Localizing Nonlinear Eigenvalues: Theory and Applications

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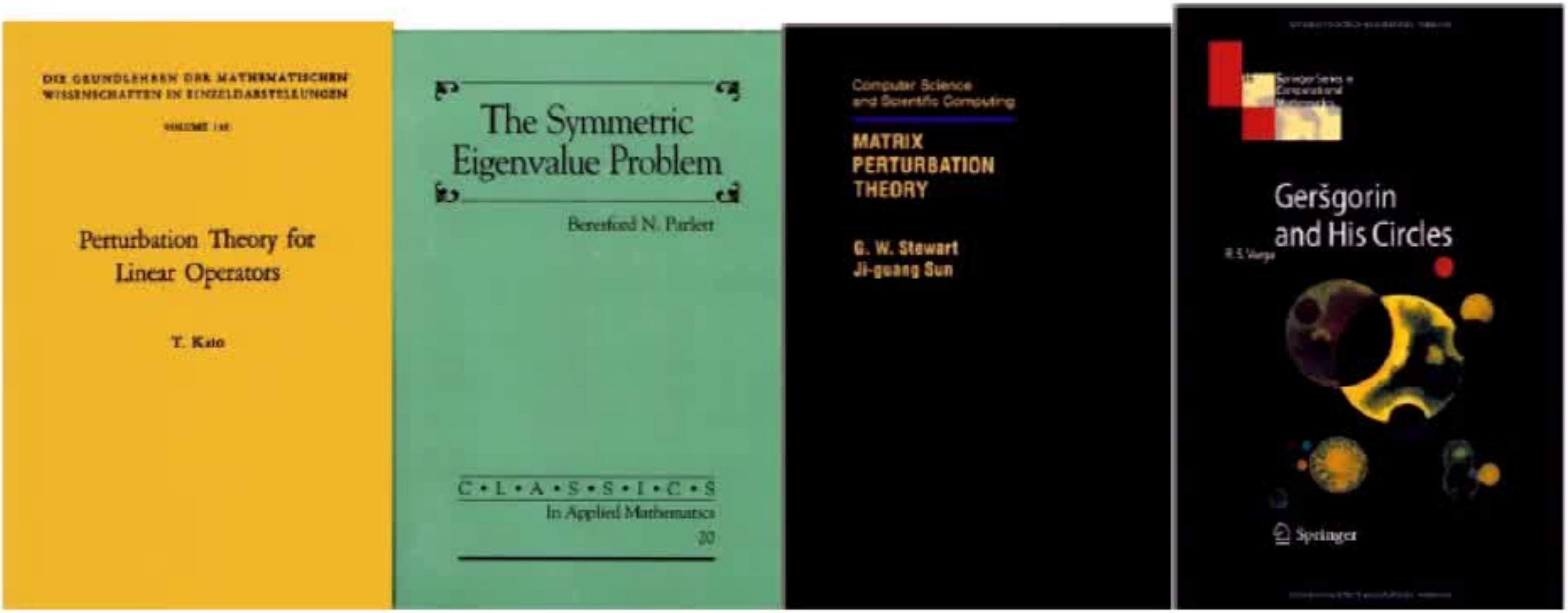


Vibrations are everywhere, and so too are the eigenvalues associated with them. As mathematical models invade more and more disciplines, we can anticipate a demand for eigenvalue calculations in an ever richer variety of contexts.

– *Beresford Parlett*, The Symmetric Eigenvalue Problem

Why Eigenvalues?

As a student in a first ODE course:

$$y' - Ay = 0 \xrightarrow{y=e^{\lambda t}v} (\lambda I - A)v = 0.$$

Me: "How do I compute this?"

$$p(\lambda) = \det(\lambda I - A) = 0.$$

Why Nonlinear Eigenvalues?

$$y' - Ay = 0 \xrightarrow{y=e^{\lambda t}v} (\lambda I - A)v = 0$$

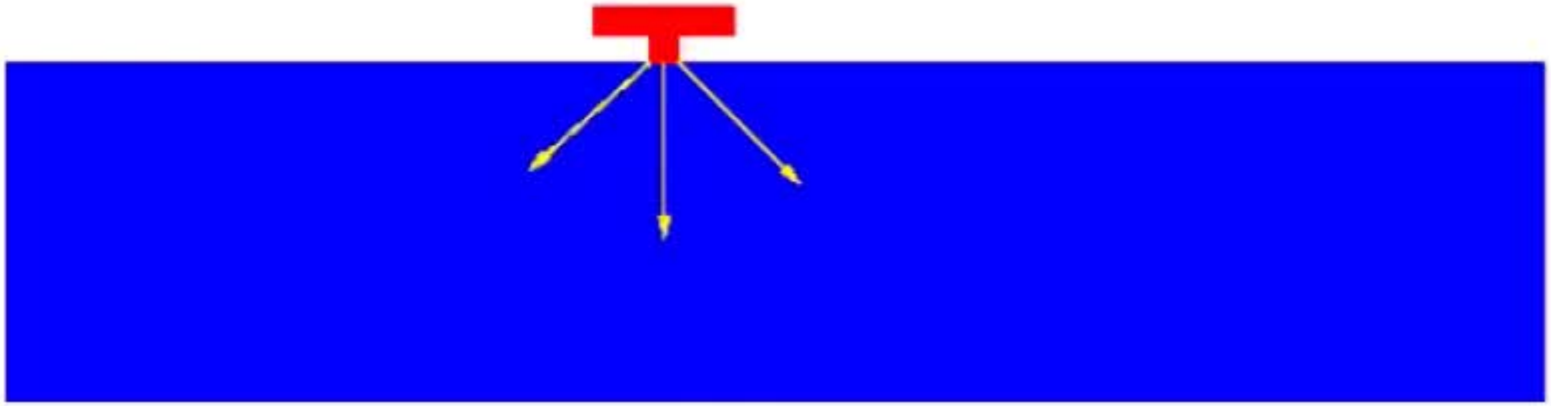
$$y'' + By' + Ky = 0 \xrightarrow{y=e^{\lambda t}v} (\lambda^2 I + \lambda B + K)v = 0$$

$$y' - Ay - By(t-1) = 0 \xrightarrow{y=e^{\lambda t}v} (\lambda I - A - Be^{-\lambda})v = 0$$

$$T(d/dt)y = 0 \xrightarrow{y=e^{\lambda t}v} T(\lambda)v = 0$$

- Higher-order ODEs
- Dynamic element formulations
- Delay differential equations
- Boundary integral equation eigenproblems
- Radiation boundary conditions

My motivation



$$T(\omega)v \equiv \left(K - \omega^2 M + G(\omega)\right) v = 0$$

Wanted: Perturbation theory justifying a terrible estimate of $G(\omega)$

Nonlinear eigenvalue problem

$$T(\lambda)v = 0, \quad v \neq 0.$$

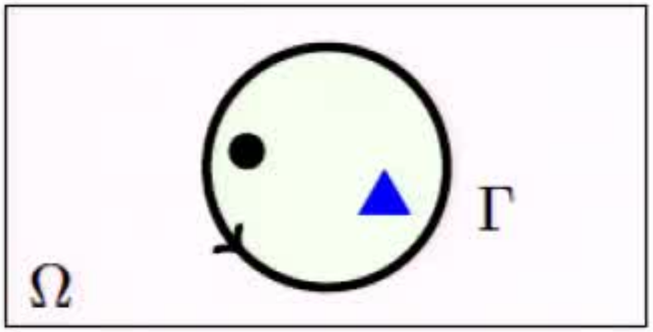
where

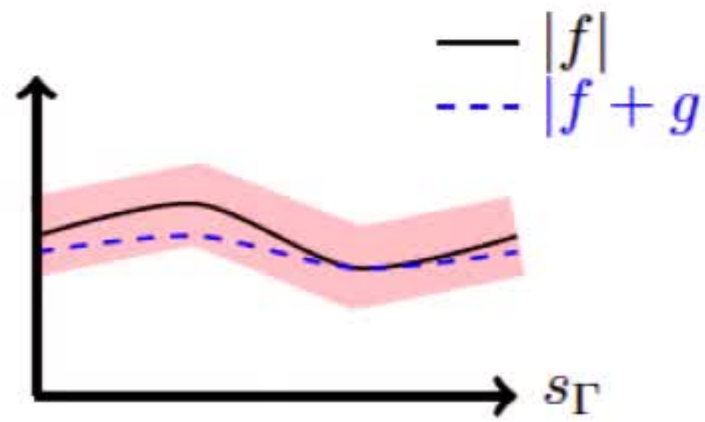
- $T : \Omega \to \mathbb{C}^{n\times n}$ analytic, $\Omega \subset \mathbb{C}$ simply connected
- Regularity: $\det(T) \not\equiv 0$

Nonlinear spectrum: $\Lambda(T) = \{z \in \Omega : T(z) \text{ singular}\}$.

Goal: Use analyticity to *compare* and to *count*

Winding, Rouché, and Gohberg-Sigal





Analytic	$f, g : \Omega \to \mathbb{C}$	$T, E : \Omega \to \mathbb{C}^{n \times n}$
Winding #	$\frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} \, dz$	$\mathrm{tr}\left(\frac{1}{2\pi i} \int_\Gamma T(z)^{-1} T'(z) \, dz\right)$
Theorem	Rouché (1862): $\lvert g\rvert < \lvert f\rvert$ on $\Gamma \implies$ same # zeros of $f, f+g$	Gohberg-Sigal (1971): $\lVert T^{-1}E\rVert < 1$ on $\Gamma \implies$ same # eigs of $T, T+E$

Comparing NEPs

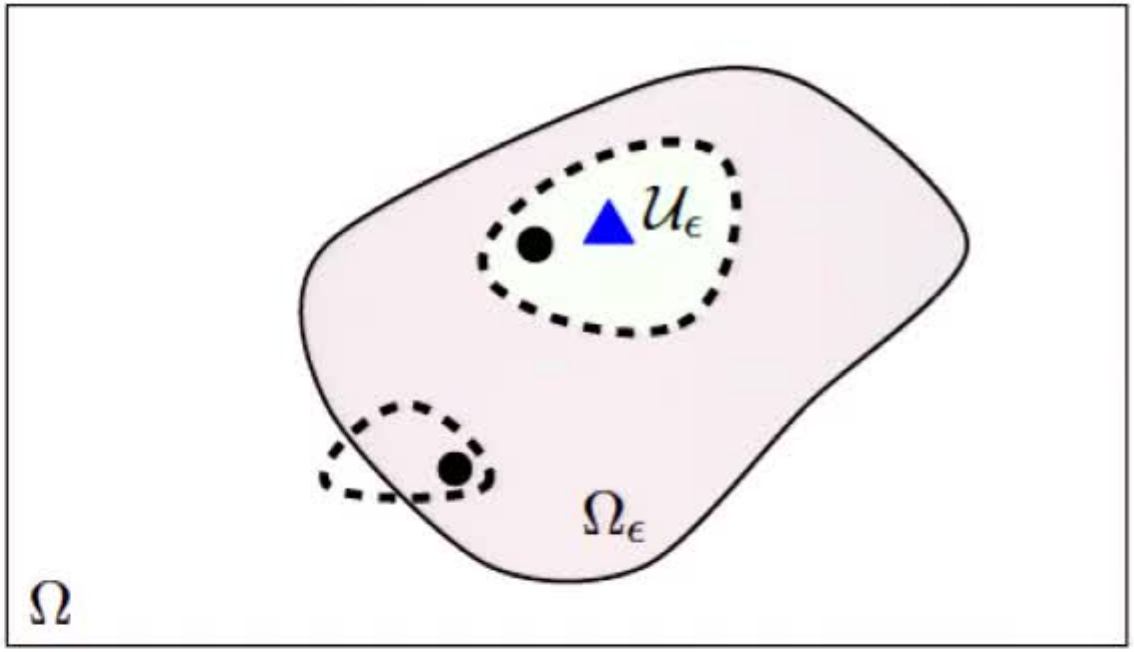
Suppose

$$T, E : \Omega \to \mathbb{C}^{n\times n} \text{ analytic}$$
$$\Gamma \subset \Omega \text{ a simple closed contour}$$
$$T(z) + sE(z) \text{ nonsingular } \forall s \in [0,1], z \in \Gamma$$

Then T and $T+E$ have the same number of eigenvalues inside Γ.

Pf: Constant winding number around Γ.

Pseudospectral comparison



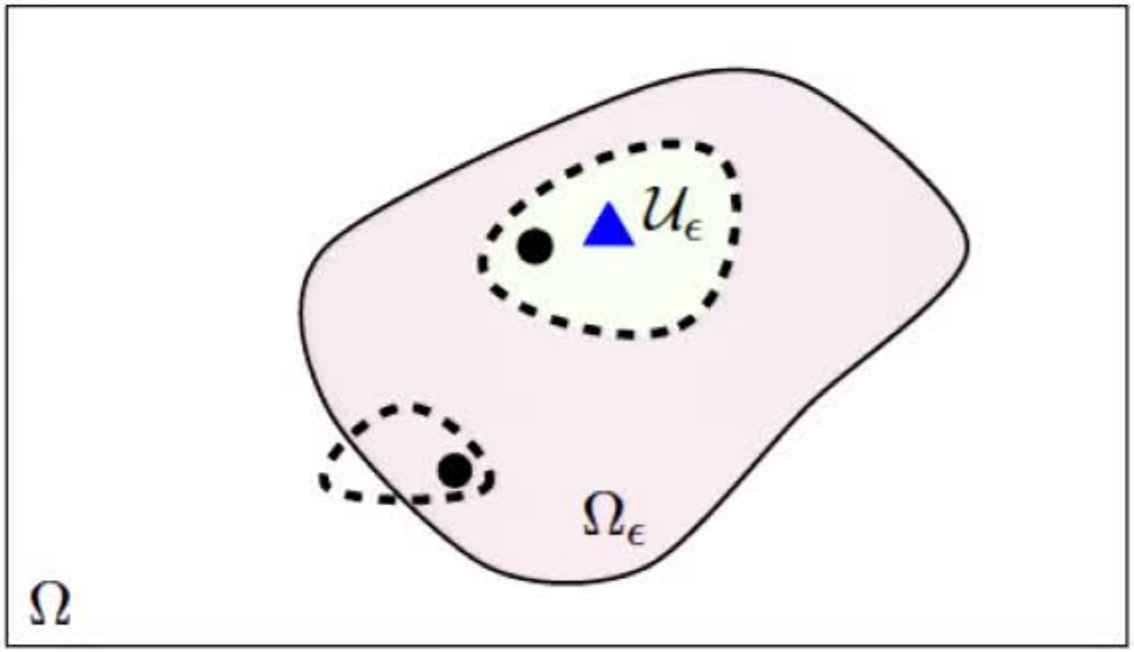
E analytic, $\|E(z)\| < \epsilon$ on Ω_ϵ. Then

$$\Lambda(T+E) \cap \Omega_\epsilon \;\subset\; \Lambda_\epsilon(T) \cap \Omega_\epsilon$$

Also, if $\mathcal{U}_\epsilon$ a component of Λ_ϵ and $\bar{\mathcal{U}}_\epsilon \subset \Omega_\epsilon$, then

$$|\Lambda(T+E) \cap \mathcal{U}_\epsilon| \;=\; |\Lambda(T) \cap \mathcal{U}_\epsilon|$$

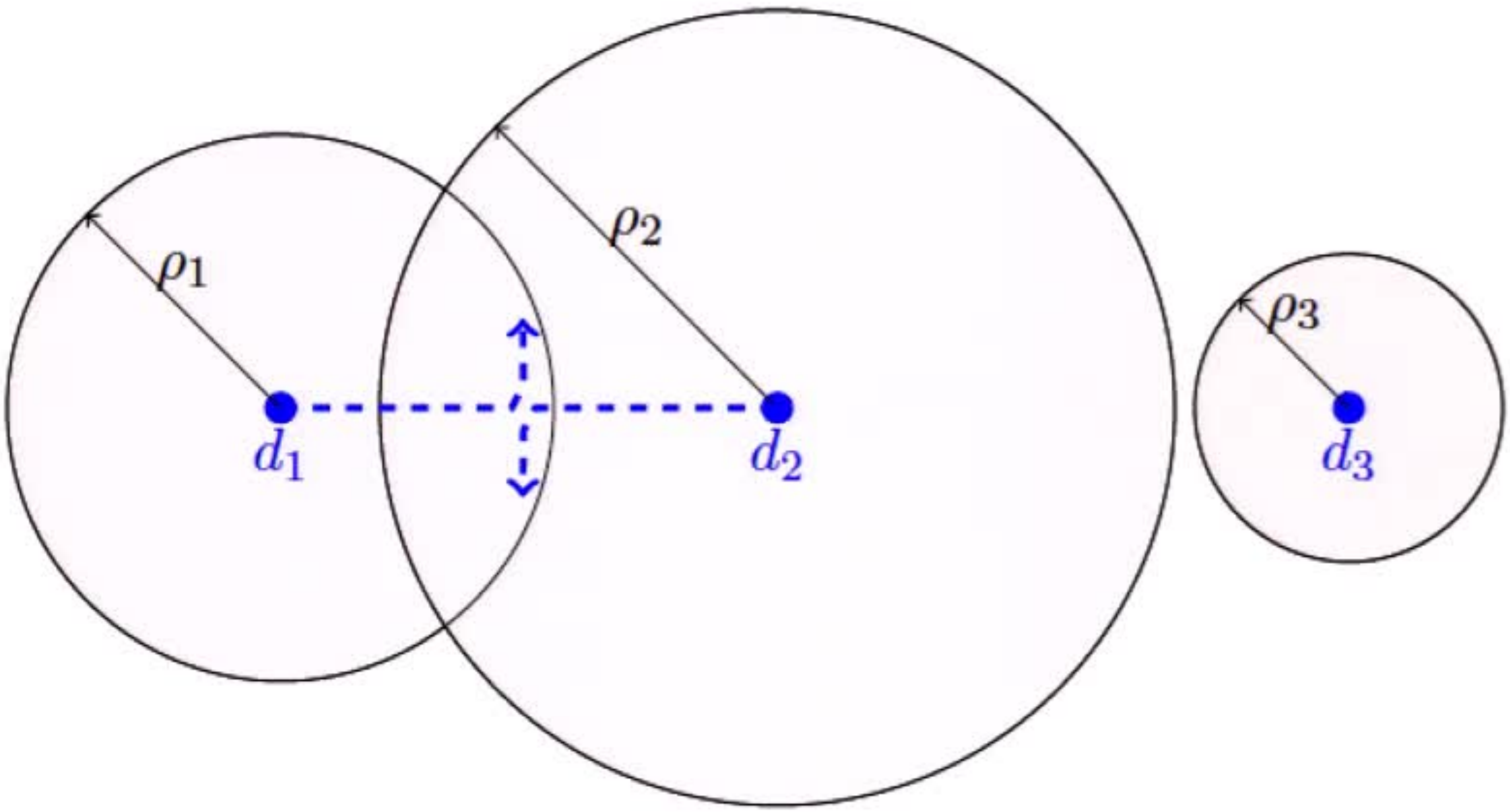
Pseudospectral comparison



- Most useful when T is linear
- Even then, can be expensive to compute!
- What about related tools?

The Gershgorin picture (linear case)

$$A = D + F, \qquad D = \mathrm{diag}(d_i), \qquad \rho_i = \sum_j |f_{ij}|$$



Gershgorin ($+\epsilon$)

Write $A = D + F$, $D = \mathrm{diag}(d_1, \ldots, d_n)$. Gershgorin disks are:

$$G_i = \left\{ z \in \mathbb{C} : |z - d_i| \le \sum_j |f_{ij}| \right\}.$$

Useful facts:

- Spectrum of A lies in $\bigcup_{i=1}^m G_i$
- $\bigcup_{i \in \mathcal{I}} G_i$ disjoint from other disks $\implies$ contains $|\mathcal{I}|$ eigenvalues.

Pf:

$A - zI$ strictly diagonally dominant outside $\bigcup_{i=1}^m G_i$.
Eigenvalues of $D - sF$, $0 \le s \le 1$, are continuous.

Nonlinear Gershgorin

Write $T(z) = D(z) + F(z)$. Gershgorin *regions* are

$$G_i = \left\{ z \in \mathbb{C} : |d_i(z)| \leq \sum_j |f_{ij}(z)| \right\}.$$

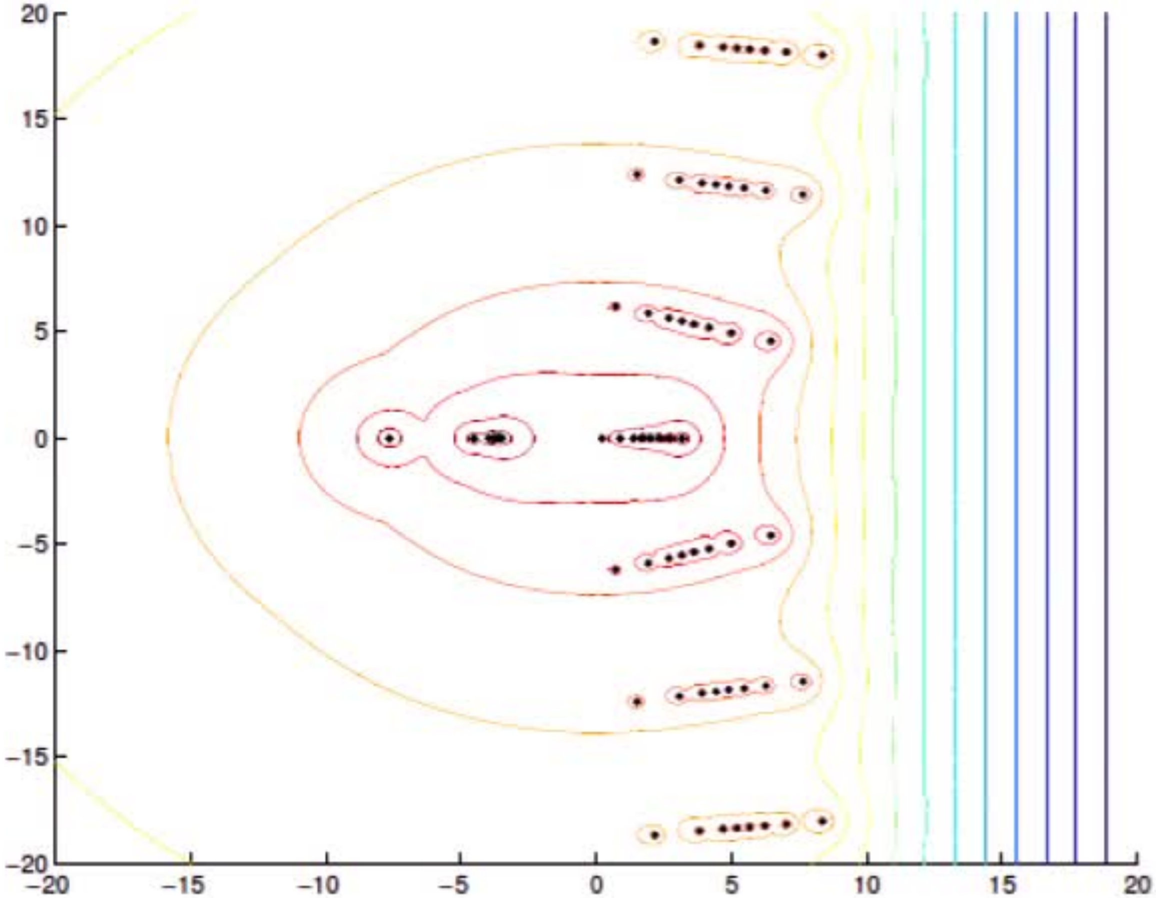
Useful facts:

- Spectrum of T lies in $\bigcup_{i=1}^m G_i$
- Bdd connected component of $\bigcup_{i=1}^m G_i$ strictly in Ω
  $\implies$ same number of eigs of D and T in component
  $\implies$ at least one eig per component of G_i involved

Pf: Strict diag dominance test + continuity of eigs

Example I: Hadeler

$$T(z) = (e^z - 1)B + z^2 A - \alpha I, \quad A, B \in \mathbb{R}^{8\times 8}$$



Comparison to simplified problem

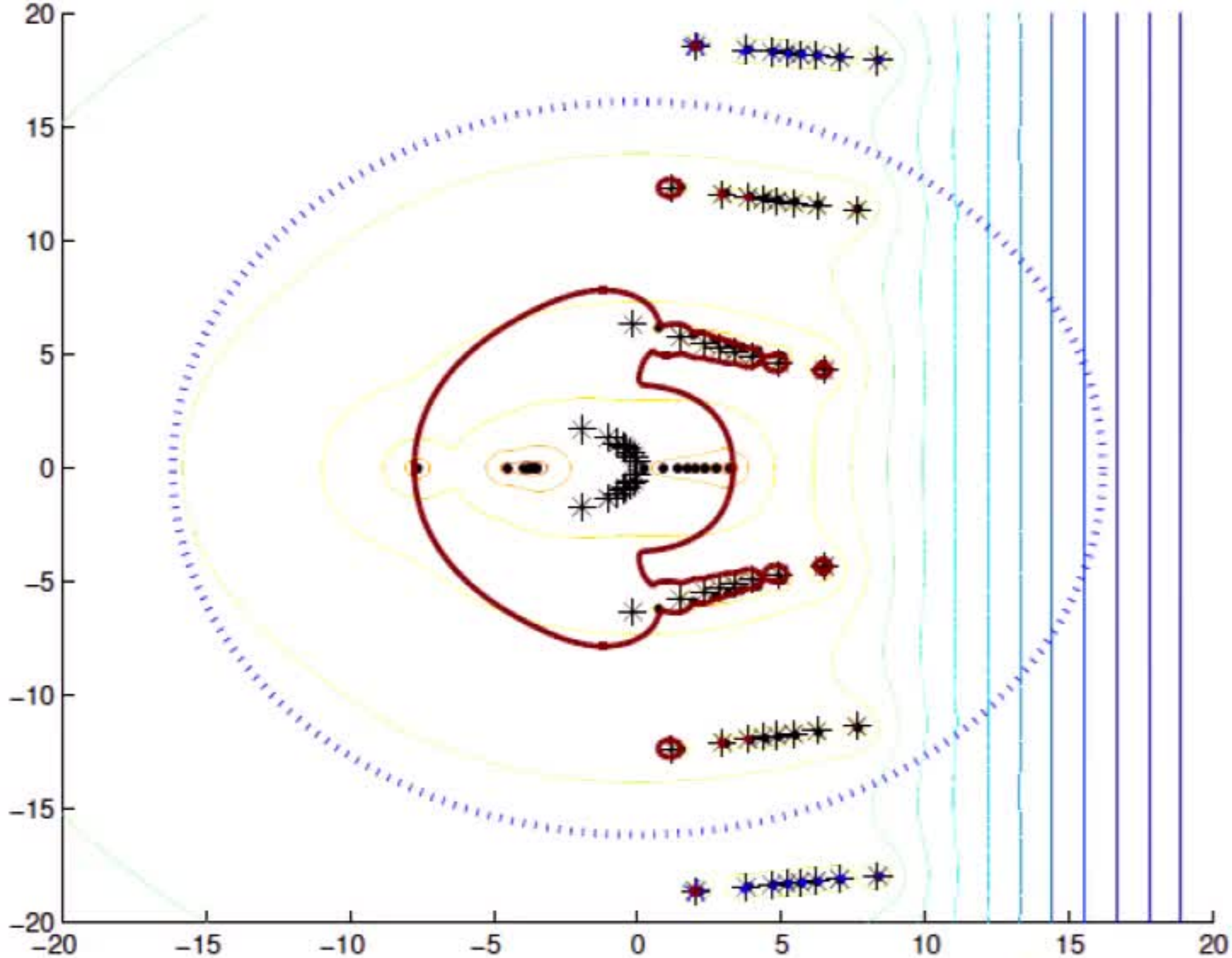
Bauer-Fike idea: apply a similarity!

$$T(z) = (e^z - 1)B + z^2 A - \alpha I$$

$$\begin{aligned}\tilde{T}(z) &= U^T T(z) U \\ &= (e^z - 1)D_B + z^2 I - \alpha E \\ &= D(z) - \alpha E\end{aligned}$$

$$G_i = \{z : |\beta_i(e^z - 1) + z^2| < \rho_i\}.$$

Gershgorin regions



A different comparison

Approximate $e^z - 1$ by a Chebyshev interpolant:

$$T(z) = (e^z - 1)B + z^2 A - \alpha I$$
$$\tilde{T}(z) = q(z)B + z^2 A - \alpha$$

$$T(z) = \tilde{T}(z) + r(z)B$$

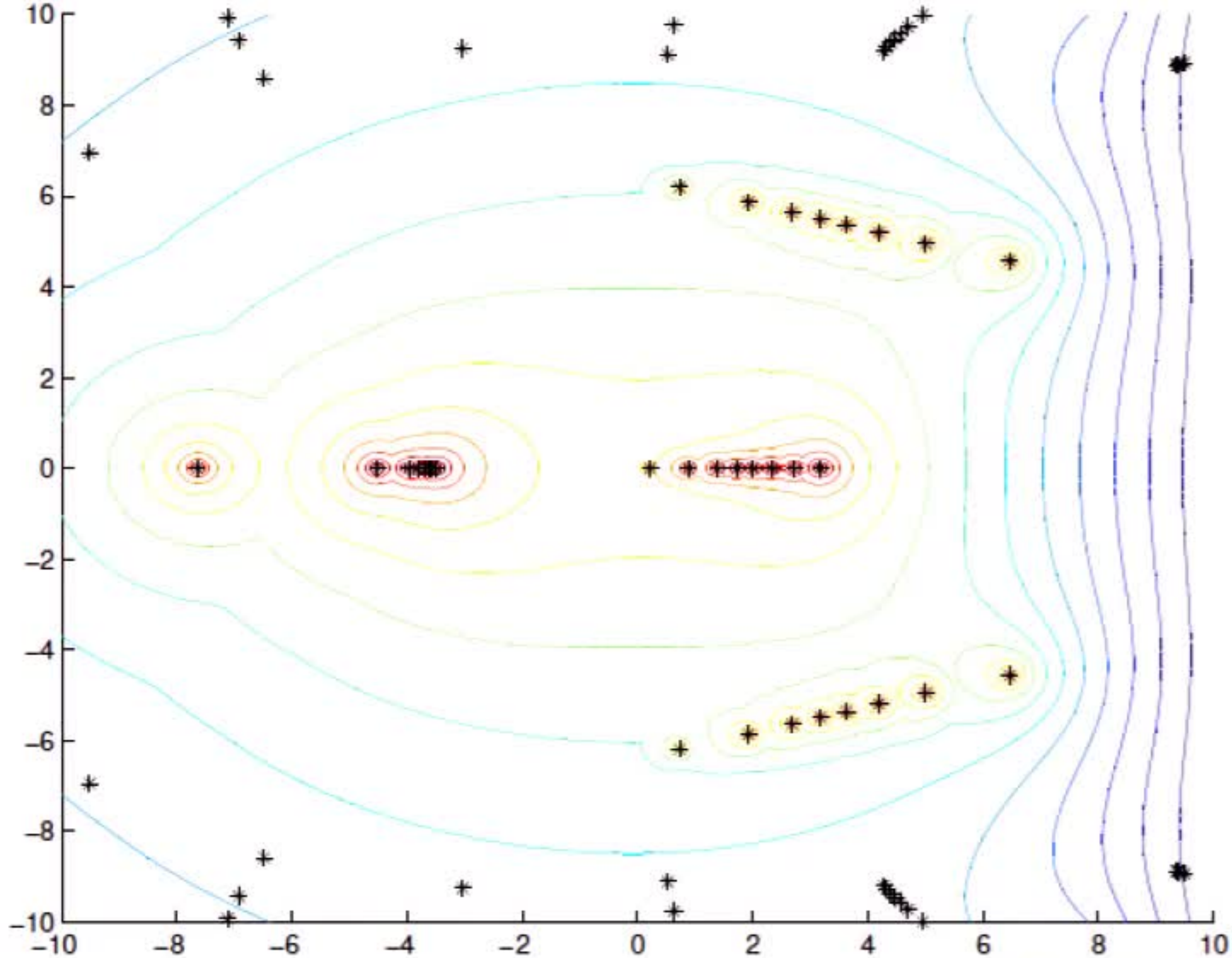
Linearize $\tilde{T}$ and transform both:

$$\tilde{T}(z) \mapsto D_C - zI$$
$$T(z) \mapsto D_C - zI + r(z)E$$

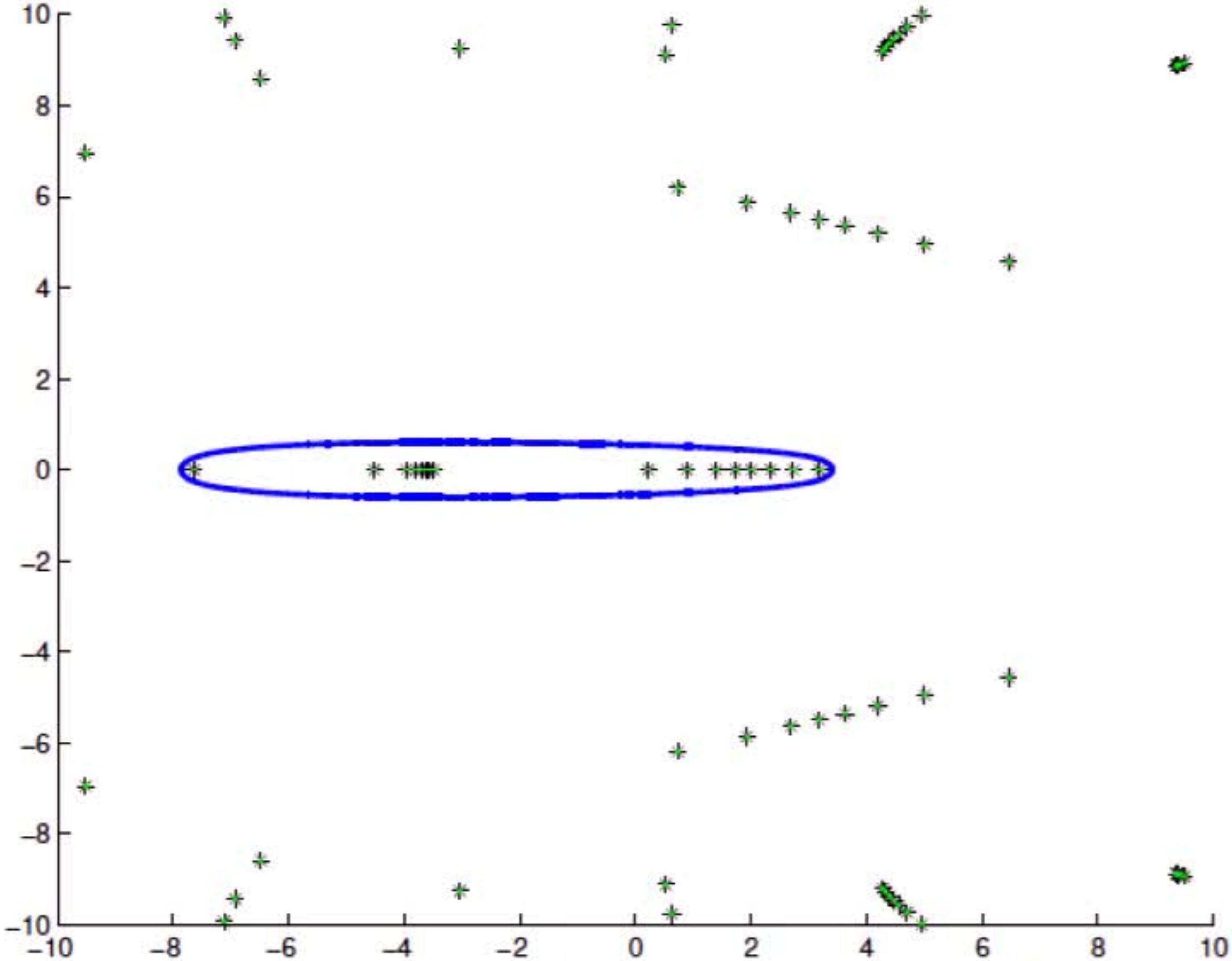
Restrict to $\Omega_\epsilon = \{z : |r(z)| < \epsilon\}$:

$$G_i \subset \hat{G}_i = \{z : |z - \mu_i| < \rho_i \epsilon\}, \qquad \rho_i = \sum_j |e_{ij}|$$

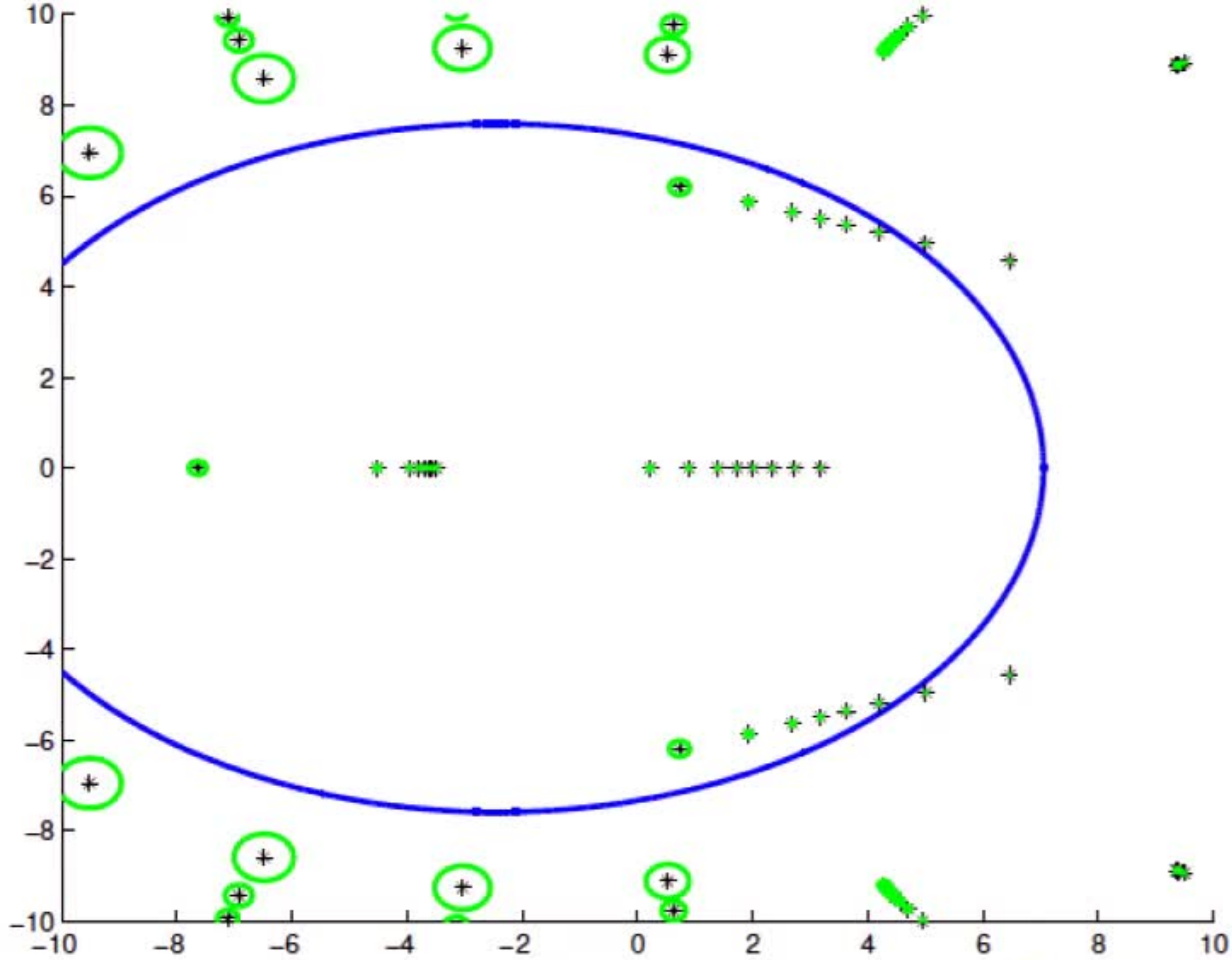
Spectrum of $\tilde{T}$



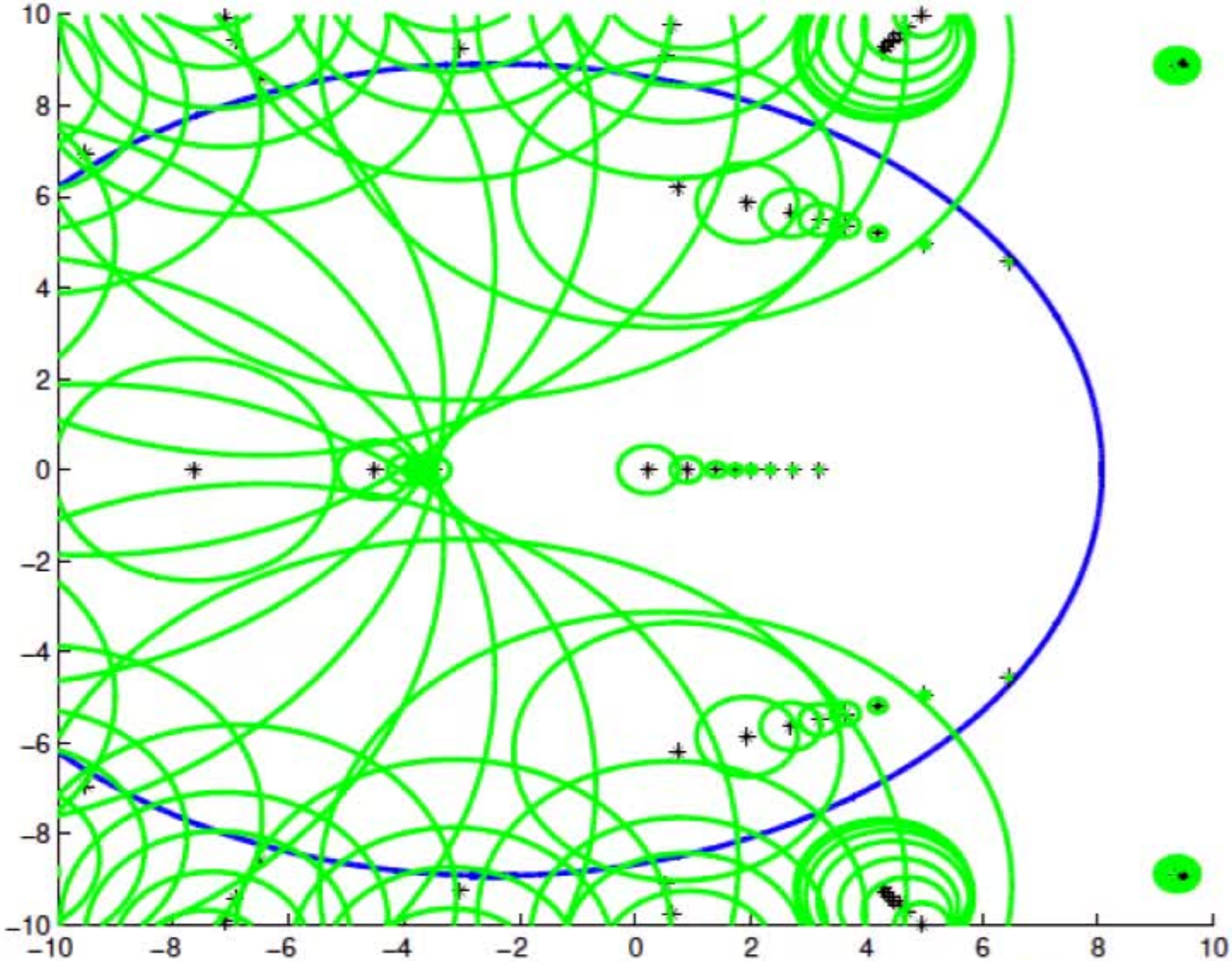
$\hat{G}_i$ for $\epsilon < 10^{-10}$



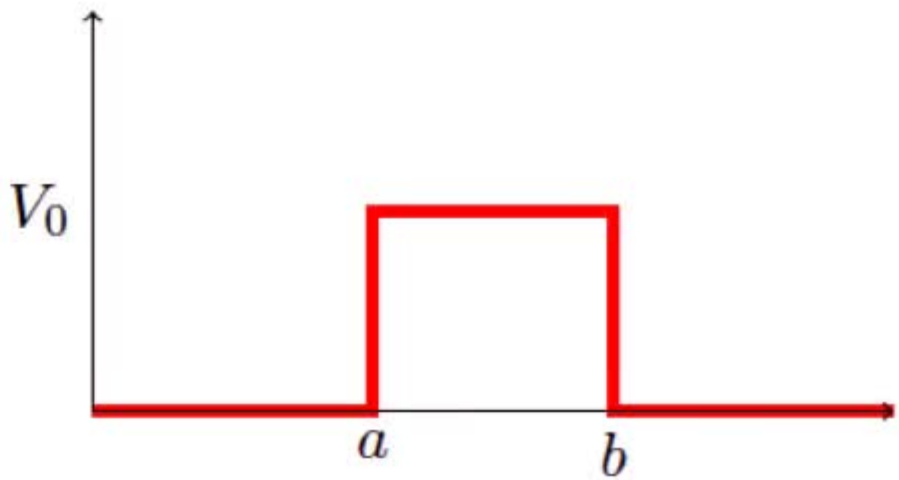
$\hat{G}_i$ for $\epsilon = 0.1$



$\hat{G}_i$ for $\epsilon = 1.6$



Example II: Resonance problem

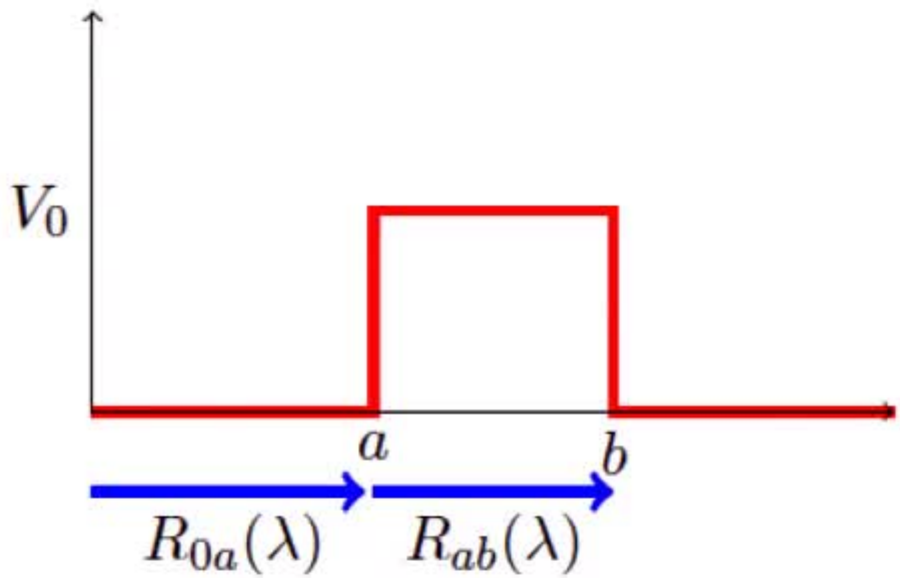


$$\psi(0) = 0$$

$$\left(-\frac{d^2}{dx^2} + V - \lambda\right)\psi = 0 \qquad \text{on } (0,b),$$

$$\psi'(b) = i\sqrt{\lambda}\psi(b),$$

Reduction via shooting



$$\psi(0) = 0,$$

$$R_{0a}(\lambda) \begin{bmatrix} \psi(0) \\ \psi'(0) \end{bmatrix} = \begin{bmatrix} \psi(a) \\ \psi'(a) \end{bmatrix},$$

$$R_{ab}(\lambda) \begin{bmatrix} \psi(b) \\ \psi'(b) \end{bmatrix} = \begin{bmatrix} \psi(b) \\ \psi'(b) \end{bmatrix},$$

$$\psi'(b) = i\sqrt{\lambda}\psi(b)$$

Reduction via shooting

First-order form:

$$\frac{du}{dx} = \begin{bmatrix} 0 & 1 \\ V-\lambda & 0 \end{bmatrix} u, \text{ where } u(x) \equiv \begin{bmatrix} \psi(x) \\ \psi'(x) \end{bmatrix}.$$

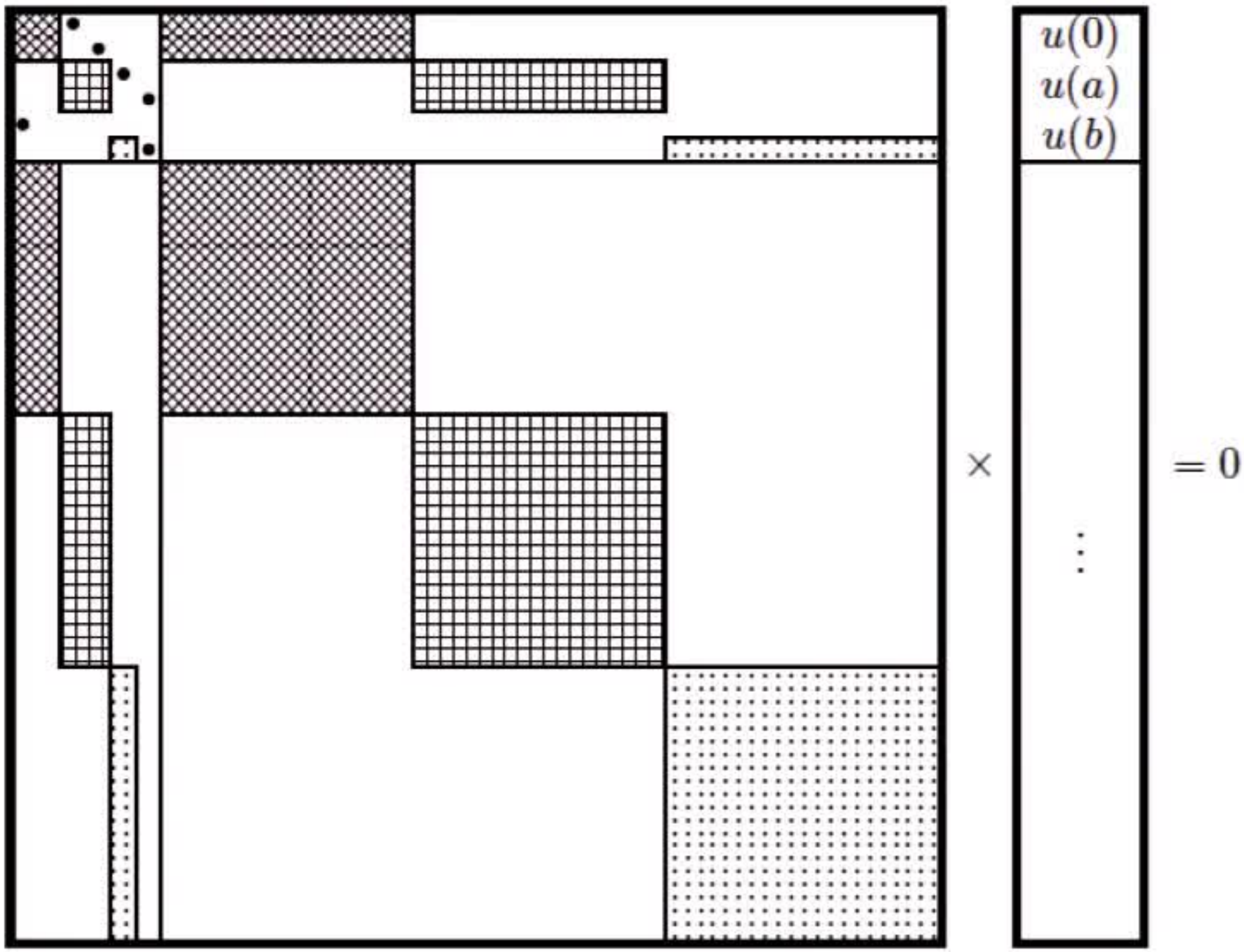
On region (c, d) where V is constant:

$$u(d) = R_{cd}(\lambda)u(c), \quad R_{cd}(\lambda) = \exp\left((d-c) \begin{bmatrix} 0 & 1 \\ V-\lambda & 0 \end{bmatrix} \right)$$

Reduce resonance problem to 6D NEP:

$$T(\lambda)u_{\text{all}} \equiv \begin{bmatrix} R_{0a}(\lambda) & -I & 0 \\ 0 & R_{ab}(\lambda) & -I \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 & 0 \\ -i\sqrt{\lambda} & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u(0) \\ u(a) \\ u(b) \end{bmatrix} = 0.$$

Expansion via rational approximation



Analyzing the expanded system

- $\hat{T}(z)$ is a Schur complement in $K - zM$
  - So $\Lambda(\hat{T})$ is easy to compute.
- Or: think $T(z)$ is a Schur complement in $K - zM + E(z)$
- Compare $\hat{T}(z)$ to $T(z)$ or compare $K - zM + E(z)$ to $K - zM$

Analyzing the expanded system

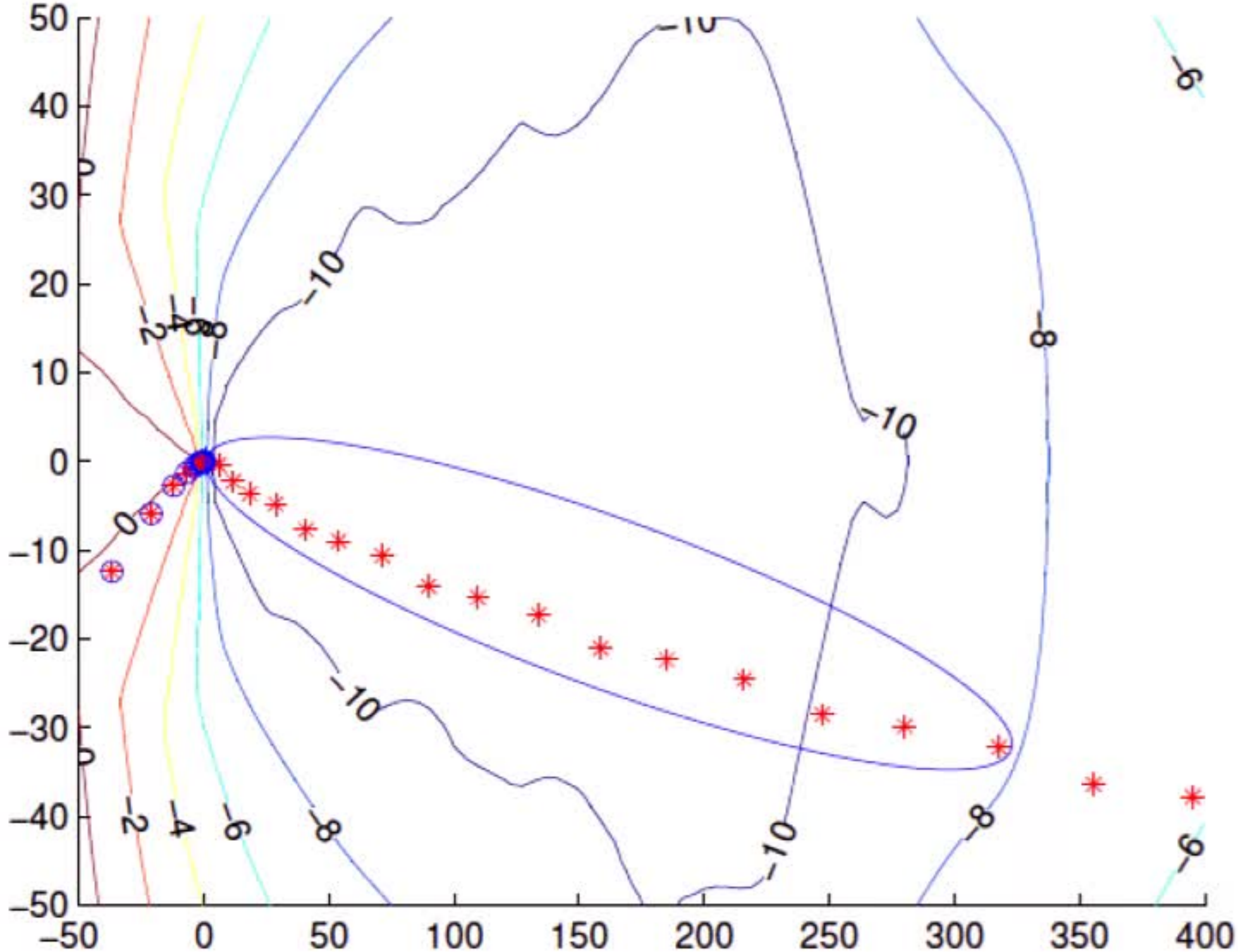
Q: Can we find all eigs in a region *not missing anything*?

Concrete plan ($\epsilon = 10^{-8}$)

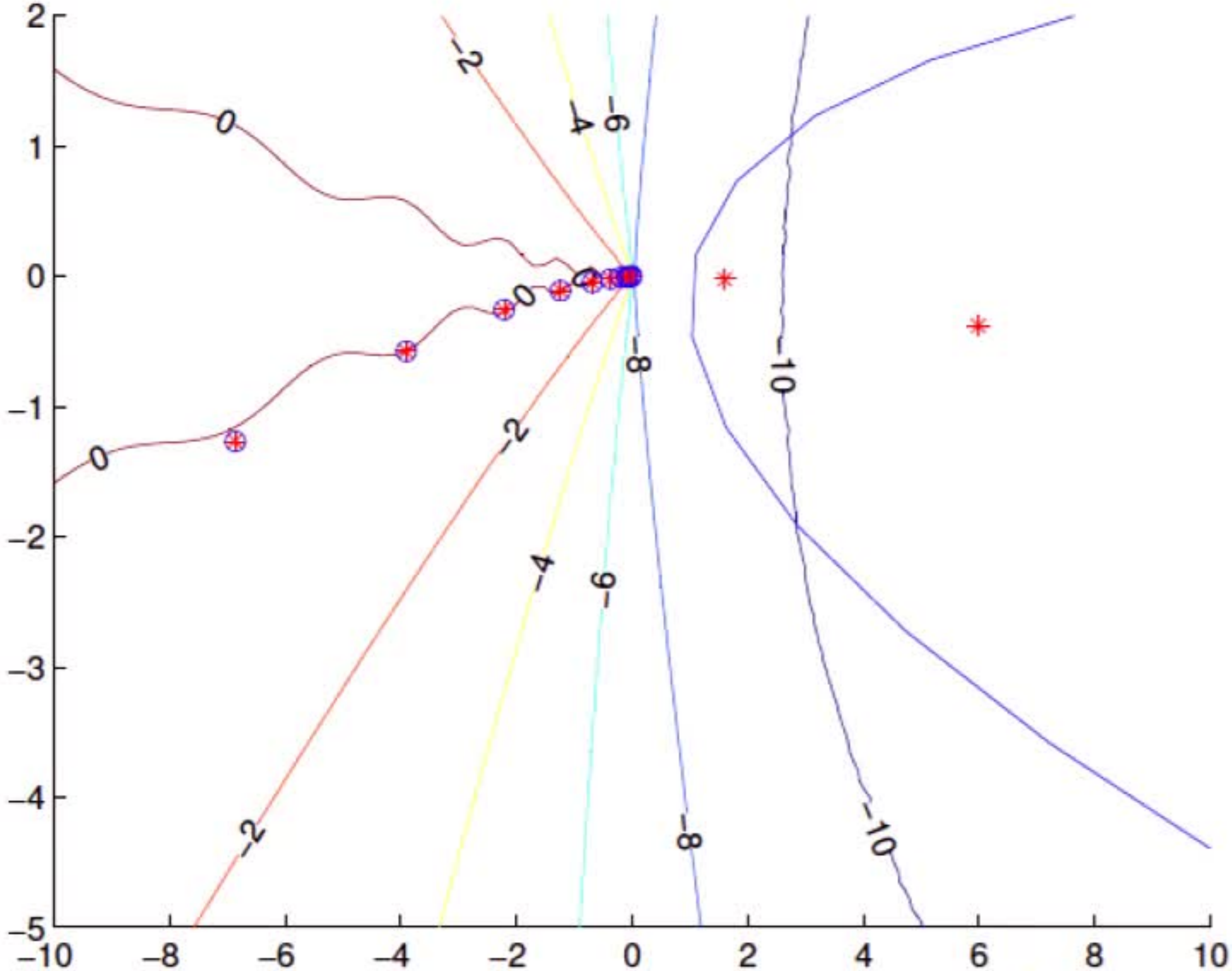
- T = shooting system
- $\hat{T}$ = rational approximation
- Find region D with boundary Γ s.t.
  - $D \subset \Omega_\epsilon$ (i.e. $\|T - \hat{T}\| < \epsilon$ on D)
  - Γ does not intersect $\Lambda_\epsilon(T)$
- $\implies$ Same eigenvalue counts for T, $\hat{T}$
- $\implies$ Eigs of $\hat{T}$ in components of $\Lambda_\epsilon(T)$
  - Converse holds if $D \subset \Omega_{\epsilon/2}$

Can refine eigs of $\hat{T}$ in D via Newton.

Resonance approximation



Resonance approximation



For more

Localization theorems for nonlinear eigenvalues.
David Bindel and Amanda Hood, SIMAX 34(4), 2013

`http://epubs.siam.org/doi/abs/10.1137/130913651`

SIREV version to appear in December issue.