Optimal shape and location of sensors or actuators in PDE models

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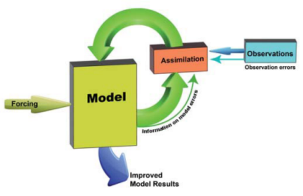
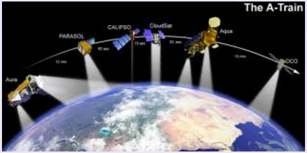
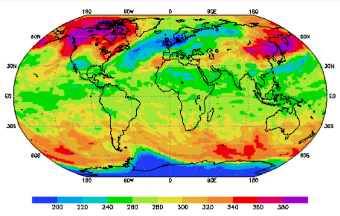
SIAM Conference on Analysis of Partial Differential Equations, 2015

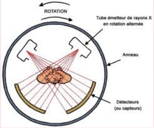


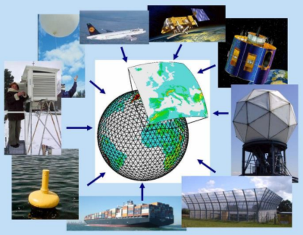


What is the best shape and placement of sensors?

- Reduce the cost of instruments.
- Maximize the efficiency of reconstruction and estimations.



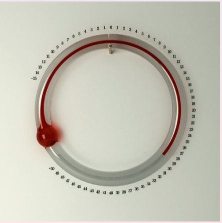




The observed system may be described by:

- wave equation $\boxed{\partial_{tt} y = \triangle y}$
  or
  Schrödinger equation $\boxed{i\partial_t y = \triangle y}$
- general parabolic equations $\boxed{\partial_t y = Ay}$ (e.g., heat or Stokes equations)

in some domain Ω, with either Dirichlet, Neumann, mixed, or Robin boundary conditions



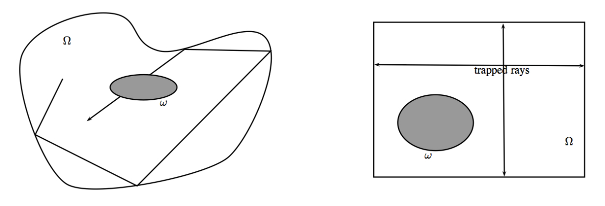
For instance, when dealing with the heat equation:

What is the optimal shape and placement of a thermometer?

Waves propagating in a cavity:

$$\partial_{tt} y - \Delta y = 0$$
$$y(t,\cdot)_{|\partial\Omega} = 0$$

Observable
$y(t,\cdot)_{|\omega}$



Observability inequality

The observability constant $C_T(\omega)$ is the largest nonnegative constant such that

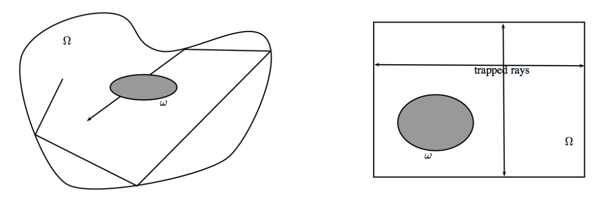
$$\forall (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \qquad C_T(\omega) \|(y^0, y^1)\|^2_{L^2 \times H^{-1}} \leq \int_0^T \int_\omega |y(t,x)|^2 \, dx \, dt$$

The system is said observable on $[0, T]$ if $C_T(\omega) > 0$ (otherwise, $C_T(\omega) = 0$).

Waves propagating in a cavity:

$$\partial_{tt} y - \Delta y = 0$$
$$y(t,\cdot)_{|\partial\Omega} = 0$$

Observable $y(t,\cdot)_{|\omega}$



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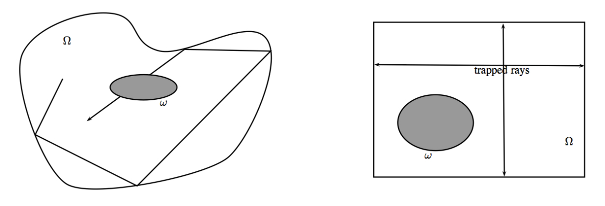
Bardos-Lebeau-Rauch (1992): Observability holds if the pair (ω, T) satisfies the Geometric Control Condition (GCC) in Ω:

Every ray of geometrical optics that propagates in Ω and is reflected on its boundary $\partial\Omega$ intersects ω in time less than T.

Waves propagating in a cavity:

$$\partial_{tt} y - \Delta y = 0$$
$$y(t,\cdot)_{|\partial\Omega} = 0$$

Observable $y(t,\cdot)_{|\omega}$



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Q: What is the "best possible" subdomain ω of fixed given measure? (say, $|\omega| = L|\Omega|$ with $0 < L < 1$)

N.B.: we want to optimize not only the placement but also the shape of ω, over *all possible measurable subsets*.
(they do not have a prescribed shape, they are not necessarily BV, etc)

Related problems

1) What is the "best domain" for achieving HUM optimal control?

$$y_{tt} - \Delta y = \chi_\omega u$$

2) What is the "best domain" domain for stabilization (with localized damping)?

$$y_{tt} - \Delta y = -k\chi_\omega y_t$$

Existing works by

- P. Hébrard, A. Henrot: theoretical and numerical results in 1D for optimal stabilization.
- A. Münch, P. Pedregal, F. Periago: numerical investigations (fixed initial data).
- S. Cox, P. Freitas, F. Fahroo, K. Ito, ...: variational formulations and numerics.
- M.I. Frecker, C.S. Kubrusly, H. Malebranche, S. Kumar, J.H. Seinfeld, ...: numerical investigations over a finite number of possible initial data.
- M. Demetriou, K. Morris, S.L. Padula, O. Sigmund, M. Van de Wal, ...: actuator placement (predefined set of possible candidates), Riccati approaches.
- A. Armaou, M. Demetriou, K. Chen, C. Rowley, K. Morris, S. Yang, W. Kang, S. King, L. Xu, ...: H_2 optimization, frequency methods, LQ criteria, Gramian approaches.
- ...

The model

Observability inequality

$$\forall y \text{ solution} \qquad C_T(\omega)\|(y(0,\cdot),\partial_t y(0,\cdot))\|^2_{L^2\times H^{-1}} \leq \int_0^T\int_\omega |y(t,x)|^2\,dx\,dt$$

Let $L\in(0,1)$ and $T>0$ fixed.

It is **a priori** natural to model the problem as:
$$\boxed{\sup_{\substack{\omega\subset\Omega\\ |\omega|=L|\Omega|}} C_T(\omega)}$$

with

$$C_T(\omega)=\inf\left\{\frac{\int_0^T\int_\omega |y(t,x)|^2\,dx\,dt}{\|(y(0,\cdot),\partial_t y(0,\cdot))\|^2_{L^2\times H^{-1}}}\quad\Big|\quad (y(0,\cdot),\partial_t y(0,\cdot))\in L^2(\Omega)\times H^{-1}(\Omega)\setminus\{(0,0)\}\right\}$$

BUT...

The model

Observability inequality

$$\forall y \text{ solution} \qquad C_T(\omega)\|(y(0,\cdot),\partial_t y(0,\cdot))\|^2_{L^2\times H^{-1}} \leq \int_0^T \int_\omega |y(t,x)|^2\,dx\,dt$$

Let $L \in (0,1)$ and $T > 0$ fixed.

It is **a priori** natural to model the problem as: $\boxed{\sup_{\substack{\omega\subset\Omega \\ |\omega|=L|\Omega|}} C_T(\omega)}$

BUT:

1. Theoretical difficulty due to crossed terms in the spectral expansion (cf Ingham inequalities).
2. In practice: many experiments, many measures. This deterministic constant is pessimistic: it gives an account for the worst case.

$\longrightarrow$ optimize shape and location of sensors in average, over a large number of measurements

$\longrightarrow$ define an averaged observability inequality

Randomized observability constant

Averaging over random initial data:

Randomized observability inequality (wave equation)

$$C_{T,\mathrm{rand}}(\omega)\,\|(y(0,\cdot),y_t(0,\cdot))\|^2_{L^2\times H^{-1}} \leq \mathbb{E}\left(\int_0^T\int_\omega |y_\nu(t,x)|^2\,dx\,dt\right)$$

where

$$y_\nu(t,x)=\sum_{j=1}^{+\infty}\left(\beta^\nu_{1,j}a_je^{i\lambda_jt}+\beta^\nu_{2,j}b_je^{-i\lambda_jt}\right)\phi_j(x)$$

with $\beta^\nu_{1,j}$, $\beta^\nu_{2,j}$ i.i.d. random variables (e.g., Bernoulli, Gaussian) of mean 0

(inspired from Burq-Tzvetkov, Invent. Math. 2008)

with $(\phi_j)_{j\in\mathbb{N}^*}$ Hilbert basis of eigenfunctions

Randomization
- generates a full measure set of initial data
- does not regularize

Randomized observability constant

Theorem

$\forall \omega$ measurable $\boxed{C_{T,\mathrm{rand}}(\chi_\omega) = T \inf_{j\in\mathbb{N}^*} \gamma_j \int_\omega \phi_j(x)^2\, dx}$

with

$$\gamma_j = \begin{cases} 1/2 & \text{for the wave equation} \\ 1 & \text{for the Schrödinger equation} \\ \frac{e^{2\lambda_j^2 T}-1}{2\lambda_j^2} & \text{for the heat equation} \end{cases}$$

with $(\phi_j)_{j\in\mathbb{N}^*}$ a fixed Hilbert basis of eigenfunctions of $\triangle$

Remark

There holds $C_{T,\mathrm{rand}}(\chi_\omega) \geqslant C_T(\chi_\omega)$.

For the wave equation, the randomized observability constant is a spectral quantity ignoring the rays' contribution.
($\longrightarrow$ spectral criterion = half of the truth!)

There are examples where the inequality is strict:

- in 1D: $\Omega = (0,\pi)$, $T \neq k\pi$.
- in multi-D: Ω stadium-shaped, ω containing the wings.

Randomized observability constant

Theorem

$$\forall \omega \text{ measurable} \qquad \boxed{C_{T,\mathrm{rand}}(\chi_\omega) = T \inf_{j\in\mathbb{N}^*} \gamma_j \int_\omega \phi_j(x)^2\, dx}$$

with

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with $(\phi_j)_{j\in\mathbb{N}^*}$ a fixed Hilbert basis of eigenfunctions of $\triangle$

Conclusion: we model the problem as

$$\boxed{\sup_{\substack{\omega\subset\Omega \\ |\omega|=L|\Omega|}} \inf_{j\in\mathbb{N}^*} \gamma_j \int_\omega \phi_j(x)^2\, dx}$$

$$\boxed{\sup_{\substack{\omega\subset\Omega\\|\omega|=L|\Omega|}} \inf_{j\in\mathbb{N}^*} \gamma_j \int_\omega \phi_j(x)^2\, dx}$$

To solve the problem, we distinguish between:

parabolic equations (e.g., heat, Stokes) $\neq$ wave or Schrödinger equations

Remarks

- requires some knowledge on the asymptotic behavior of ϕ_j^2
- $\mu_j = \phi_j^2\, dx$ is a probability measure
  $\Rightarrow$ strong difference between $\gamma_j \sim e^{\lambda_j T}$ (parabolic) and $\gamma_j = 1$ (hyperbolic)

Parabolic equations

(e.g.: heat, Stokes, anomalous diffusions)

We assume that Ω is piecewise C^1

Theorem

There exists a unique optimal domain ω^*

- Quite difficult proof, requiring in particular: Hartung minimax theorem; fine lower estimates of ϕ_j^2 by J. Apraiz, L. Escauriaza, G. Wang, C. Zhang (JEMS 2014)
- Algorithmic construction of the best observation set ω^*: to be followed (further)

Wave and Schrödinger equations

Optimal value

Under appropriate spectral assumptions:

$$\sup_{\substack{\omega\subset\Omega\\ |\omega|=L|\Omega|}} \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2\,dx = L$$

- Proof: 1) convexification (relaxation), 2) no-gap (not obvious because not lsc).
- Main spectral assumption:

  QUE (Quantum Unique Ergodicity): the whole sequence $\phi_j^2\,dx \rightharpoonup \dfrac{dx}{|\Omega|}$ vaguely.

  true in 1D, but in multi-D?

Wave and Schrödinger equations

Optimal value

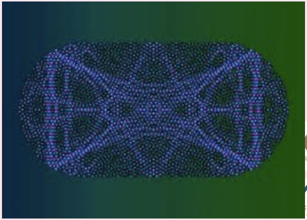
Under appropriate spectral assumptions:

$$\sup_{\substack{\omega\subset\Omega \\ |\omega|=L|\Omega|}} \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2\,dx = L$$

Relationship to quantum chaos theory:

what are the possible (weak) limits of the probability measures $\mu_j = \phi_j^2\,dx$?
(quantum limits, or semi-classical measures)

- See also Shnirelman theorem: ergodicity implies Quantum Ergodicity (QE; but possible gap to QUE!)
- If QUE fails, we may have scars
- QUE conjecture (negative curvature)



Wave and Schrödinger equations

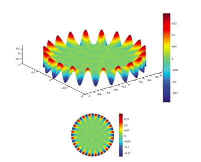
Optimal value

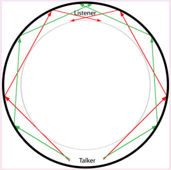
Under appropriate spectral assumptions:

$$\sup_{\substack{\omega\subset\Omega \\ |\omega|=L|\Omega|}} \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2\, dx = L$$

Remark: The above result holds true as well in the disk. Hence the spectral assumptions are not sharp.

(proof: requires the knowledge of all quantum limits in the disk, Privat Hillairet Trélat)





$$\mu_{j_k} \rightharpoonup \delta_{r=1}$$

(this is one QL: *whispering galleries*)

Wave and Schrödinger equations

Optimal value

Under appropriate spectral assumptions:

$$\sup_{\substack{\omega\subset\Omega\\|\omega|=L|\Omega|}} \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2\, dx = L$$

- Supremum reached? Open problem in general.
  - in 1D: reached $\Leftrightarrow L = 1/2$ (infinite number of optimal sets)
  - in 2D square: reached over Cartesian products $\Leftrightarrow L \in \{1/4, 1/2, 3/4\}$

    Conjecture: Not reached for generic domains Ω and generic values of L.
- Construction of a maximizing sequence (by a kind of homogenization)

Spectral approximation

Following Hébrard-Henrot (SICON 2005), we consider the finite-dimensional spectral approximation:

$$\sup_{\substack{\omega\subset\Omega \\ |\omega|=L|\Omega|}} \min_{1\leq j\leq N} \gamma_j \int_\omega \phi_j^2(x)\,dx$$

Theorem

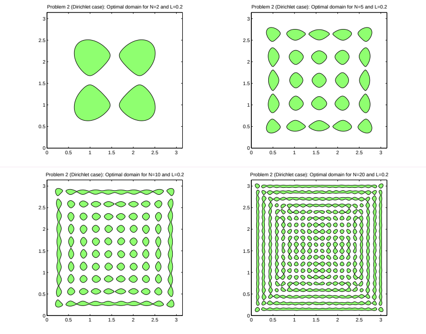
The problem has a unique solution ω^N.

Moreover, ω^N is semi-analytic and thus has a finite number of connected components.

Wave and Schrödinger equations

The complexity of ω^N is increasing with N.

Spillover phenomenon: the best domain ω^N for the N first modes is the worst possible for the $N+1$ first modes.



$\boxed{\Omega = (0,\pi)^2}$ $\quad L = 0.2$

$4, 25, 100, 500$ eigenmodes

Parabolic equations

(e.g., heat, Stokes, anomalous diffusions)

Under a slight spectral assumption:
(satisfied, e.g., by $(-\Delta)^\alpha$ with $\alpha > 1/2$)

The sequence of optimal sets ω^N is stationary:

$$\exists N_0 \mid \forall N \geq N_0 \quad \omega^N = \omega^{N_0} = \omega^*$$

with ω^* the optimal set for all modes.
In particular, ω^* is semi-analytic and thus has a finite number of connected components.

Wave and Schrödinger equations

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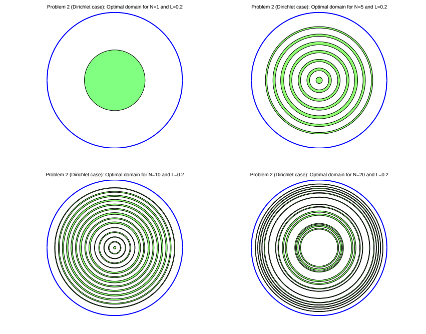

$\boxed{\Omega = \text{unit disk}}$ $\quad L = 0.2$

$1, 25, 100, 400$ eigenmodes

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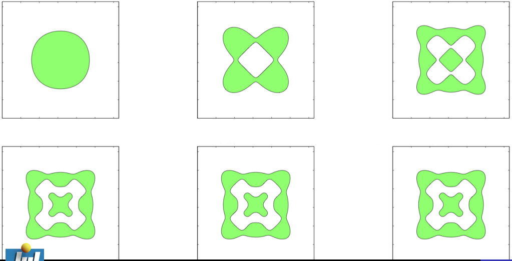
$\rightarrow$ no fractal set!



$\boxed{\Omega = (0,\pi)^2}$

1, 4, 9, 16, 25, 36 eigenmodes

$L = 0.2,\ T = 0.05$

$\rightarrow$ optimal thermometer in a square

Conclusion and perspectives

- Same kind of analysis for the optimal design of the control domain.
- Intimate relations between domain optimization and quantum chaos (quantum ergodicity properties).
- Optimal design for boundary observability (P. Jounieaux' PhD):

$$\sup_{|\omega|=L|\partial\Omega|} \inf_{j\in\mathbb{N}^*} \gamma_j \int_\omega \frac{1}{\lambda_j}\left(\frac{\partial\phi_j}{\partial\nu}\right)^2 d\mathcal{H}^{n-1}$$

- Strategies to avoid spillover?
- Discretization issues: do the numerical optimal designs converge to the continuous optimal design as the mesh size tends to 0?

Y. Privat, E. Trélat, E. Zuazua,

- *Optimal observation of the one-dimensional wave equation*, J. Fourier Analysis Appl. (2013).
- *Optimal location of controllers for the one-dimensional wave equation*, Ann. Inst. H. Poincaré (2013).
- *Complexity and regularity of maximal energy domains for the wave equation with fixed initial data*, Discrete Contin. Dyn. Syst. (2015).
- *Optimal shape and location of sensors for parabolic equations with random initial data*, Arch. Ration. Mech. Anal. (2015).
- *Optimal observability of the multi-dimensional wave and Schrödinger equations in quantum ergodic domains*, to appear in J. Europ. Math. Soc. (2015).

Conclusion and perspectives

- What can be said for the classical (deterministic) observability constant?

A result for the wave observability constant:
(Humbert Privat Trélat, ongoing)

$$\lim_{T\to+\infty}\frac{C_T(\omega)}{T}=\frac{1}{2}\min\left(\underbrace{\inf_{j\in\mathbb{N}^*}\int_\omega \phi_j^2\,dx}_{\text{spectral}},\ \underbrace{\lim_{T\to+\infty}\inf_{\{\gamma\ \text{ray}\}}\frac{1}{T}\int_0^T \chi_\omega(\gamma(t))\,dt}_{\text{geometric (rays)}}\right)$$

Two quantities: spectral geometric (rays)

$\downarrow$

randomized obs. constant

Remark: another way of arriving at the criterion (wave equation)

Averaging in time:
Time asymptotic observability inequality:

$$C_\infty(\chi_\omega)\|(y(0,\cdot),y_t(0,\cdot))\|^2_{L^2\times H^{-1}} \leq \lim_{T\to+\infty}\frac{1}{T}\int_0^T\int_\omega |y(t,x)^2|\,dx\,dt,$$

with

$$C_\infty(\chi_\omega)=\inf\left\{\lim_{T\to+\infty}\frac{1}{T}\frac{\int_0^T\int_\omega |y(t,x)|^2\,dx\,dt}{\|(y(0,\cdot),y_t(0,\cdot))\|^2_{L^2\times H^{-1}}}\mid (y(0,\cdot),y_t(0,\cdot))\in L^2\times H^{-1}\setminus\{(0,0)\}\right\}.$$

Theorem

If the eigenvalues of $\triangle_g$ are simple then $C_\infty(\chi_\omega)=\dfrac{1}{2}\inf_{j\in\mathbb{N}^*}\displaystyle\int_\omega \phi_j(x)^2\,dx=\dfrac{1}{2}J(\chi_\omega)$.

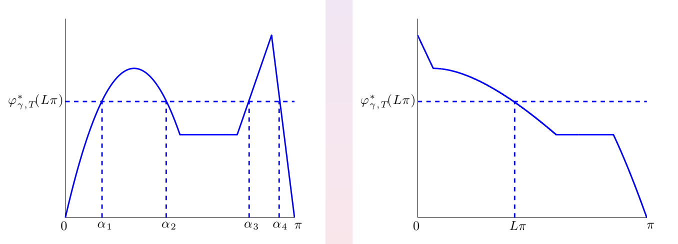
Remarks

- $C_\infty(\chi_\omega)\leq \dfrac{1}{2}\inf_{j\in\mathbb{N}^*}\displaystyle\int_\omega \phi_j(x)^2\,dx.$
- $\limsup_{T\to+\infty}\dfrac{C_T(\chi_\omega)}{T}\leq C_\infty(\chi_\omega)$. There are examples where the inequality is strict.

A remark for fixed initial data

If we maximize $\omega \mapsto \int_0^T \int_\omega |y(t,x)|^2\,dx\,dt$ with **fixed initial data**, then, using a decreasing rearrangement argument:

> *There always exists (at least) one optimal set ω.*
> *The regularity of ω depends on the initial data: it may be a Cantor set of positive measure, even for C^∞ data.*



$\longrightarrow$ In our model, we consider an infimum over all initial data.

A remark on the class of subdomains

Let $A > 0$ fixed. If we restrict the search to

$$\{\omega \subset \Omega \mid |\omega| = L|\Omega| \text{ and } P_\Omega(\omega) \leq A\} \qquad \text{(perimeter)}$$

or

$$\{\omega \subset \Omega \mid |\omega| = L|\Omega| \text{ and } \|\chi_\omega\|_{BV(\Omega)} \leq A\} \qquad \text{(total variation)}$$

or

$$\{\omega \subset \Omega \mid |\omega| = L|\Omega| \text{ and } \omega \text{ satisfies the } 1/A\text{-cone property}\}$$

or

ω ranges over some finite-dimensional (or "compact") prescribed set...

then *there always exists (at least) one optimal set ω.*

$\longrightarrow$ but then...
- the complexity of ω may increase with A
- we want to know if there is a "very best" set (over all possible measurable)

Remedies (wave and Schrödinger equations)

1. Existence of a maximizer

Ensured if $\mathcal{U}_L$ is replaced with any of the following choices:

$$\mathcal{V}_L = \{\chi_\omega \in \mathcal{U}_L \mid P_\Omega(\omega) \leq A\} \qquad \text{(perimeter)}$$
$$\mathcal{V}_L = \{\chi_\omega \in \mathcal{U}_L \mid \|\chi_\omega\|_{BV(\Omega)} \leq A\} \qquad \text{(total variation)}$$
$$\mathcal{V}_L = \{\chi_\omega \in \mathcal{U}_L \mid \omega \text{ satisfies the } 1/A\text{-cone property}\}$$

where $A > 0$ is fixed.

Remedies (wave and Schrödinger equations)

2. Weighted observability inequality

$$C_{T,\sigma}(\chi_\omega)\left(\|(y^0,y^1)\|^2_{L^2\times H^{-1}}+\sigma\|y^0\|^2_{H^{-1}}\right)\leq\int_0^T\int_\omega |y(t,x)|^2\,dx\,dt,$$

where $\sigma\geq 0$: weight.

Note that $C_{T,\sigma}(\chi_\omega)\leq C_T(\chi_\omega)$.

Randomization $\Rightarrow 2\,C_{T,\sigma,\mathrm{rand}}(\chi_\omega)=TJ_\sigma(\chi_\omega)$, where

$$J_\sigma(\chi_\omega)=\inf_{j\in\mathbb{N}^*}\sigma_j\int_\omega \phi_j(x)^2\,dx,$$

with $\sigma_j=\frac{\lambda_j^2}{\sigma+\lambda_j^2}$.

Remedies (wave and Schrödinger equations)

Theorem

Assume that L^∞-QUE holds. If $\sigma_1 < L < 1$ then there exists $N \in \mathbb{N}^*$ such that

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \sigma_j \int_\omega \phi_j^2 = \max_{\chi_\omega \in \mathcal{U}_L} \inf_{1 \leq j \leq n} \sigma_j \int_\omega \phi_j^2 \leq \sigma_1 < L,$$

for every $n \geq N$. In particular there is a unique solution χ_{ω^N}. Moreover if M is analytic then ω^N is semi-analytic and has a finite number of connected components.

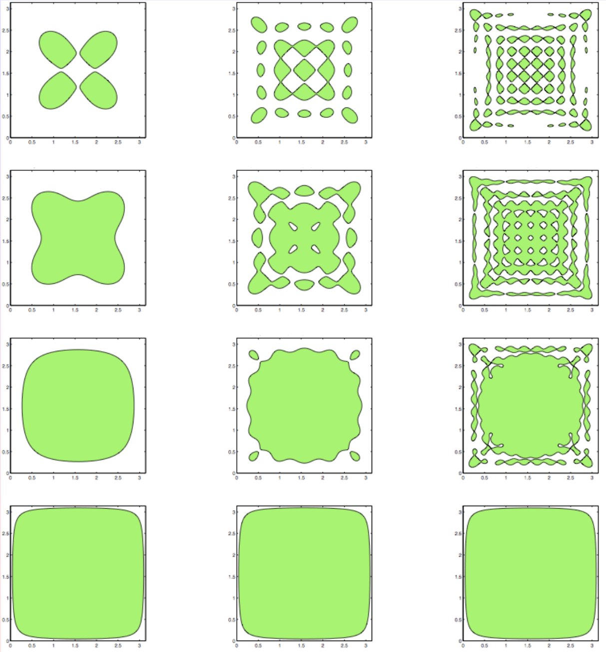
- The condition $\sigma_1 < L < 1$ seems optimal (see numerical simulations).
- This result holds as well in any torus, or in the Euclidean n-dimensional square for Dirichlet or mixed Dirichlet-Neumann conditions.

$L = 0.2$

$L = 0.4$

$L = 0.6$

$L = 0.9$



An additional remark

Anomalous diffusion equations, Dirichlet: $\partial_t y + (-\triangle)^\alpha y = 0$ ($\alpha > 0$ arbitrary)
with a surprising result:

In the square $\Omega = (0,\pi)^2$, with the usual basis (products of sine): the optimal domain ω^* has a finite number of connected components, $\forall \alpha > 0$.

In the disk $\Omega = \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$, with the usual basis (Bessel functions), the optimal domain ω^* is radial, and

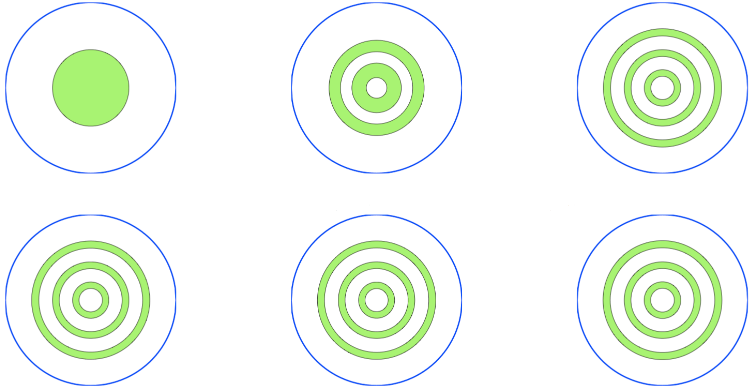
- $\alpha > 1/2 \;\Rightarrow\; \omega^* =$ finite number of concentric rings (and $d(\omega, \partial\Omega) > 0$)
- $\alpha < 1/2 \;\Rightarrow\; \omega^* =$ infinite number of concentric rings accumulating at $\partial\Omega$!
  (or $\alpha = 1/2$ and T small enough)

The proof is long and very technical. It uses in particular the knowledge of quantum limits in the disk.

(L. Hillairet, Y. Privat, E.Trélat)

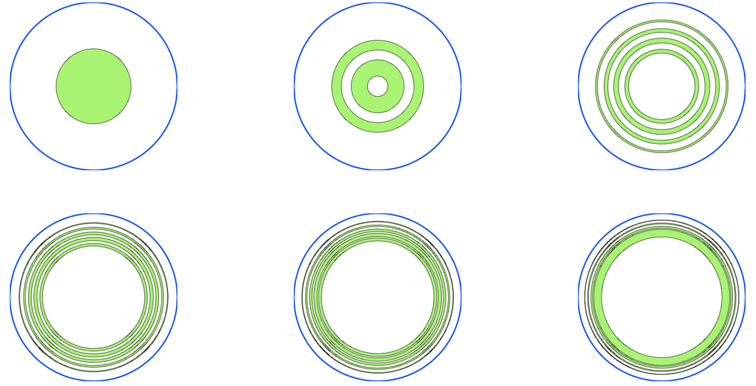
$\boxed{\Omega = \text{unit disk}}$ 1, 4, 9, 16, 25, 36 eigenmodes

$L = 0.2$, $T = 0.05$, $\alpha = 1$



$\boxed{\Omega = \text{unit disk}}$ 1, 4, 25, 100, 144, 225 eigenmodes

$L = 0.2$, $T = 0.05$, $\alpha = 0.15$



Comparison

$$\boxed{\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \gamma_j \int_\omega \phi_j^2}$$

	square	disk
wave or Schrödinger	relaxed solution $a = L$ $\exists \omega$ for $L \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ $\nexists$ otherwise (conjecture)	relaxed solution $a = L$ $\exists \omega$ for $L \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ $\nexists$ otherwise (conjecture)
diffusion $(-\triangle)^\alpha$	$\exists! \omega \quad \forall L \quad \forall \alpha > 0$ $\# c.c.(\omega) < +\infty$	$\exists! \omega$ (radial) $\quad \forall L \quad \forall \alpha > 0$ if $\alpha > 1/2$ then $\# c.c.(\omega) < +\infty$ if $\alpha < 1/2$ then $\# c.c.(\omega) = +\infty$