Two-velocity formulation of the degenerate Navier-Stokes equations

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SIAM PDEs 2015, Scottsdale, Arizona, December 7th, 2015

Equations of motion of compressible fluid

We consider a set of conservation laws:

$$\begin{cases} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \mathrm{div}\, \mathbf{S} + \nabla p = \mathbf{0}, \\ \partial_t(\varrho E) + \mathrm{div}(\varrho E \mathbf{u}) + \mathrm{div}\, \mathbf{Q} + \mathrm{div}\,(p\mathbf{u}) + \mathrm{div}\,(\mathbf{S}\mathbf{u}) = 0. \end{cases}$$

- The unknowns: $\varrho, \mathbf{u}$ and ϑ.
- The pressure: $p = R\varrho\vartheta$.
- The energy: $\varrho E = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e, \qquad e = c_v\vartheta$.
- The heat flux: $\mathbf{Q} = -k(\varrho, \vartheta)\nabla\vartheta$.
- The stress tensor: $\mathbf{S} = -2\mu(\varrho)\mathbf{D}(\mathbf{u}) - \lambda(\varrho)\,\mathrm{div}\,\mathbf{u}\mathbf{I}$, $\lambda(\varrho) = 2(\mu'(\varrho)\varrho - \mu(\varrho))$.

Rescaled System

We rescale the system using the Mach number $\varepsilon = \frac{u_{ref}}{c_{ref}}$

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ \partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbf{S} + \frac{1}{\varepsilon^2} \nabla p = \mathbf{0}, \\ \partial_t (\varrho e) + \operatorname{div}(\varrho e \mathbf{u}) - \operatorname{div}(k \nabla \vartheta) - \varepsilon^2 \mathbf{S} : \nabla \mathbf{u} + p \operatorname{div} \mathbf{u} = 0. \end{cases}$$

Denoting $\gamma = 1 + R/c_v$ and using definition of p and e the last equation writes

$$\frac{1}{\gamma - 1} (\partial_t p + \operatorname{div}(p \mathbf{u})) - \operatorname{div} \mathbf{k} \nabla \vartheta - \varepsilon^2 \mathbf{S} : \nabla \mathbf{u} + p \operatorname{div} \mathbf{u} = 0.$$

For ε small, we formally justify that

$$p = P(t) + \Pi(t, x) \varepsilon^2 + o(\varepsilon^2).$$

Low Mach Number System

For $P(t) = P_0 = const$, the system reads

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ \partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2 \operatorname{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho) \operatorname{div} \mathbf{u}) + \nabla \Pi = \mathbf{0}, \\ \gamma P_0 \operatorname{div} \mathbf{u} = (\gamma - 1) \operatorname{div}(k(\varrho, \vartheta) \nabla \vartheta), \\ P_0 = R \varrho \vartheta, \end{cases}$$

or equivalently

$$[LM] \begin{cases} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ \partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2 \operatorname{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho) \operatorname{div} \mathbf{u}) + \nabla \Pi = \mathbf{0}, \\ \operatorname{div} \mathbf{u} = -2\kappa \Delta \varphi(\varrho) \end{cases}$$

with φ – an increasing function of ϱ and $\kappa > 0$.

T. Alazard. Low Mach number limit of the full Navier–Stokes equations, ARMA 2006.

Literature

- Local strong solutions
  - Beirão Da Veiga '82, Secchi '82, Danchin & Liao '12 (in critical Besov spaces).
- Global in time solutions
  - Kazhikov & Smagulov '77 (for modified convective term and small κ existence of generalized solution which is unique in 2d), Lions '98 (weak solutions for $\varphi = -1/\varrho$ and small perturbation of a constant density), Secchi '88 (2d, unique solution for small κ) Danchin & Liao '12 (small perturbation of constant density and small initial velocity).
- No smallness assumption
  - Bresch, Essoufi & Sy '07 global weak solutionds for special relation

$$\varphi'(s) = \mu'(s)/s, \quad \kappa = 1,$$

Cai, Liao & Sun '12 (uniqueness in 2d), Liao '14 (global strong solution in 2d, critical Besov spaces).

Existence for $0 < \kappa < 1$

T1: Let $0 < T < \infty$, $\Omega = \mathbb{T}^d$, $\varphi'(\varrho) = \mu'(\varrho)/\varrho$, $0 < \kappa < 1$. For the initial conditions satisfying

$$\sqrt{(1-\kappa)\kappa}\varrho_0 \in H^1(\Omega), \quad 0 < r \le \varrho_0 \le R < \infty, \quad \mathbf{u_0} + \kappa\nabla\varphi(\rho_0) \in H,$$

$\mu(\varrho)$ such that $\mu(\varrho) \in C^1([r,R])$, $\mu'(\varrho) > 0$, $\mu \ge c > 0$ on $[r,R]$, and

$$\left(\frac{1-d}{d}\mu(\varrho) + \mu'(\varrho)\varrho\right) \ge c > 0.$$

There exists a global in time weak solution* to [LM].

T2: For $\kappa \to 0$ this solution converges to the weak solution of the non-homogenous incompressible N-S equations; for $\kappa \to 1$ (and $\kappa\varrho_\kappa^0 \in H^1(\Omega)$) it converges to the weak solutions of the Kazhikhov-Smagulov system.

D. Bresch, V. Giovangigli, E.Z., JMPA 2015.

General case

For $T < \infty$, $\Omega = \mathbb{T}^3$ and the general low Mach number system

$$[LMG] \begin{cases} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ \partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2 \operatorname{div}(\mu(\varrho) D(\mathbf{u})) + \nabla \Pi_1 = \mathbf{0}, \\ \operatorname{div} \mathbf{u} = -\Delta \tilde{\varphi}(\varrho), \end{cases}$$

with $\tilde{\varphi}'(s) = \tilde{\mu}'(s)/s$, we have:

T3: Under previous assumptions on the data and, for $\mu(\cdot) \in C^1([r, R])$, $\mu'(\cdot) > 0$, $\mu \geq c > 0$ on $[r, R]$ and $\tilde{\varphi}(\cdot) \in C^1([r, R])$ and $\mu(\varrho)$, $\tilde{\mu}(\varrho)$ related by

$$c \leq \min_{\varrho \in [r,R]} \left(\mu(\varrho) - \tilde{\mu}(\varrho)\right),$$

$$c' \leq \tilde{\mu}'(\varrho)\varrho + \frac{1-d}{d} \tilde{\mu}(\varrho),$$

$$\max_{\varrho \in [r,R]} \frac{(\mu(\varrho) - \tilde{\mu}(\varrho) - \xi \tilde{\mu}(\varrho))^2}{2\left(\mu(\varrho) - \tilde{\mu}(\varrho)\right)} \leq \xi \min_{\varrho \in [r,R]} \left(\tilde{\mu}'(\varrho)\varrho + \frac{1-d}{d} \tilde{\mu}(\varrho) \right).$$

for some positive constants c, ξ. There exists global weak solution to [LMG].

D. Bresch, V. Giovangigli, E.Z., JMPA 2015.

Conclusions

- For $\mu(\varrho) = \varrho$ (i.e. $\varphi(\varrho) = \log \varrho$) we recover the pollutant model.
- For $\mu(\varrho) = \log(\varrho)$ (i.e. $\varphi(\varrho) = -1/\varrho$) we recover the combustion model.
- For $\mu(\varrho) = \varrho, \quad \tilde{\mu}(\varrho) = \log \varrho \quad i.e. \ \ \kappa\varphi(\varrho) = -1/\varrho$. There exists a non-empty interval $[\tilde{r}, \tilde{R}]$ such that if

$$0 < \tilde{r} \leq \varrho_0 \leq \tilde{R} < 0,$$

then the weak solution to [LM] exists globally in time. This corresponds to the dense gas approximation.

S. Chapman & T.G. Cowling: The mathematical theory of non-uniform gases, Cambridge Univ. Press, 1970.

Reformulation of the [LM] system

We introduce the solenoidal vector field $\mathbf{w}$ such that

$$\begin{aligned} \mathbf{w} &= \mathbf{u} + 2\kappa\nabla\varphi(\varrho), \\ \operatorname{div}\mathbf{w} &= 0. \end{aligned}$$

Then, system [LM] can be rewritten as

$$\begin{gathered} \partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) = 0, \\ \partial_t\left(\varrho\mathbf{w}\right) + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{w}) - 2\operatorname{div}(\mu(\varrho)D(\mathbf{u})) + 2\kappa\operatorname{div}(\mu(\varrho)\nabla^t\mathbf{u}) + \nabla\Pi_{\mathbf{1}} = \mathbf{0}, \end{gathered}$$

where

$$\Pi_{\mathbf{1}} = \Pi + 2(\mu'(\varrho)\varrho - \mu(\varrho))\operatorname{div}\mathbf{u} - \lambda(\varrho)\operatorname{div}\mathbf{u}.$$

Equivalently, the "momentum" equation can be rewritten as

$$\partial_t\left(\varrho\mathbf{w}\right) + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{w}) - 2(1-\kappa)\operatorname{div}(\mu(\varrho)D(\mathbf{u})) - 2\kappa\operatorname{div}(\mu(\varrho)A(\mathbf{u})) + \nabla\Pi_{\mathbf{1}} = \mathbf{0},$$

where $D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla^t\mathbf{u})$ and $A(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} - \nabla^t\mathbf{u})$.

Generalzed energy estimate

Multiplying momentum equation by $\mathbf{w}$ and integrating by parts:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \varrho|\mathbf{w}|^2\,\mathrm{d}x + 2(1-\kappa)\int_\Omega \mu(\varrho)|D(\mathbf{u})|^2\,\mathrm{d}x + 2\kappa\int_\Omega \mu(\varrho)|A(\mathbf{u})|^2\,\mathrm{d}x + 4(1-\kappa)\kappa\int_\Omega \mu(\varrho)\nabla\mathbf{u} : \nabla^2\varphi(\varrho)\,\mathrm{d}x = 0, \tag{1}$$

but when $\kappa \neq 1$ the last term does not vanish. From the continuity equation we obtain

$$\partial_t\left(\varrho\nabla\varphi(\varrho)\right) + \mathrm{div}(\varrho\mathbf{u}\otimes\nabla\varphi(\varrho)) + \mathrm{div}(\mu(\varrho)\nabla^t\mathbf{u}) + \nabla\left(((\mu'(\varrho)\varrho - \mu(\varrho))\,\mathrm{div}\,\mathbf{u}\right) = \mathbf{0}$$

and the corresponding energy is

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \varrho\frac{|\nabla\varphi|^2}{2}\,\mathrm{d}x - \int_\Omega(\mu'(\varrho)\varrho - \mu(\varrho))\,\mathrm{div}\,\mathbf{u}\Delta\varphi\,\mathrm{d}x - \int_\Omega \mu(\varrho)\nabla\mathbf{u} : \nabla^2\varphi(\varrho)\,\mathrm{d}x = 0. \tag{2}$$

Multiplying (2) by $4(1-\kappa)\kappa$ and adding it to (1):

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \varrho\left(\frac{|\mathbf{w}|^2}{2} + (1-\kappa)\kappa\frac{|2\nabla\varphi|^2}{2}\right)\mathrm{d}x + 2(1-\kappa)\int_\Omega \mu(\varrho)|D(\mathbf{u})|^2\,\mathrm{d}x + 2\kappa\int_\Omega \mu(\varrho)|A(\mathbf{u})|^2\,\mathrm{d}x + 2(1-\kappa)\kappa^2\int_\Omega(\mu'(\varrho)\varrho - \mu(\varrho))|\Delta\varphi|^2\,\mathrm{d}x = 0.$$

Extension to the compressible brotropic system

The generalized κ- entropy inequality:

$$\sup_{t\in[0,T]}\Big[\int_\Omega \varrho\left(\frac{|\mathbf{u}+2\kappa\nabla\varphi(\varrho)|^2}{2}+(1-\kappa)\kappa\frac{|2\nabla\varphi(\varrho)|^2}{2}\right)(t)\,\mathrm{d}x+\int_\Omega \varrho e(\varrho)(t)\,\mathrm{d}x\Big]$$
$$+2\kappa\int_0^T\int_\Omega \mu(\varrho)|A(\mathbf{u})|^2\,\mathrm{d}x\,\mathrm{d}s+2\kappa\int_0^T\int_\Omega \frac{\mu'(\varrho)p'(\varrho)}{\varrho}|\nabla\varrho|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$+2(1-\kappa)\int_0^T\Big[\int_\Omega \mu(\varrho)|D(\mathbf{u})|^2\,\mathrm{d}x+\int_\Omega(\mu'(\varrho)\varrho-\mu(\varrho))|\operatorname{div}\mathbf{u}|^2\,\mathrm{d}x\Big]\,\mathrm{d}s$$
$$\le\int_\Omega \varrho\left(\frac{|\mathbf{u}+2\kappa\nabla\varphi(\varrho)|^2}{2}+(1-\kappa)\kappa\frac{|2\nabla\varphi(\varrho)|^2}{2}\right)(0)\,\mathrm{d}x+\int_\Omega \varrho_0 e(\varrho_0)\,\mathrm{d}x$$

where we have introduced $\frac{\varrho^2\mathrm{d}e(\varrho)}{\mathrm{d}\varrho}=p(\varrho)$.

- For $\kappa=1$ this is the so called Bresch-Desjardins inequality.
- In general for $0<\kappa<1$ mixing parameter

$$\int_\Omega \varrho\left(\frac{|\mathbf{u}+2\kappa\nabla\varphi(\varrho)|^2}{2}+(1-\kappa)\kappa\frac{|2\nabla\varphi(\varrho)|^2}{2}\right)(t)\,\mathrm{d}x$$
$$=\int_\Omega \varrho\left((1-\kappa)\frac{|\mathbf{u}|^2}{2}+\kappa\frac{|\mathbf{u}+2\nabla\varphi(\varrho)|^2}{2}\right)(t)\,\mathrm{d}x.$$

S.M. Shugrin. Two-velocity hydrodynamics and thermodynamics, J. Applied Mech. and Tech. Physics. 1994.

Possible modifications for $\lambda(\varrho) = 2(\varrho\mu'(\varrho) - \mu(\varrho))$

1. Compressible N-S with singular pressure:

$$[NS_P] \left\{ \begin{array}{l} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div}(2\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho) \operatorname{div} \mathbf{u}) + \nabla p(\varrho) = 0, \end{array} \right.$$

where the pressure p is singular close to zero density

$$p'(\varrho) = \left\{ \begin{array}{ll} c_1 \varrho^{-\gamma^- - 1} & \text{for } \varrho \leq \varrho^*, \\ c_2 \varrho^{\gamma^+ - 1} & \text{for } \varrho > \varrho^* \end{array} \right.$$

such that

$$\gamma^+ > 1, \quad \gamma^- > \frac{2n(3m-2)}{4m-3} - 1$$

for

$$m > 3/4, \quad 2/3 < n < \frac{\gamma^- + 1}{2},$$

and

$$\text{for all} \quad s < \varrho^*, \quad \mu(s) \geq c_0 s^n \quad \text{and} \quad 3\lambda(s) + 2\mu(s) \geq s^n,$$

$$\text{for all} \quad s \geq \varrho^*, \quad c_1 s^m \leq \mu(s) \leq \frac{s^m}{c_1} \quad \text{and} \quad c_1 s^m \leq 3\lambda(s) + 2\mu(s) \leq \frac{s^m}{c_1}.$$

2. Compressible N-S with a drag term:

$$[NS_D]\left\{\begin{array}{l}\partial_t\varrho + \mathrm{div}\,(\varrho\mathbf{u}) = 0,\\ \partial_t\,(\varrho\mathbf{u}) + \mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \mathrm{div}\,(2\mu(\varrho)D(\mathbf{u})) - \nabla(\lambda(\varrho)\,\mathrm{div}\ \mathbf{u}) + r_{\mathbf{1}}\varrho|\mathbf{u}|\mathbf{u} + \nabla p(\varrho) = 0,\end{array}\right.$$

with viscosity coefficients satisfying

$$\mu'(\varrho) \geq \nu, \qquad \mu(0) \geq 0,$$

$$|\lambda'(\varrho)| \leq \frac{1}{\nu}\mu'(\varrho), \qquad \nu\mu(\varrho) \leq 2\mu(\varrho) + 3\lambda(\varrho) \leq \frac{1}{\nu}\mu(\varrho).$$

Moreover

$$\mu(\varrho) \leq C\varrho^{\mathbf{2}/\mathbf{3}+\mathbf{1}/(\mathbf{3}\eta)}, \text{ with } \eta \in (1/\gamma, 1) \text{ when } \varrho \geq 1.$$

A. Mellet and A. Vasseur. On the barotropic compressible Navier-Stokes equations. Comm. PDE, 2007.

A pair of functions $(\varrho, \mathbf{u})$ is a weak solution to $[\mathrm{NS}_P]$ or $[\mathrm{NS}_D]$ if it satisfies the weak formulation of the continuity and the momentum equations and the κ-entropy estimate.

D. Bresch, B. Desjardins, E. Z., JMPA 2015.

Construction of solutions

We introduce augmented system with two velocities $\mathbf{w} = \mathbf{u} + 2\kappa\nabla\varphi$ and $\mathbf{v} = 2\nabla\varphi(\varrho)$:

$$
\begin{aligned}
&\partial_t \varrho + \operatorname{div}(\varrho[\mathbf{w}]_\delta) - 2\kappa \operatorname{div}\left([\mu'(\varrho)]_\alpha \nabla\varrho\right) = 0,\\
&\partial_t(\varrho\mathbf{w}) + \operatorname{div}((\varrho[\mathbf{w}]_\delta - 2\kappa[\mu'(\varrho)]_\alpha\nabla\varrho)\otimes\mathbf{w}) - \nabla\left[\left(\lambda(\varrho) - 2\kappa(\mu'(\varrho)\varrho - \mu(\varrho))\right)\operatorname{div}(\mathbf{w}-\kappa\mathbf{v})\right]\\
&\qquad - 2(1-\kappa)\operatorname{div}(\mu(\varrho)D(\mathbf{w})) - 2\kappa\operatorname{div}(\mu(\varrho)A(\mathbf{w})) + \varepsilon\Delta^{2s}\mathbf{w} - \varepsilon\operatorname{div}((1+|\nabla\mathbf{w}|^2)\nabla\mathbf{w}) + \nabla p(\varrho)\\
&\quad = -2\kappa(1-\kappa)\operatorname{div}(\mu(\varrho)\nabla\mathbf{v}),\\
&\partial_t(\varrho\mathbf{v}) + \operatorname{div}((\varrho[\mathbf{w}]_\delta - 2\kappa[\mu'(\varrho)]_\alpha\nabla\varrho)\otimes\mathbf{v}) - 2\kappa\operatorname{div}(\mu(\varrho)\nabla\mathbf{v}) + 2\nabla\left[(\mu'(\varrho)\varrho - \mu(\varrho))\operatorname{div}(\mathbf{w}-\kappa\mathbf{v})\right]\\
&\quad = -2\operatorname{div}(\mu(\varrho)\nabla^t\mathbf{w}).
\end{aligned}
$$

with the following κ-entropy equality

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \varrho\left(\frac{|\mathbf{w}|^2}{2} + (1-\kappa)\kappa\frac{|\mathbf{v}|^2}{2}\right)\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \varrho e(\varrho)\,\mathrm{d}x + 2(1-\kappa)\int_\Omega \mu(\varrho)|D(\mathbf{w}) - \kappa\nabla\mathbf{v}|^2\,\mathrm{d}x\\
&\quad + 2(1-\kappa)\int_\Omega (\mu'(\varrho)\varrho - \mu(\varrho))\,(\operatorname{div}\,\mathbf{w} - \kappa\operatorname{div}\,\mathbf{v})^2\,\mathrm{d}x + 2\kappa\int_\Omega \mu(\varrho)|A(\mathbf{w})|^2\,\mathrm{d}x\\
&\quad + 2\kappa\int_\Omega \frac{\mu'(\varrho)p'(\varrho)}{\varrho}|\nabla\varrho|^2\,\mathrm{d}x + \varepsilon\int_\Omega\left(|\Delta^s\mathbf{w}|^2 + (1+|\nabla\mathbf{w}|^2)|\nabla\mathbf{w}|^2\right)\,\mathrm{d}x\\
&\quad = -\int_\Omega p(\varrho)\operatorname{div}\left([\mathbf{w}]_\delta - \mathbf{w}\right)\,\mathrm{d}x.
\end{aligned}
$$

Generalizations

- The κ-entropy equality may be seen as a nonlinear version of the so-called *hypocoercivity property* which is strongly used in the framework of strong solutions

  C. Villani. Hypocoercivity, Mem. Amer. Math. Soc. 2009.

- The two-velocity structure allows to introduce the generalised κ-temperature. The obtained system does not coincide with the usual N-S-Fourier system, but is consistent with the low Mach number limit

  D. Bresch, B. Desjardins, E. Z., JMPA 2015.

- Relative κ-entropy and its application to the weak-strong uniqueness and studies of singular limits

  D. Bresch, P. Noble and J.-P. Vila, preprint 2015.

Thank you for your attention !