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Tomography with Random and Optimized Sources and Detectors

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Overview

1. Diffuse Optical Tomography and Nonlinear Parametric Inversion
2. A Stochastic Approach Using Simultaneous Random Sources and Detectors
3. Improving the Stochastic Approach Using Optimized Sources and Detectors
4. Conclusions and Future Work

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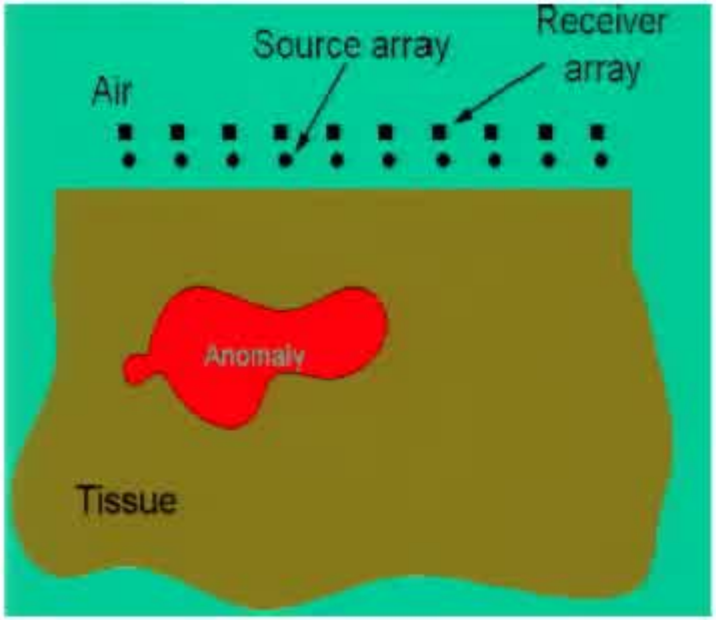
Diffuse Optical Tomography

- Tissue illuminated by near-infrared, frequency modulated light
- Light detected in array(s)
- Tumors have different optical properties than healthy tissue
- Recover images of optical properties, $D(\mathbf{x})$ and $\mu(\mathbf{x})$, in $\nabla \cdot (D(\mathbf{x})\nabla\eta) - (\mu(\mathbf{x}) + \frac{\imath\omega}{\nu}\tilde{I})\eta = b_j$
- Problem is ill-posed and underdetermined, and data is noisy

- Sources 2mm apart: 2500 sources
- Detectors $\approx$ 0.5mm apart. Implies $> O(10^6)$ forward unknowns
- m_s sources and m_ω frequency modulations: solution of $m_s \cdot m_\omega$ discretized PDEs for function evaluation. Additional solves for Jacobian.



Parametric Inversion

- Parameterize D and μ by expansion in compactly supported radial basis functions (CSRBF) and map to near bivalued function using approximate Heavyside function and level set [1]
- Solve for (relatively) modest number of parameters
- This parameterization yields regularized solution
- Assume for simplicity a single frequency. Nonlinear residual:

$$\begin{bmatrix} \mathbf{r}_1(\mathbf{p}) \\ \vdots \\ \mathbf{r}_{m_s}(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} \mathbf{C}^T \mathbf{A}^{-1}(\mathbf{p})\mathbf{b}_1 - \mathbf{d}_1 \\ \vdots \\ \mathbf{C}^T \mathbf{A}^{-1}(\mathbf{p})\mathbf{b}_{m_s} - \mathbf{d}_{m_s} \end{bmatrix}$$

- Nonlinear Least Squares Problem:

$$\min_{\mathbf{p}} \frac{1}{2} ||\mathbf{r}(\mathbf{p})||_2^2$$

[1] Aghassi/Kilmer/Miller'11

Nonlinear Problem

The Gauss-Newton (GN) approximation at $\mathbf{p}$

$$m_{GN}(\mathbf{p}+\delta) \approx \frac{1}{2}\mathbf{r}^T\mathbf{r} + \mathbf{r}^T\mathbf{J}\delta + \frac{1}{2}\delta^T\mathbf{J}^T\mathbf{J}\delta$$

where Jacobian: $\frac{\partial \mathbf{r}(\mathbf{p})}{\partial \mathbf{p}_k} = \left[\begin{array}{ccc} \frac{\partial \mathbf{r}(\mathbf{p})}{\partial \mathbf{p}_1} & \dots & \frac{\partial \mathbf{r}(\mathbf{p})}{\partial \mathbf{p}_{m_p}} \end{array}\right]$,

and $\mathbf{J}_{jk}(\mathbf{p}) = \frac{\partial}{\partial \mathbf{p}_k}(\mathbf{C}^T\mathbf{A}^{-1}(\mathbf{p})\mathbf{b}_j) = \mathbf{C}^T\mathbf{A}^{-1}(\mathbf{p})\frac{\partial}{\partial \mathbf{p}_k}\mathbf{A}(\mathbf{p})\mathbf{A}^{-1}(\mathbf{p})\mathbf{b}_j$

Trust region algorithm with REGularized model Solution[2] (TREGS) solves Nonlinear Least-Squares Problem:

- Ill-conditioned Jacobians
- Gauss-Newton step: Regularized solve for $\mathbf{J}\delta\mathbf{p} \approx -\mathbf{r}$
- TSVD of $\mathbf{J}$ for the large singular values
- Combined with selecting large spectral components in $\mathbf{r}$

[2] EdS/Kilmer'11

Recast as a Stochastic Problem

$$\min_{\mathbf{p}} \|\mathbf{r}(\mathbf{p})\|_2^2 \;=\; \min_{\mathbf{p}} \sum_{j=1}^{m_s} \|\mathbf{C}^T\mathbf{A}^{-1}(\mathbf{p})\mathbf{b}_j - \mathbf{d}_j\|_2^2 = \min_{\mathbf{p}} \|\mathbf{C}^T\mathbf{A}^{-1}(\mathbf{p})\mathbf{B} - \mathbf{D}\|_F^2$$

Matrix Form: $\mathbf{R}(\mathbf{p}) = \mathbf{C}^{\mathsf{T}}\mathbf{A}(\mathbf{p})^{-1}\mathbf{B} - \mathbf{D}$

Stochastic optimization problem (Haber, Chung, Herrmann SIOPT'12):

$$\min_{\mathbf{p}} \mathrm{E}_{\mathbf{w}}\left(\mathbf{w}^T\mathbf{R}(\mathbf{p})^T\mathbf{R}(\mathbf{p})\mathbf{w}\right) \;=\; \min_{\mathbf{p}} \mathrm{trace}\; \mathbf{R}(\mathbf{p})^T\mathbf{R}(\mathbf{p}) = \min_{\mathbf{p}} \|\mathbf{R}(\mathbf{p})\|_F^2$$

Rademacher distribution: $\mathbf{w}_j \in \{-1, +1\}$ i.i.d. with equal probability $(1/2)$

Drastically reduces the number of solves in function evaluation but not in Jacobian.

$$\mathbf{J}_{jk}(\mathbf{p}) = \mathbf{C}^T\mathbf{A}^{-1}(\mathbf{p})\frac{\partial}{\partial \mathbf{p}_k}\mathbf{A}(\mathbf{p})\mathbf{A}^{-1}(\mathbf{p})\mathbf{b}_j$$

Simultaneous Random Sources & Detectors

Similar approach for detectors:

$$\min_{\mathbf{p}} \mathrm{E}\left(\mathbf{v}^T \mathbf{R}(\mathbf{p})\mathbf{w}\right)^2 = \min ||\mathbf{R}(\mathbf{p})||_F^2$$

Columns of $\mathbf{B}$ correspond to m_s sources; rows of $\mathbf{C}^T$ to m_d detectors

Replace columns of $\mathbf{B}$ by a few random linear combinations $\mathbf{Bw}_j$ (SAA)
Replace columns of $\mathbf{C}$ by a few random linear combinations $\mathbf{Cv}_j$

$$\mathbf{V}^T \left(\mathbf{C}^T \mathbf{A}(\mathbf{p})^{-1}\mathbf{B} - \mathbf{D}\right)\mathbf{W} = \mathbf{V}^T \mathbf{R}(\mathbf{p})\mathbf{W}$$

Simultaneous Random Sources & Detectors

In vector form

$$\min_{\mathbf{p}} \|\mathbf{V}^T \mathbf{R}(\mathbf{p}) \mathbf{W}\|_F^2 = \min_{\mathbf{p}} \| \left(\mathbf{W}^T \otimes \mathbf{V}^T \right) \mathbf{r}(\mathbf{p}) \|_2^2$$

With simultaneous sources $\mathbf{W}$ and detectors $\mathbf{V}$
Regularized solve for $\mathbf{J}\delta\mathbf{p} \approx -\mathbf{r}$ and the Gauss-Newton step becomes

$$\left(\mathbf{W}^T \otimes \mathbf{V}^T \right) \mathbf{r}(\mathbf{p} + \delta\mathbf{p}) \approx \left(\mathbf{W}^T \otimes \mathbf{V}^T \right) (\mathbf{r}(\mathbf{p}) + \mathbf{J}\delta\mathbf{p})$$

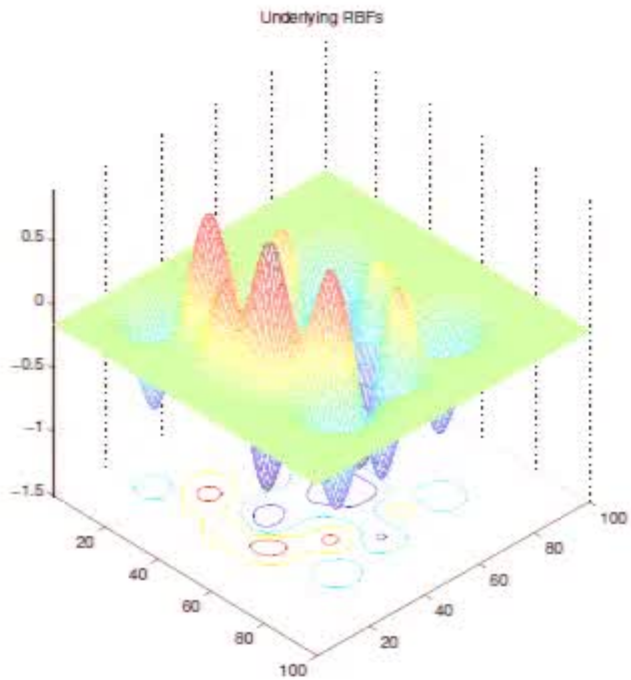
Hence

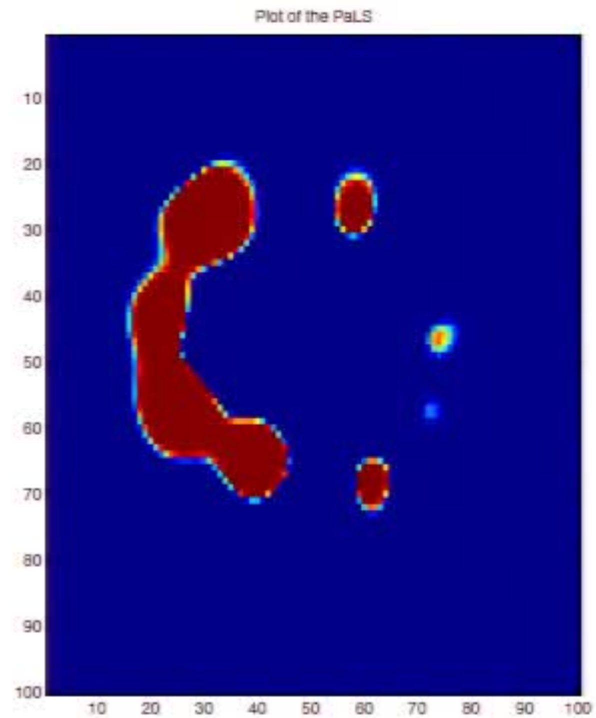
$$(\mathbf{W}^T \otimes \mathbf{V}^T)\mathbf{J}\delta\mathbf{p} \approx -(\mathbf{W}^T \otimes \mathbf{V}^T)\mathbf{r}$$

Use TREGS

A modest number of random sources and detectors is typically enough to get a reasonable solution

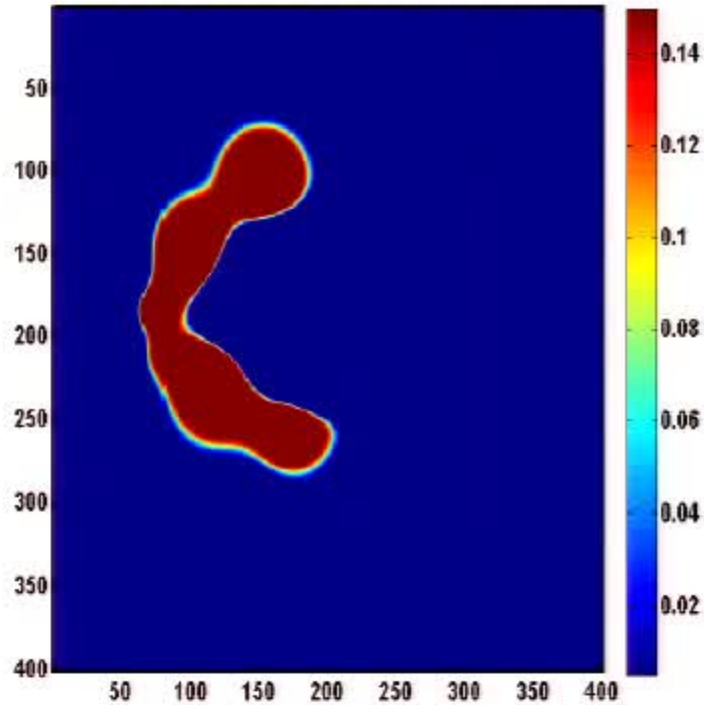
Weighted CSRBFs





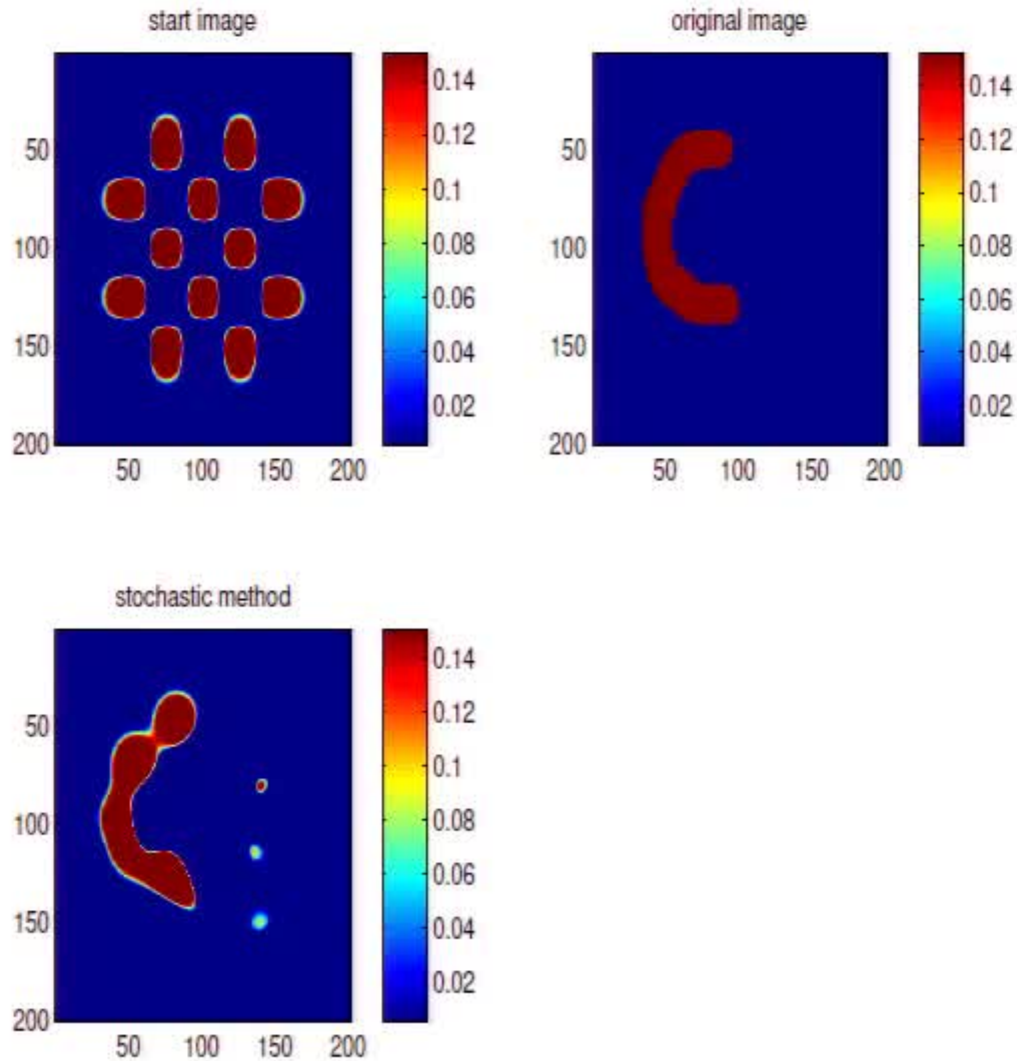
Full Order Model Result

Figure: Image reconstruction results with all sources and all detectors $m_s = m_d = 32$



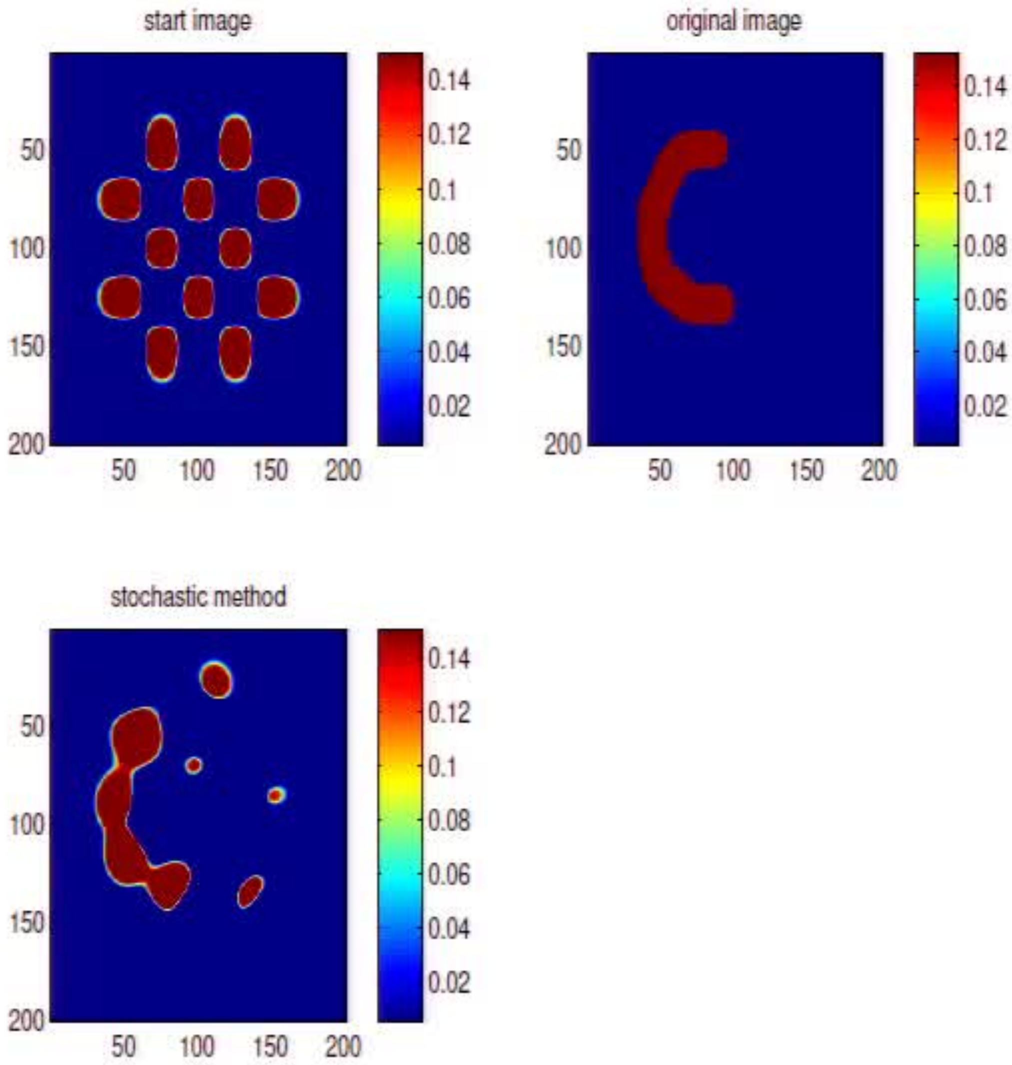
Stochastic Results

Figure: Example 1: Image reconstruction results with $\ell_s = \ell_d = 10$



Stochastic Results

Figure: Example 2: Image reconstruction results with $\ell_s = \ell_d = 10$



How to Improve the Stochastic Approach?

Good localization from random sources and detectors in modest # steps.
Replace few sources and detectors by optimized sources and detectors.

Compute full Jacobian and residual
Replace a few sources/detectors to maximize Frobenius norm

$$\max \|(\mathbf{W}^T \otimes \mathbf{V}^T)\mathbf{J}\|_F^2 \qquad \text{(think spaces not vectors)}$$

TREGS prefers large sin. values (also link with opt. design of experiments)

Drop: $\mathbf{V}^{m_d \times \ell_d}$ by $\tilde{\mathbf{V}}^{m_d \times \ell_{d-1}}$ such that $\mathcal{R}(\tilde{\mathbf{V}}) \subset \mathcal{R}(\mathbf{V})$

Find unitary $\mathbf{\Gamma}^{\ell_d \times \ell_{d-1}}$ such that $\tilde{\mathbf{V}} = \mathbf{V}\mathbf{\Gamma}^{\ell_d \times \ell_{d-1}}$ and

$$\max_{\mathbf{\Gamma}} \|(\mathbf{W}^T \otimes \tilde{\mathbf{V}}^T)\mathbf{J}\|_F^2$$

Drop Random Sources & Detectors (optimal fashion)

For simplicity consider a single simultaneous source and ℓ_d detectors

$$\left[\mathbf{w}_1^T \otimes \mathbf{V}^T\right] \mathbf{J} = \mathbf{V}^T \widehat{\mathbf{J}}$$

Goal: $\max ||\tilde{\mathbf{V}}^T \widehat{\mathbf{J}}||_F^2$ i.e. $\max(\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_{\ell_d - 1}^2)$

$$\max ||\tilde{\mathbf{V}}^T \widehat{\mathbf{J}}||_F^2 = \max ||\mathbf{\Gamma}^T \mathbf{V}^T \widehat{\mathbf{J}}||_F^2$$

$$\widehat{\mathbf{V}}^T \widehat{\mathbf{J}} = \mathbf{\Phi \Omega \Psi}^T (\mathrm{SVD})$$

Keep the largest $(\ell_d - 1)$ left singular vectors of $\mathbf{V}^T \widehat{\mathbf{J}}$

$$\mathbf{\Gamma} = [\phi_1 \ \ \phi_2 \ \ \cdots \ \ \phi_{\ell_d - 1}]$$

We can also remove one (or more) simultaneous sources

Adding Optimized Sources & Detectors

Find unit $\mathbf{q} \perp \mathbf{V}$ that maximizes

$$\left\| \begin{bmatrix} \mathbf{w}_1^T \otimes \mathbf{V}^T \\ \mathbf{w}_1^T \otimes \mathbf{q}^T \\ \mathbf{w}_2^T \otimes \mathbf{V}^T \\ \mathbf{w}_2^T \otimes \mathbf{q}^T \\ \vdots \\ \mathbf{w}_{\ell_s} \otimes \mathbf{q}^T \end{bmatrix} \mathbf{J} \right\|_F^2 = \left\| \begin{bmatrix} \mathbf{V}^T \\ \mathbf{q}^T \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{J}}_1 & \dots & \widehat{\mathbf{J}}_{\ell_s} \end{bmatrix} \right\|_F^2$$

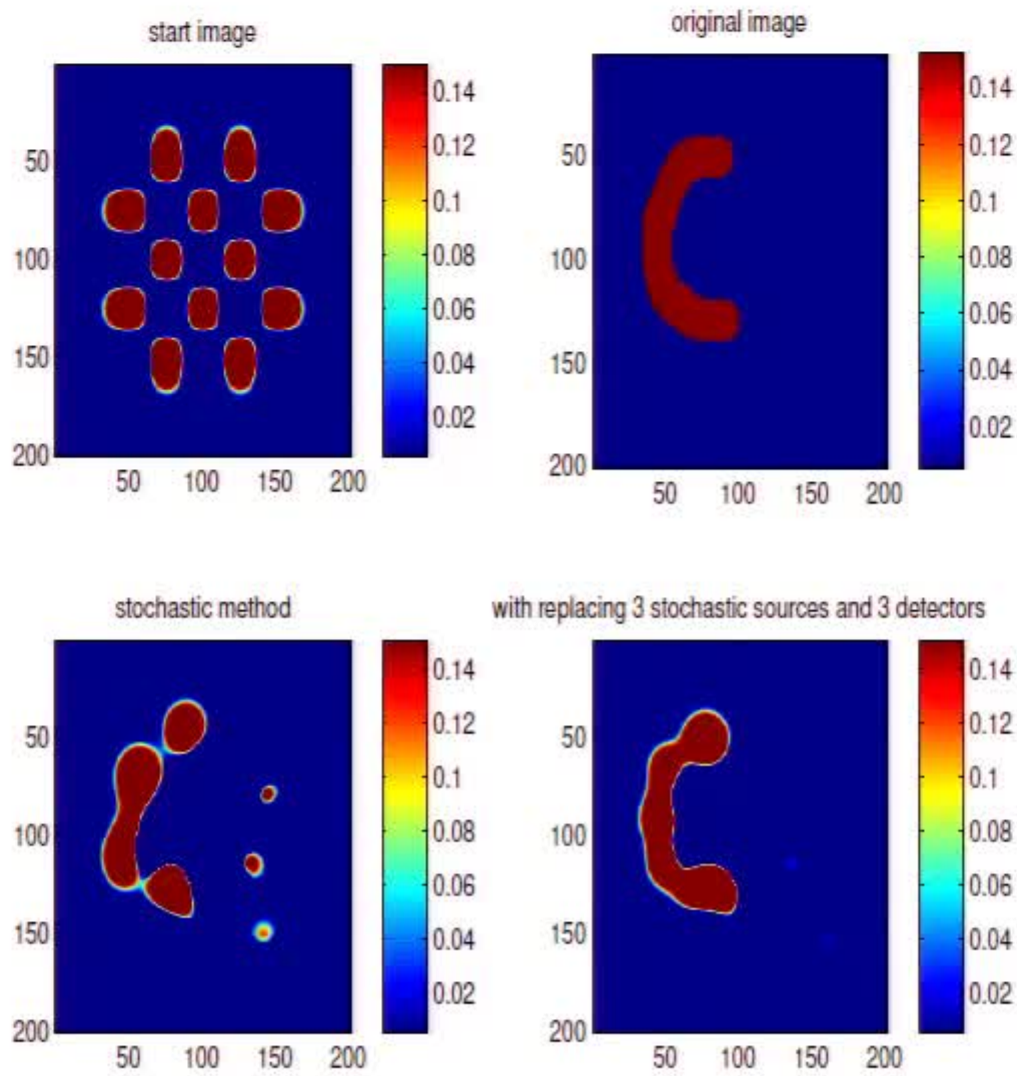
New part: $\left\| \mathbf{q}^T \left[\widehat{\mathbf{J}}_1 \ \dots \ \widehat{\mathbf{J}}_{\ell_s} \right] \right\|_2^2$ and taking $\mathbf{q} = \mathbf{V}_c \mathbf{y}$ $(\mathbf{V}_c \perp \mathbf{V})$ gives

$$\max_{\mathbf{y}} \left\| \mathbf{y}^T \left[\mathbf{V}_c^T \widehat{\mathbf{J}}_1 \ \dots \ \mathbf{V}_c^T \widehat{\mathbf{J}}_{\ell_s} \right] \right\|_2^2$$

Standard problem, can be solved by SVD (also for multiple new det.s)

Examples

Figure: Example 1: Image reconstruction results with $\ell_s = \ell_d = 10$



Results in Iterations

Average number of iterations, Function evaluations, Jacobian evaluations

	Iter	Fevals	Jevals	tol	PDE solves
Only Stochastic Method	12	13	6	3e-7	190
+ Replacing 1 src/1 det	10	9	6	3e-10	214
+ Replacing 2 src/2 det	10	9	5	3e-10	204
Only Stochastic Method	(149)	(150)	(119)	(3e-10)	2690
All srcs/ All dets	16	17	9	3e-10	1120