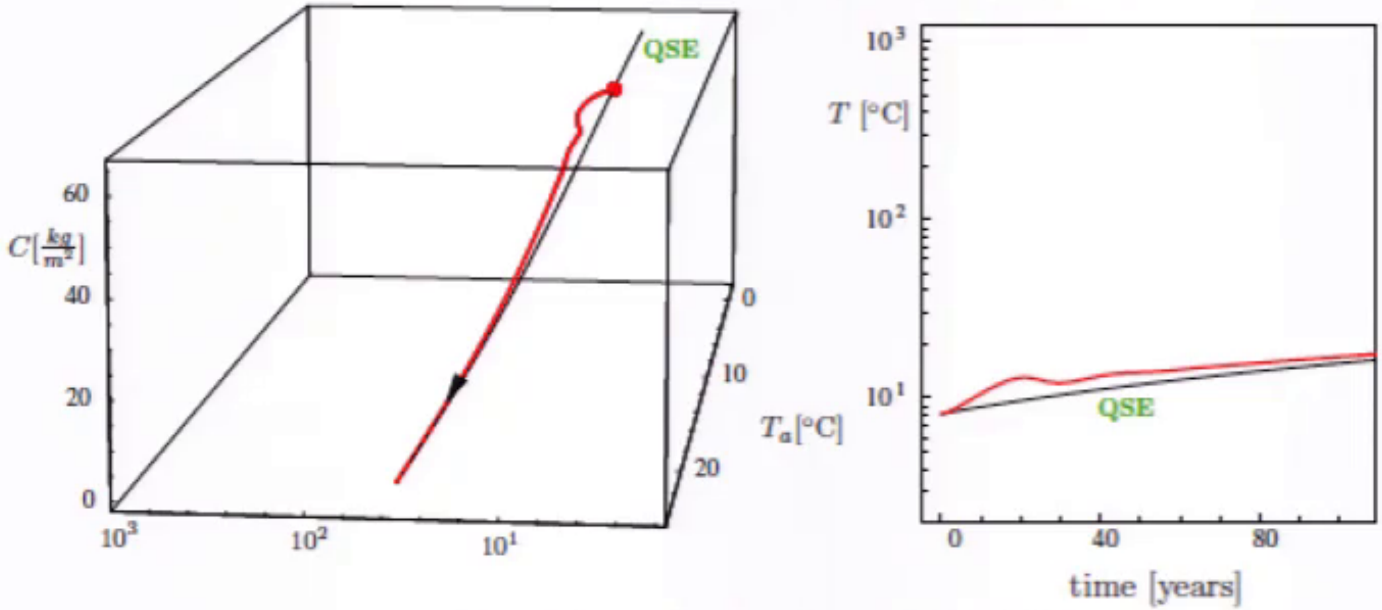
C is soil carbon content
T is soil temperature
T_a is atmospheric temperature

[C. Luke and P. Cox, *Eur. J. Soil Sci.*, 62, 5-12 (2011)]

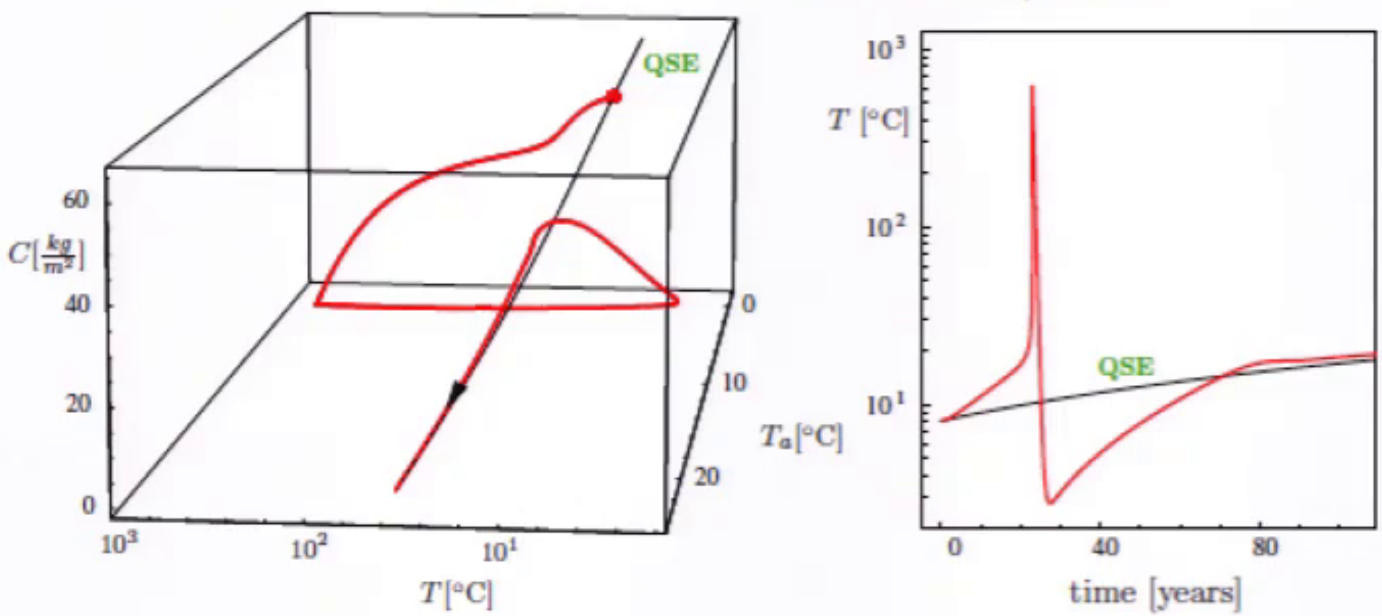
[S.Wieczorek, P. Ashwin, C.M. Luke, P. Cox, *Proc. Roy. Soc. A* 467 (2011) 1243]

The Compost-Bomb Instability

below the critical rate r = 0.75 °C/decade



above the critical rate r = 0.9 °C/decade

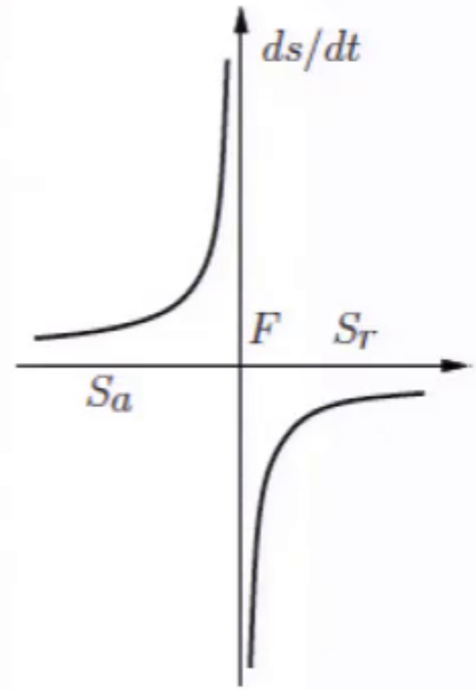


Mathematical Techniques for Proving Existence of Critical Rates and Thresholds

$$\epsilon \frac{d}{dt} x = f(x, y, \lambda(rt), \epsilon), \quad \frac{d}{dt} y = g(x, y, \lambda(rt), \epsilon), \quad \frac{d}{dt} t = 1$$

Desingularisation:

$dt = -ds\, (\partial f/\partial x)|_S$



The desingularised system $(\epsilon = 0)$

folded singularities become regular equilibria

$$\frac{d}{ds} x = [g\, \partial f/\partial y + (\partial f/\partial \lambda)(d\lambda/dt)]\,|_S$$

$$\frac{d}{ds} t = -(\partial f/\partial x)|_S$$

Blow-up $(0 < \epsilon \ll 1)$

[**Szmolyan & Wechselberger** *J. Diff. Eq.* **177 (2001) 419**]

Existence of Critical Rates and Thresholds

$$\epsilon \frac{d}{dt} x = f(x, y, \lambda(rt), \epsilon), \quad \frac{d}{dt} y = g(x, y, \lambda(rt), \epsilon), \quad \frac{d}{dt} t = 1$$

Theorem (Existence of critical rates)

A critical rate requires that projections of $F(\lambda)$ and QSE(λ) onto the slow y-direction intersect on an open set.

Suppose the system satisfies the folded singularity condition:

$$\left[g \frac{\partial f}{\partial y} + \frac{\partial f}{\partial \lambda} \frac{d\lambda}{dt} \right]_F = 0$$

Then there is a critical rate $r = r_c$

Theorem (Existence of non-obvious thresholds)

The threshold is guaranteed if a folded saddle is the only folded singularity.

[C. Perryman and S.Wieczorek, *Proc. Roy Soc. A* **470 20140226 (2014)]**

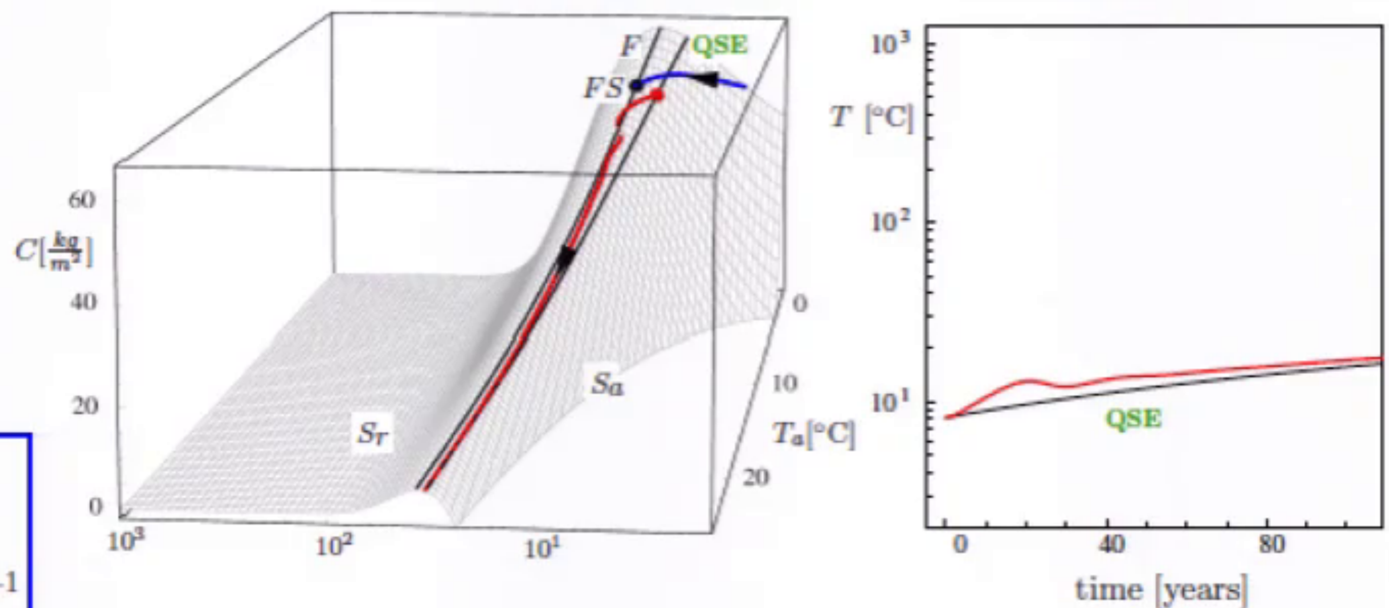
Critical Rates and Non-obvious Thresholds

The critical rate:

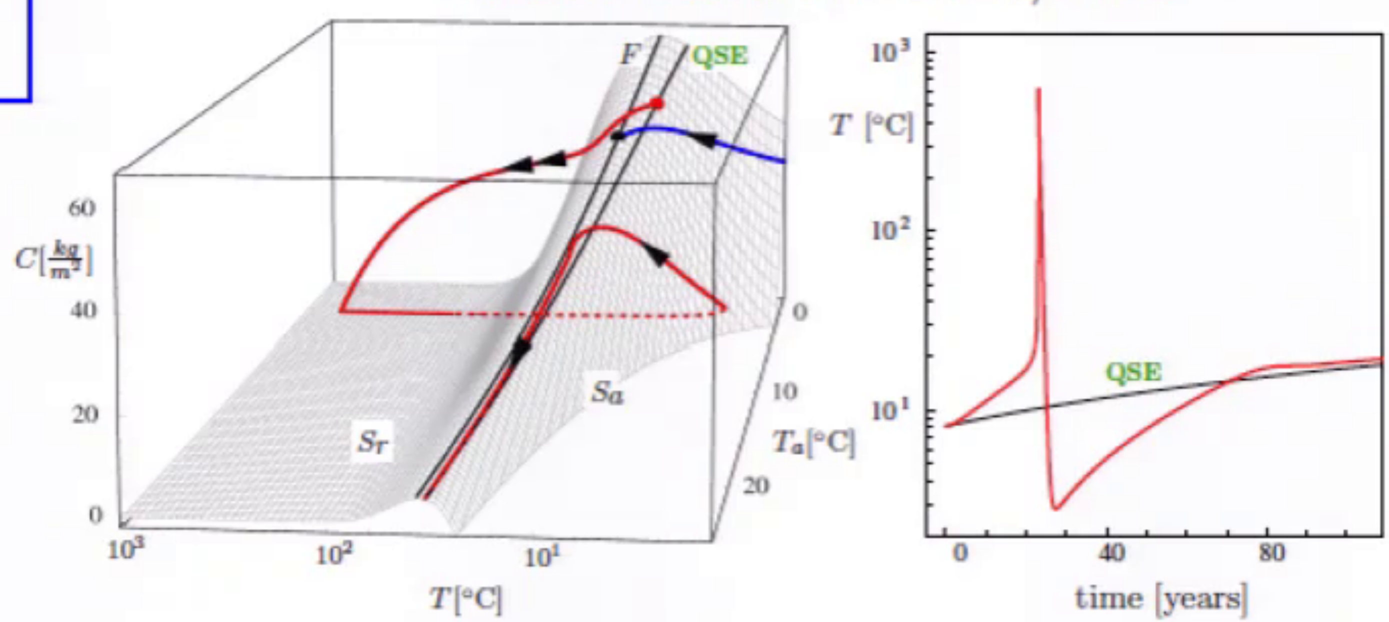
$$r_c = \frac{r_0(1 - \alpha A \Pi/\lambda)}{\alpha} e^{\alpha T_a^0 + 1}$$

$r_c \approx 0.8$ °C/decade

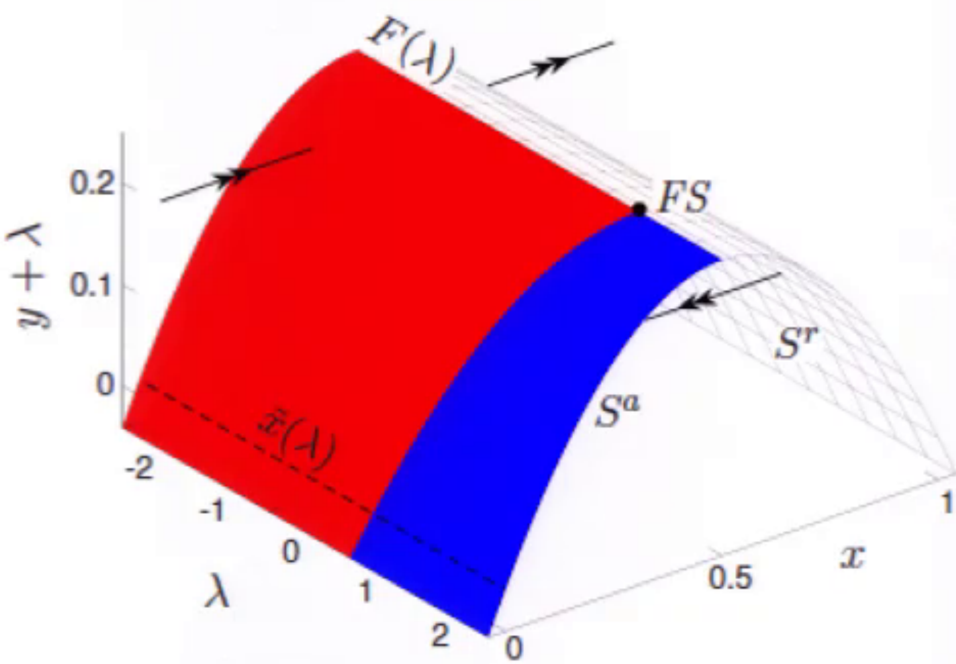
below the critical rate r = 0.75 °C/decade

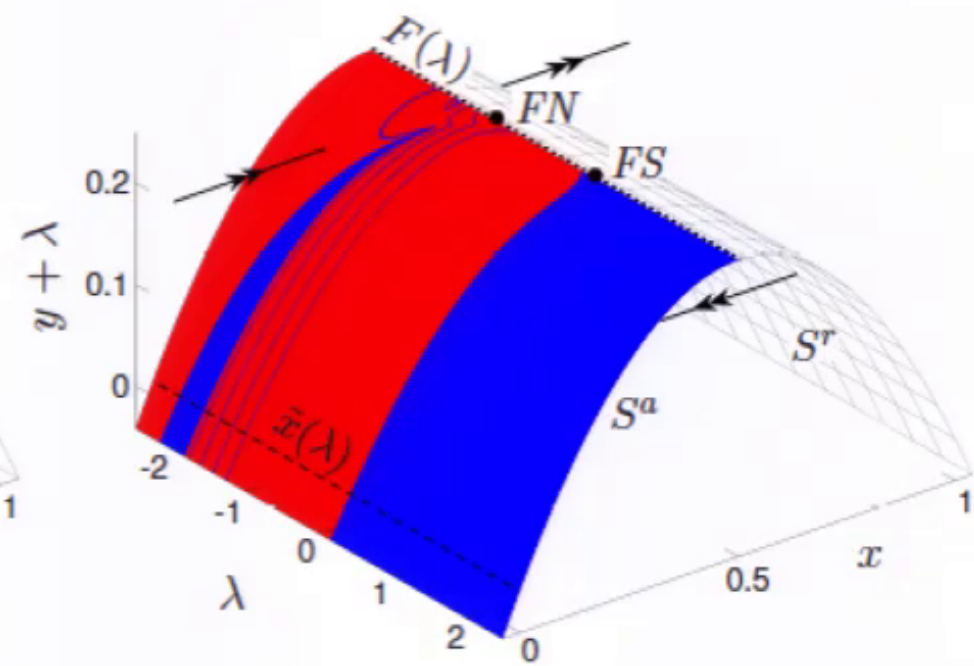


above the critical rate r = 0.9 °C/decade



Non-obvious Threshold in Slow-fast Systems

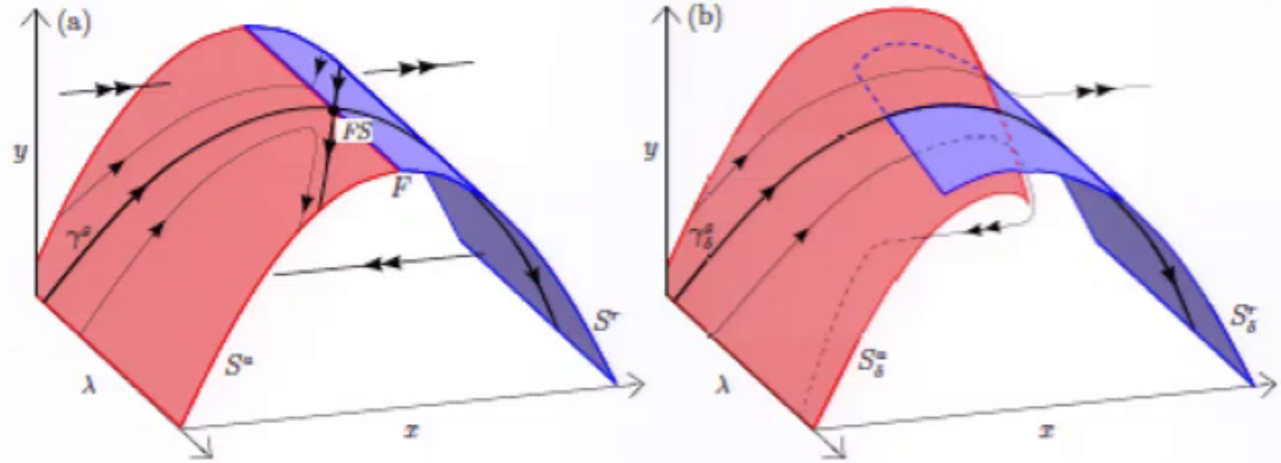




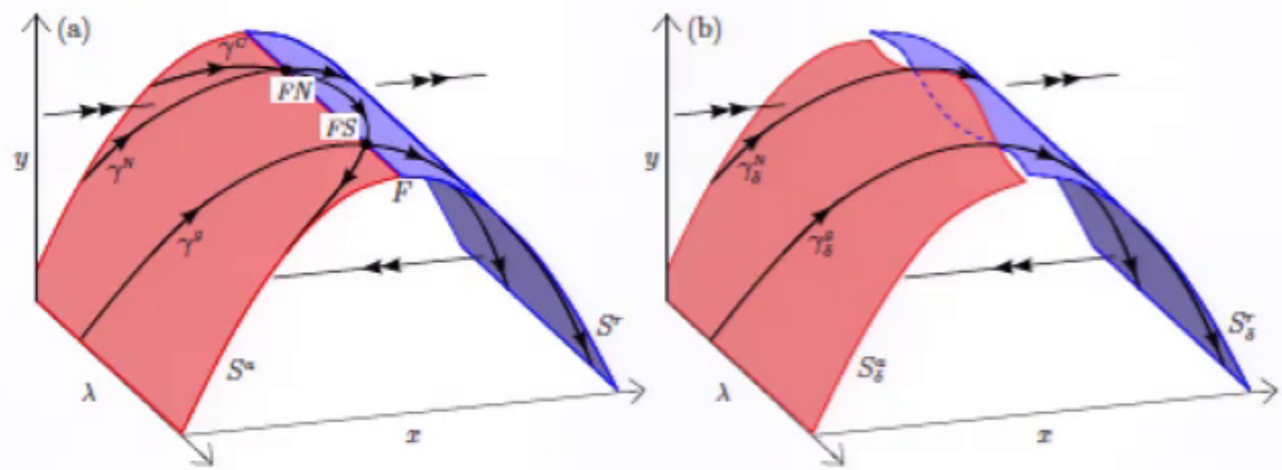
[**C. Perryman and S.Wieczorek,** *Proc. Roy Soc. A* **470 20140226 (2014)**]

Non-obvious Thresholds & Intersecting Slow Manifolds

Simple Threshold (Folded Saddle)

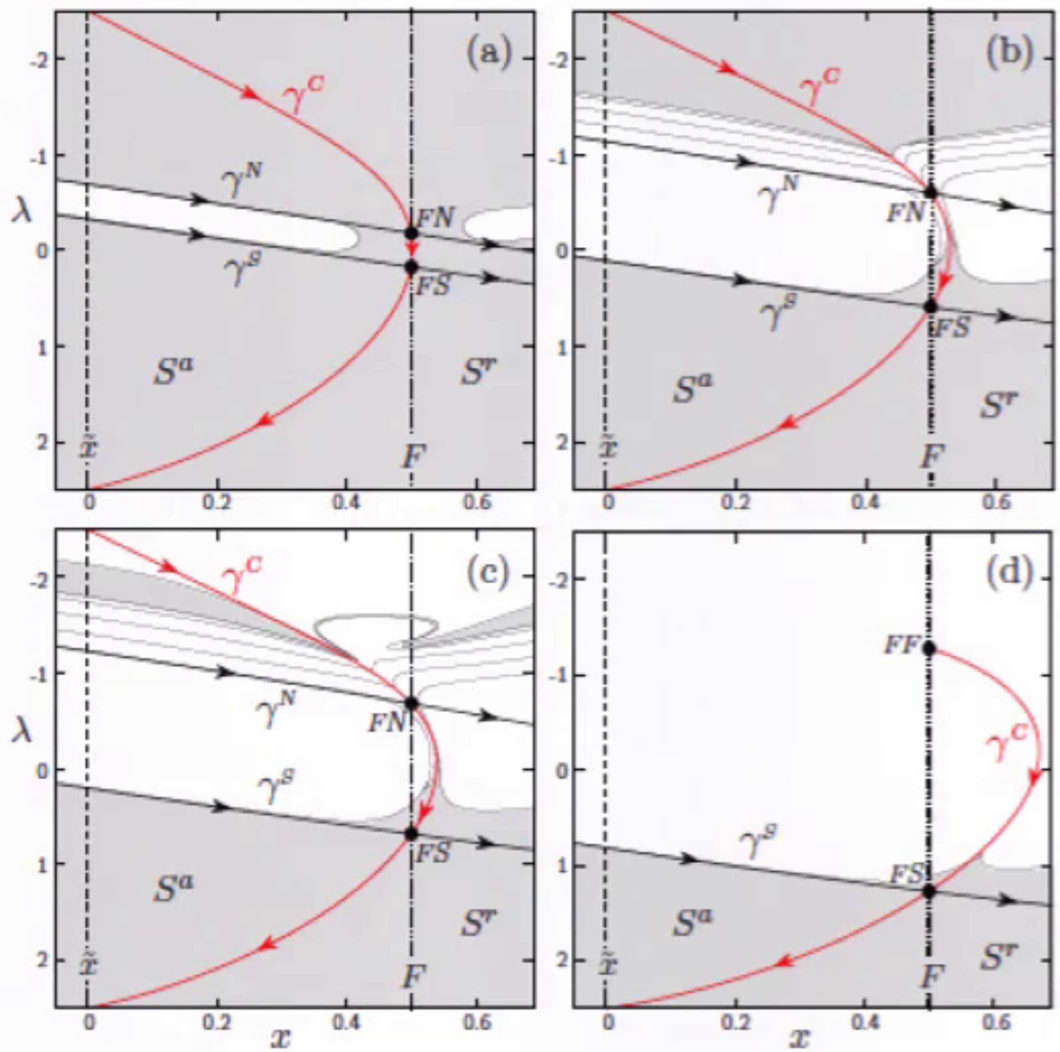


Complicated Threshold (Folded Saddle-Node)



Non-obvious Threshold in Slow-fast Systems

Complicated Threshold



Non-obvious Threshold & Composite Canards