Photoacoustic tomography and thermodynamic attenuation

Sebastian Acosta
Baylor College of Medicine
Texas Children's Hospital

Collaboration with C. Montalto, U of Washington.

OUTLINE

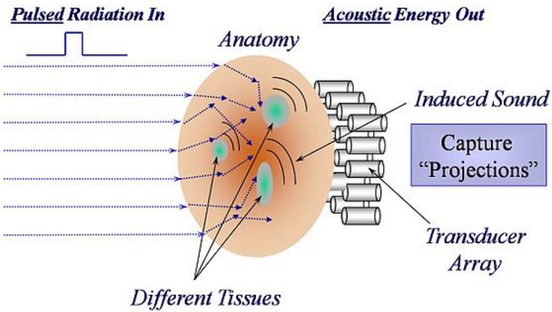
INTRO

ANALYSIS

NUMERICS

PHOTOACOUSTIC EFFECT [1]

1. Short pulse of radiation (at appropriate wavelength).
2. Energy deposited in tissue $a(x)I(x,t)$.
3. Thermal expansion proportional to $a(x)I(x,t)$.
4. Propagation of acoustic (pressure) waves

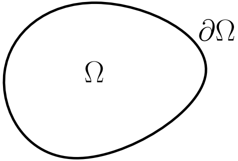


[1] A.G.Bell 1880, Bowen 1981

PHOTOACOUSTIC TOMOGRAPHY (PAT)

Since the incoming radiation is a very short pulse,
$I(x,t) \approx I_o(x)\delta(t)$,

$$\begin{aligned} \partial_t^2 p - c^2 \Delta p &= 0 \\ p|_{t=0} &\sim a(x) I_o(x) \\ \partial_t p|_{t=0} &= 0 \end{aligned}$$



Inverse Problem: Recover the absorption coefficient $a(x)$ or the product $(a(x)I_o(x))$ from knowledge the acoustic pressure p at the boundary $\partial\Omega$.

ATTENUATION

Recent efforts to account for attenuation:

- In the frequency domain (dissipation and dispersion relations)
- In the time domain (fractional time derivatives, visco-elastic terms, etc.)

Recent developments:

- Kowar 2010 , Treeby et al 2010 , Cook et al 2011 , Huang et al 2012 , Ammari et al 2012 ,
- Kowar-Scherzer 2012 , Kalimeris-Scherzer 2013 , Homan 2013 , Kowar 2014, Palacios 2016 ...
- ... others

THERMODYNAMIC ATTENUATION

PAT is based on the photoacoustic effect, which consists of two transformations of energy:

- EM/optical radiation is absorbed and transformed into heat.
- Conversion from heat into mechanical energy (due to thermal expansion).

However, due to thermodynamic interaction between temperature (entropy) and pressure, the reverse transformation occurs:

- Conversion from mechanical energy to heat.

and heat dissipates, thus the pressure wave is attenuated.

THERMOELASTIC COUPLING

Mathematically described by the system:

$$\rho\partial_t^2\mathbf{u} - \nabla\left(\lambda\,\mathrm{div}\,\mathbf{u}\right) - \mathrm{div}\,\mu\left(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}\right) + \beta K\,\nabla\theta = 0,$$
$$\rho c_{\mathrm{p}}\partial_t\theta - \kappa\Delta\theta + \theta_{\mathrm{ref}}\beta K\,\mathrm{div}\,\partial_t\mathbf{u} = 0,$$

Elastic variables:

$\mathbf{u}$ displacement.

K bulk modulus ; λ, μ Lame coeff; ρ density

Thermal variables

θ temperature deviation from reference θ_{ref}.

ρ density ; c_{p} specific heat; κ heat conductivity.

Coupling

β thermal expansion coeff. (Gruneisen $G = \beta K/\rho c_{\mathrm{p}}$.)

THERMODYNAMIC ATTENUATION

How strong is the thermoelastic attenuation ?

We write equations in unitless form and introduce pressure $p = -(\lambda + 2\mu)\text{div}\,\mathbf{u}$, to arrive at a scalar system:

$$\partial_t^2 p - c^2 \Delta p - \epsilon\, c^2 \Delta \theta = 0, \tag{1.1}$$

$$\partial_t \theta - \alpha \Delta \theta - \epsilon\, \sigma\, \partial_t p = 0. \tag{1.2}$$

where, for soft biological tissues:

$c \approx 1$ unitless wave speed

$\alpha \ll 1$ unitless thermal diffusivity

$\sigma \approx 1$

$\epsilon = \beta\theta_{\text{ref}}$ unitless coupling $\in (0.05, 0.1)$

RAPID DEPOSITION OF HEAT

If heat deposition is much faster than pressure relaxation:

$$p|_{t=0} = p_0$$

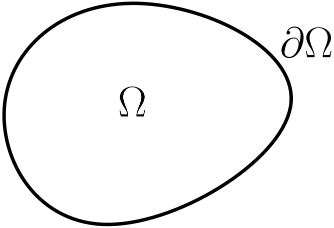
$$\partial_t p|_{t=0} = 0$$

and if heat deposition is much faster than thermal diffusion:

$$\theta|_{t=0} = \epsilon p_0$$

MAIN MATHEMATICAL QUESTIONS

$$\begin{aligned} \partial_t^2 p - c^2 \Delta p - \epsilon c^2 \Delta \theta &= 0 \\ \partial_t \theta - \alpha \Delta \theta - \epsilon \partial_t p &= 0 \\ p|_{t=0} = p_0, \quad \partial_t p|_{t=0} &= 0 \\ \theta|_{t=0} &= \epsilon p_0 \end{aligned}$$



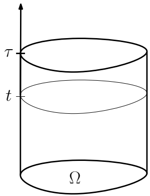
- Uniqueness & Stability
- Reconstruction algorithm

Answers may depend on

- Full or partial data on boundary $\partial\Omega$
- Shape of domain Ω
- Constant or variable wavespeed $c(x)$

PURE ACOUSTICS

$$
\begin{aligned}
\partial_t^2 p - c(x)^2 \Delta p &= 0 \\
p|_{t=0} &= p_o \\
\partial_t p|_{t=0} &= 0
\end{aligned}
$$



Conditions for time-reversal approaches [2] [3] [4]:

- Non trapping condition for variable medium.
- Energy decay for either forward or backward problem.
- Well-posedness for backward problem.

[2] Stefanov-Uhlmann 2009, 2011; Qian et al 2011
[3] Grun et al 2007; Hristova et al 2008, 2009; Wang 2009; Homan '13
[4] Acosta-Montalto '15, Stefanov-Yang '15; Nguyen-Kunyansky '15

... WHAT ABOUT THE THERMOACOUSTIC SYSTEM ?

$$
\begin{aligned}
\partial_t^2 p - c^2 \Delta p - \epsilon c^2 \Delta \theta &= 0 \\
\partial_t \theta - \alpha \Delta \theta - \epsilon \partial_t p &= 0 \\
p|_{t=0} = p_0, \quad \partial_t p|_{t=0} &= 0 \\
\theta|_{t=0} &= \epsilon p_0
\end{aligned}
$$

Unfortunately, the backward problem is ill-posed due to heat equation.

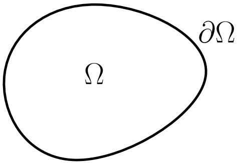
Alternatives:

- Other approximate inverse (Neumann series)
- Fredholm alternative (compactness)
- Coercivity (normal equation)

NON-TRAPPING CONDITION

From math viewpoint, a geometric setting is natural to study waves in variable media.

$$\partial_t^2 p - \Delta_g p = 0$$



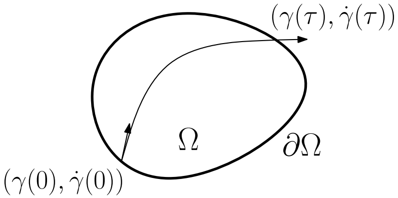
Δ_g: Laplace-Beltrami operator for manifold (Ω, g) with Riemannian metric g. In local coordinates,

$$\Delta_g u = \frac{1}{|g|^{1/2}} \frac{\partial}{\partial x_i} \left(|g|^{1/2} g^{ij} \frac{\partial}{\partial x_j} u \right)$$

where $g^{ij} = g_{ij}^{-1}$ and $|g| = \det g$.

NON-TRAPPING CONDITION

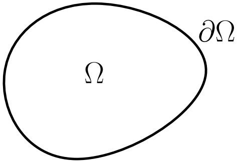
From math viewpoint, a geometric setting is natural to study waves in variable media.



The manifold (Ω, g) is non-trapping if all bi-characteristic geodesics $\gamma(t)$ reach the boundary $\partial\Omega$ in finite time.

TIME REVERSAL FOR PURE ACOUSTICS

$$\begin{aligned} \partial_t^2 p - \Delta_g p &= 0 \\ p|_{t=0} &= p_\mathrm{o} \\ \partial_t p|_{t=0} &= 0 \end{aligned}$$



Theorem

If the manifold (Ω, g) is non-trapping, then p_o can be uniquely and stably reconstructed from measurements on $\partial\Omega$ on a finite window of time.

- In free-space: Stefanov-Uhlmann 2009
- In an enclosure: Acosta-Montalto 2015, Stefanov-Yang 2015, Kunyansky-Nguyen 2015.

Our goal: Prove similar theorem for thermoacoustic system.

PAT WITH THERMOELASTIC ATTENUATION

$$
\begin{aligned}
\partial_t^2 p - c^2 \Delta p - \epsilon c^2 \Delta \theta &= 0 \\
\partial_t \theta - \alpha \Delta \theta - \epsilon \partial_t p &= 0 \\
p|_{t=0} = p_0, \quad \partial_t p|_{t=0} &= 0 \\
\theta|_{t=0} &= \epsilon p_0
\end{aligned}
$$

Measurement map $\mathcal{M}$: $p_0 \mapsto p|_{(0,\tau)\times\partial\Omega}$

Our goal: Prove that normal operator $(\mathcal{M}^*\mathcal{M})$ is coercive, so that $p_0 = (\mathcal{M}^*\mathcal{M})^{-1}\mathcal{M}^* p|_{(0,\tau)\times\partial\Omega}$.

PAT WITH THERMODYNAMIC ATTENUATION

Theorem (Main Result)

If the manifold $(\Omega, c^{-2}\delta)$ *is non-trapping and* ϵ *is sufficiently small, then* $\mathcal{M}$ *is injective and*

$$\|p_0\|_{H^1(\Omega)} \leq C\|\mathcal{M}p_0\|_{H^1((0,\tau)\times\partial\Omega)}$$

Corollary

If the manifold $(\Omega, c^{-2}\delta)$ *is non-trapping and* ϵ *is sufficiently small, then* $(\mathcal{M}^*\mathcal{M})$ *is coercive and*

$$p_0 = (\mathcal{M}^*\mathcal{M})^{-1}\mathcal{M}^*p|_{(0,\tau)\times\partial\Omega}$$

PAT WITH THERMODYNAMIC ATTENUATION

Proof.

We combine three inequalities:

$$\begin{aligned}
\|\Delta\theta\|^2_{H^0((0,\tau);H^0(\Omega))} &\leq \epsilon^2 C \left(\|\partial_t p\|^2_{H^0((0,\tau);H^0(\Omega))} + \|\nabla p_0\|^2_{H^0(\Omega)} \right), \\
\|\partial_t p\|^2_{H^0((0,\tau);H^0(\Omega))} &\leq (1+\epsilon^2)\|\nabla p_0\|^2_{H^0(\Omega)}, \\
\|p_0\|^2_{H^1(\Omega)} &\leq C \left(\epsilon^2 \|\Delta\theta\|^2_{H^0((0,\tau);H^0(\Omega))} + \|p\|^2_{H^1((0,\tau)\times\partial\Omega)} \right).
\end{aligned}$$

□

$$\|p_0\|_{H^1(\Omega)} \leq C\|\mathcal{M}p_0\|_{H^1((0,\tau)\times\partial\Omega)}$$

RECONSTRUCTION METHOD

$$
\begin{aligned}
\partial_t^2 p - c^2 \Delta p - \epsilon c^2 \Delta \theta &= 0 \\
\partial_t \theta - \alpha \Delta \theta - \epsilon \partial_t p &= 0 \\
p|_{t=0} = p_0, \quad \partial_t p|_{t=0} &= 0 \\
\theta|_{t=0} &= \epsilon p_0
\end{aligned}
$$

Theorem (Reconstruction Method)

Under the nontrapping condition and small ϵ, the solution to PAT is given by

$$p_{\mathrm{o}} = (\mathcal{M}^* \mathcal{M})^{-1} \mathcal{M}^* p|_{(0,\tau)\times\Gamma}$$

where $(\mathcal{M}^* \mathcal{M})$ is coercive on $H^s(\Omega)$, so can be approximated with conjugate gradient method.

CONJUGATE GRADIENT ALGORITHM

Governing equation $(\mathcal{M}^*\mathcal{M})\phi = \zeta$

$$\phi_{k+1} = \phi_k + \alpha_k s_k \qquad \text{where} \qquad \alpha_k = \|r_k\|^2/\|\mathcal{M}s_k\|^2$$
$$r_{r+1} = \zeta - (\mathcal{M}^*\mathcal{M})\phi_{k+1}$$
$$s_{k+1} = r_{k+1} + \beta_k s_k \qquad \text{where} \qquad \beta_k = \|r_{k+1}\|^2/\|r_k\|^2$$

starting with initial guess ϕ_0, and $r_0 = \zeta - (\mathcal{M}^*\mathcal{M})\phi_0$, and $s_0 = r_0$. The algorithm is convergent in a Hilbert setting:

$$\|\phi_* - \phi_k\| \leq e^{-\sigma k}\|\phi_* - \phi_0\|, \qquad k \geq 0, \qquad \text{some } \sigma > 0.$$

NUMERICAL SIMULATION

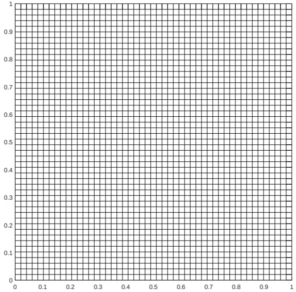
$$\begin{aligned} \partial_t^2 p - c^2 \Delta p - \epsilon c^2 \Delta \theta &= 0 \\ \partial_t \theta - \alpha \Delta \theta - \epsilon \partial_t p &= 0 \\ p|_{t=0} = p_0, \quad \partial_t p|_{t=0} &= 0 \\ \theta|_{t=0} &= \epsilon p_0 \end{aligned}$$

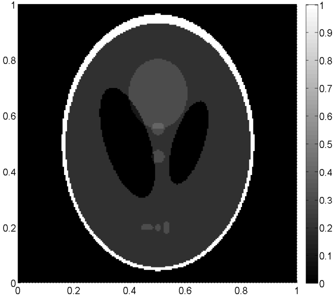
The operator $\mathcal{M} : p_0 \mapsto p|_{(0,\tau)\times\partial\Omega}$ can be approximated using numerical methods for PDEs.

Similarly, the $\mathcal{M}^*$ is associated with an adjoint PDE system.

NUMERICAL SIMULATION

Finite difference method. Refinement (space and time) is consistent and CFL stable.

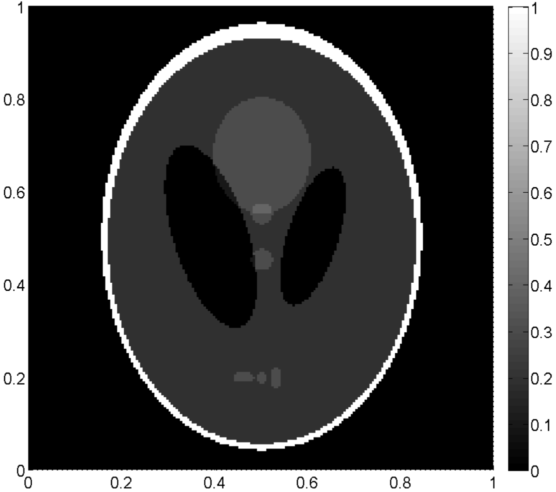




Numerically evaluate the operators $\mathcal{M}$ and $\mathcal{M}^*$, to apply the conjugate gradient method.

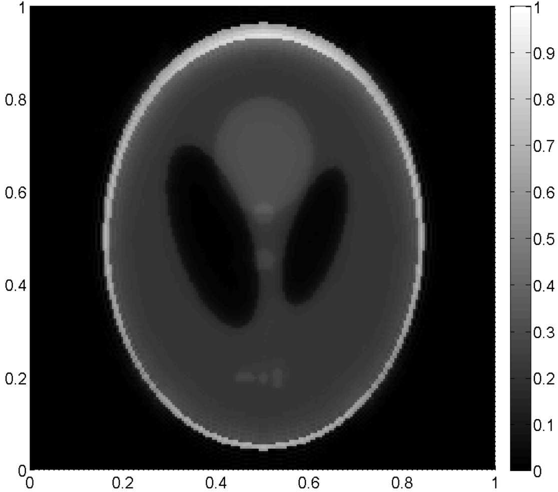
NUMERICAL SIMULATION

Exact solution:



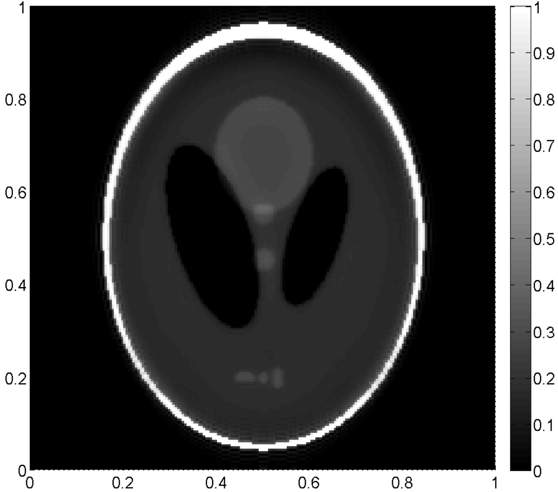
NUMERICAL SIMULATION

Iteration $n = 0$:



NUMERICAL SIMULATION

Iteration $n = 1$:



NUMERICAL SIMULATION

Wave speed $c = 1$
Diffusivity $\alpha = 0.01$
Coupling $\epsilon = 0.1$

Iter	$H^1(\Omega)$-norm	$H^0(\Omega)$-norm
0	52.6 %	31.1 %
1	19.8 %	12.8 %
2	10.6 %	5.7 %
3	6.3 %	4.4 %
4	4.5 %	3.8 %
5	3.8 %	3.1 %

SUMMARY

We have analyzed the PAT problem taking into account attenuation due to thermodynamic dissipation.

Under the non-trapping condition and weak thermoacoustic coupling, we showed

- Uniqueness
- Stability
- A reconstruction methods

THANK YOU