Weak vs. strong solutions to complete fluid systems

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Motivation

Compressible Euler system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0$$

Energy (entropy) inequality

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) \mathbf{u} + p(\varrho) \mathbf{u} \right] \leq 0$$

$$P(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2} \, \mathrm{d}z$$

Result of Chiodaroli, DeLellis, Kreml [2013]

There exist Lipschitz initial data such that the compressible Euler system admits infinitely many admissible (entropy) weak solutions.

Riemann problem

Riemann initial data

$$\varrho(0, x_1, \dots, x_N) = \begin{cases} \varrho_L \text{ if } x_1 \leq 0 \\ \\ \varrho_R \text{ if } x_1 > 0 \end{cases}$$

$$u^1(0, x_1, \dots, x_N) = \begin{cases} u_L^1 \text{ if } x_1 \leq 0 \\ \\ u_R^1 \text{ if } x_1 > 0 \end{cases}$$

$$u^k(0, x_1, \dots, x_N) = 0, \ \ k = 2, \dots, N$$

Wild solutions

The wild solutions emanate from the 1D Riemann data but the velocity admits non-zero second component

Shock free solutions

Geometry, pressure

$$\Omega = (-a, a) \times \mathcal{T}^1 \text{ (periodic in } x_2)$$

$$p(0) = 0,\ p'(r) > 0 \text{ for } r > 0,\ p \text{ convex}$$

Theorem EF, O.Kreml [2014]

Let $\tilde{\varrho} = \tilde{\varrho}(x_1/t)$, $\tilde{\mathbf{u}} = [\tilde{u}^1(x_1/t), 0]$ be the self-similar solution to the Riemann problem consisting of rarefaction waves (locally Lipschitz for $t > 0$) and such that

$$\mathrm{ess}\inf_{(0,t)\times R} \tilde{\varrho} > 0.$$

Let $[\varrho, \mathbf{u}]$ be a bounded admissible weak solution such that

$$\varrho \geq 0 \text{ a.a. in } (0, T) \times \Omega.$$

Then

$$\varrho \equiv \tilde{\varrho},\ \mathbf{u} \equiv \tilde{\mathbf{u}} \text{ in } (0, T) \times \Omega.$$

Method of relative energy

Relative energy

$$\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r) \right) \, \mathrm{d}x$$

Relative energy inequality

$$\left[\mathcal{E}\left(\varrho, \mathbf{u} \middle| \tilde{\varrho}, \tilde{\mathbf{u}}\right) \right]_{t=0}^{t=\tau}$$

$$\leq - \int_0^\tau \int_\Omega \left[\varrho|u^1 - \tilde{u}^1|^2 + p(\varrho) - p'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - p(\tilde{\varrho}) \right] \partial_{x_1} \tilde{u}^1 \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \text{“other terms”}$$

Full Euler system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \varrho\vartheta = 0$$

Energy balance

$$\partial_t \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + c_v \varrho\vartheta \right] + \mathrm{div}_x \left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + c_v \varrho\vartheta + \varrho\vartheta \right) \mathbf{u} \right] = 0$$

Entropy inequality

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) \geq 0, \ \ s = s(\varrho, \vartheta) \equiv \log \left(\frac{\vartheta^{c_v}}{\varrho} \right)$$

Riemann problem

Geometry

$$\Omega = R^1 \times \mathcal{T}^1, \text{ where } \mathcal{T}^1 \equiv [0,1]_{\{0,1\}} \text{ is the ``flat'' sphere}$$

Initial data

$$\varrho(0, x_1, x_2) = R_0(x_1), \ R_0 = \begin{cases} R_L \text{ for } x_1 \leq 0 \\ R_R \text{ for } x_1 > 0 \end{cases}$$

$$\vartheta(0, x_1, x_2) = \Theta_0(x_1), \ \Theta_0 = \begin{cases} \Theta_L \text{ for } x_1 \leq 0 \\ \Theta_R \text{ for } x_1 > 0 \end{cases}$$

$$u^1(0, x_1, x_2) = U_0(x_1), \ U_0 = \begin{cases} U_L \text{ for } x_1 \leq 0, \\ U_R \text{ for } x_1 > 0 \end{cases} \quad u^2(0, x_1, x_2) = 0.$$

Shock free Riemann solutions

Solution class

$$0 < \varrho \leq \overline{\varrho},\ 0 < \vartheta \leq \overline{\vartheta},\ |s(\varrho, \vartheta)| < \overline{s},\ |\mathbf{u}| < \overline{u}$$

Isentropic solutions

- the entropy S is *constant* in $[0, T] \times \Omega$
- $\Theta = R^{\frac{1}{c_v}} \exp\left(\frac{1}{c_v} S\right)$
- $R = R(t, x_1)$ and $U = U(t, x_1)$ represent a rarefaction wave solution of the 1-D *isentropic* system

$$\partial_t R + \partial_{x_1}(RU) = 0,\ R\left[\partial_t U + U \partial_{x_1} U\right] + \exp\left(\frac{1}{c_v} S\right) \partial_{x_1} R^{\frac{c_v+1}{c_v}} = 0$$

Uniqueness

Theorem, EF, O.Kreml, A.Vasseur [2014]

Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Euler system in $(0, T) \times \Omega$ originating from the Riemann data. Suppose in addition that the Riemann data give rise to the shock-free solution $[R, \Theta, U]$ of the 1-D Riemann problem.
Then
$$\varrho = R, \ \vartheta = \Theta, \ \mathbf{u} = [U, 0] \text{ a.a. in } (0, T) \times \Omega$$

Relative energy

Relative energy (entropy) functional

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \,\middle|\, \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}\right)$$

$$= \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u} - \tilde{\mathbf{u}}|^2 + H_{\tilde{\vartheta}}(\varrho, \vartheta) - \frac{\partial H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta})}{\partial \varrho}(\varrho - \tilde{\varrho}) - H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta}) \right] \, \mathrm{d}x$$

Ballistic free energy

$$H_{\tilde{\vartheta}}(\varrho, \vartheta) = \varrho \left(c_v \vartheta - \tilde{\vartheta} s(\varrho, \vartheta) \right).$$

$$\varrho \mapsto H_{\tilde{\vartheta}}(\varrho, \tilde{\vartheta}) \text{ convex}$$

$$\vartheta \mapsto H_{\tilde{\vartheta}}(\varrho, \vartheta) \begin{cases} \text{decreasing for } \vartheta < \tilde{\vartheta} \\ \text{increasing for } \vartheta > \tilde{\vartheta} \end{cases}$$

Relative energy inequality, dissipative solutions

Relative energy inequality

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}} \right) \right]_{t=0}^{t=\tau} \leq \int_0^\tau \mathcal{R} \left(\varrho, \vartheta, \mathbf{u}, \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}} \right) \ \mathrm{d}t$$

Test functions

$$\tilde{\varrho} > 0,\ \ \tilde{\vartheta} > 0, \ \left\{ \begin{array}{c} \tilde{\varrho} = R_L, \tilde{\vartheta} = \Theta_L,\ \ \tilde{u}^1 = U_L,\ \ \tilde{u}^2 = 0 \text{ if } x_1 < -A, \\ \\ \tilde{\varrho} = R_R, \tilde{\vartheta} = \Theta_R,\ \ \tilde{u}^1 = U_R,\ \ \tilde{u}^2 = 0 \text{ if } x_1 > A \end{array} \right\}$$

Remainder

Remainder in the relative energy inequality

$$\mathcal{R}\left(\varrho, \vartheta, \mathbf{u}, \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}\right)$$

$$= \int_\Omega \Big[\varrho(\tilde{\mathbf{u}} - \mathbf{u}) \cdot \partial_t \tilde{\mathbf{u}} + \varrho(\tilde{\mathbf{u}} - \mathbf{u}) \otimes \mathbf{u} : \nabla_x \tilde{\mathbf{u}} + (\tilde{\varrho}\tilde{\vartheta} - \varrho\vartheta)\mathrm{div}_x \tilde{\mathbf{u}} \Big] \ \mathrm{d}x$$

$$- \int_\Omega \Big[\varrho\Big(s(\varrho, \vartheta) - s(\tilde{\varrho}, \tilde{\vartheta}) \Big) \partial_t \tilde{\vartheta} + \varrho\Big(s(\varrho, \vartheta) - s(\tilde{\varrho}, \tilde{\vartheta}) \Big) \mathbf{u} \cdot \nabla_x \tilde{\vartheta} \Big] \ \mathrm{d}x$$

$$+ \int_\Omega \left[\left(1 - \frac{\varrho}{\tilde{\varrho}} \right) \partial_t (\tilde{\varrho}\tilde{\vartheta}) + \left(\tilde{u} - \frac{\varrho}{\tilde{\varrho}} \right) \mathbf{u} \cdot \nabla_x (\tilde{\varrho}\tilde{\vartheta}) \right] \ \mathrm{d}x$$

Robustness of 1D viscosity solutions

Navier-Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}),$$

Pressure, viscous stress

$$p(\varrho) = a\varrho^\gamma, \ a > 0, \ \gamma > 1,$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{N} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0.$$

1D problem

1D Navier-Stokes system

$$\partial_t R + \partial_y (RV) = 0,$$

$$\partial_t (RV) + \partial_y (RV^2) + \partial_y p(R) = \left[2\mu \left(1 - \frac{1}{N} \right) + \eta \right] \partial^2_{y,y} V.$$

Stability of 1D solutions - hypotheses

Theorem EF, Y.Sun [2015]

$$\gamma > \frac{N}{2},\ q > \max\{2, \gamma'\},\ \frac{1}{\gamma} + \frac{1}{\gamma'} = 1 \text{ if } N = 2$$

$$q > \max\left\{3, \frac{6\gamma}{5\gamma - 6}\right\} \text{ if } N = 3$$

Let $[R, V]$ be a (strong) solution of the one-dimensional problem, with the initial data belonging to the class

$$R_0 \in W^{1,q}(0,1),\ R_0 > 0,\ V_0 \in W_0^{1,q}(0,1)$$

Let $[\varrho, \mathbf{u}]$ be a finite energy weak solution to the Navier-Stokes system in

$$(0, T) \times \Omega,\ \Omega = (0,1) \times \mathcal{T}^{N-1},$$

with the initial data

$$\varrho_0 \in L^\infty(\Omega),\ \varrho_0 > 0,\ \mathbf{u}_0 \in L^2(\Omega; R^3).$$

Stability of 1D solutions - conclusion

Conclusion

Then

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{V}|^2 + P(\varrho) - P'(R)(\varrho - R) - P(R) \right] (\tau,\cdot)\,\mathrm{d}x$$

$$\leq c(T)\int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0-\mathbf{V}_0|^2 + P(\varrho_0) - P'(R_0)(\varrho_0 - R_0) - P(R_0) \right]\,\mathrm{d}x$$

for a.a. $\tau \in (0,T)$,

$$P(\varrho) = \frac{a}{\gamma - 1}\varrho^\gamma.$$

Full Navier-Stokes-Fourier system

Navier-Stokes-Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) - \lambda \mathbf{u}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

$$\sigma = \frac{1}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right),$$

Slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Constitutive relations - scaling

Pressure

$$p(\varrho,\vartheta) = p_M(\varrho,\vartheta) + p_R(\varrho,\vartheta),\ p_M = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right),\ p_R(\varrho,\vartheta) = \frac{\boxed{a}}{3}\vartheta^4$$

Viscous stress

$$\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) = \boxed{\nu}\left[\mu(\vartheta)\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x\mathbf{u}\mathbb{I}\right]$$

Heat flux

$$\mathbf{q} = -\boxed{\omega}\kappa(\vartheta)\nabla_x\vartheta$$

Brinkman type "damping"

$$D = -\boxed{\lambda}\mathbf{u}$$

Target system

Full Euler system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p_M(\varrho, \vartheta) = 0$$

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e_M(\varrho, \vartheta) \right)$$

$$+\mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e_M(\varrho, \vartheta) \right) \mathbf{u} + p_M(\varrho, \vartheta) \mathbf{u} \right] = 0$$

Slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Dissipative solutions

Relative energy

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \Big| r, \Theta, \mathbf{U}\right)$$

$$= \int_\Omega \left[\frac{1}{2}\varrho |\mathbf{u} - \mathbf{U}|^2 + H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r, \Theta) \right] \ \mathrm{d}x$$

Relative energy inequality

$$\left[\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \Big| r, \Theta, \mathbf{U}\right) \right]_{t=0}^{t=\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \lambda \int_0^\tau \int_\Omega |\mathbf{u}|^2 \ \mathrm{d}x \ \mathrm{d}t \leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \mathrm{d}t$$

$$r, \Theta > 0, \ \ \mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Dissipative solutions - remainder

Remainder

$$\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) = \int_\Omega \varrho(\mathbf{u} - \mathbf{U}) \cdot \nabla_x \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x$$

$$+ \int_\Omega \left[\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta + \lambda \mathbf{u} \cdot \mathbf{U} \right] \ \mathrm{d}x$$

$$+ \int_\Omega \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \Big(\mathbf{U} - \mathbf{u} \Big) \cdot \nabla_x \Theta \ \mathrm{d}x$$

$$+ \int_\Omega \varrho \Big(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x - \int_\Omega p(\varrho, \vartheta) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x$$

$$- \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{U} \cdot \nabla_x \Theta \right) \ \mathrm{d}x$$

$$+ \int_\Omega \left(\left(1 - \frac{\varrho}{r} \right) \partial_t p(r, \Theta) - \frac{\varrho}{r} \mathbf{u} \cdot \nabla_x p(r, \Theta) \right) \ \mathrm{d}x$$

Vanishing dissipation limit

Theorem EF [2015]

Let $[\varrho_E, \vartheta_E, \mathbf{u}_E]$ be the classical solution of the Euler system in a time interval $(0, T)$, with the initial data $[\varrho_{0,E}, \vartheta_{0,E}, \mathbf{u}_{0,E}]$. Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak (dissipative) solution of the Navier-Stokes-Fourier system, with the initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$.
Then

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \Big| \varrho_E, \vartheta_E, \mathbf{u}_E\right)(\tau)$$

$$\leq c_1(T, \mathrm{data}) \mathcal{E}\left(\varrho_0, \vartheta_0, \mathbf{u}_0 \Big| \varrho_{0,E}, \vartheta_{0,E}, \mathbf{u}_{0,E}\right)$$

$$+ c_2(T, \mathrm{data}) \max \left\{ a, \nu, \omega, \lambda, \frac{\nu}{\sqrt{a}}, \frac{\omega}{a}, \left(\frac{a}{\sqrt{\nu^3 \lambda}} \right)^{1/3} \right\}$$

for a.a. $\tau \in (0, T)$.