Stability from Partial Data in Current Density Impedance Imaging

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Work supported by NSF

SIAM Conference in Imaging Sciences, Albuquerque, NM, May 23-26, 2016

Outline

In motivation

Hybrid methods in Inverse Problems

Current density based EIT

Acquiring the interior data

Stability results

Injectivity regions and stability subregions

What can be stably determined when only $|\mathbf{J}|$ is available?

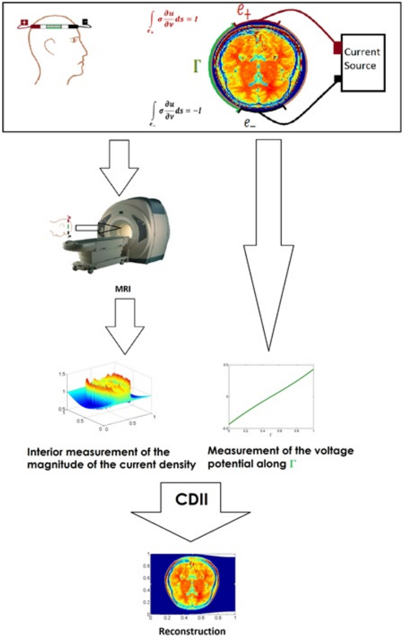
The range of interior data

Stability of 1-harmonic maps with fixed trace

Coupled Physics Imaging Methods in EIT

Combine high contrast & high resolution

- MREIT (B_z-methods): Seo et al. since 2003
- Current density impedance imaging **CDII**: Joy& Nachman since 2002, Seo et al. 2002
- Ultrasound modulated EIT: Capdebosq et at. 2008, Bal et al. 2009
- Impedance acoustic: Scherzer et al. 2009
- Lorentz force driven EIT: Ammari et al. since 2013, Kunyansky


$\int_{e_+} \sigma \frac{\partial u}{\partial \nu} ds = 1$
e_+
Γ
Current Source
$\int_{e_-} \sigma \frac{\partial u}{\partial \nu} ds = -1$
e_-
MRI
Interior measurement of the magnitude of the current density
Measurement of the voltage potential along Γ
CDII
Reconstruction

Current density fields can be traced inside an object using MRI

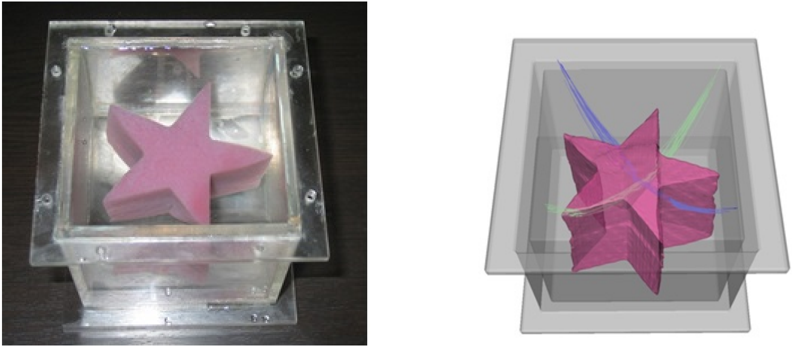
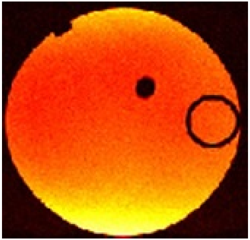


Figure : Courtesy: Joy's group, U Toronto

Recovery of the longitudinal component of the magnetic field

Magnetic resonance data: $M : \Omega \to \mathbb{C}$



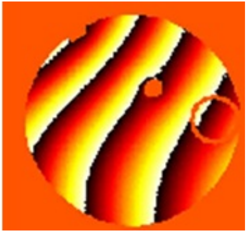


Figure : $M_{\pm}(x, y, z_0) = M(x, y, z_0) \exp(\pm i\gamma B_z(x, y, z_0)T + i\varphi_0)$

One MR scan $\Rightarrow$ longitudinal component B_z (along gantry) of the magnetic field $\mathbf{B} = (B_x, B_y, B_z)$

$$B_z(x, y, z_0) = \frac{1}{2\gamma T} \mathbb{I}\mathrm{m}\, log \left(\frac{M_+(x, y, z_0)}{M_-(x, y, z_0)} \right)$$

A very brief history of Magnetic resonance aided EIT

- MREIT (Seo at al. since 2003): Does B_z uniquely determine the electrical conductivity? In general, not known.
- Current Density based Impedance Imaging (CDII): $+$two rotations $\Rightarrow \mathbf{B} \Rightarrow \mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{B}$
  - Unique determination and reconstruction (Nachman et al since 2002, Seo (2002)); Using one $|\mathbf{J}|$ (Nachman et al. since 2007)
  - Stability:
    - Linearization: :Kuchment&Steinhauer (2012) and Bal (2013)
    - Local stability: Montalto&Stefanov (2013) and Kim & Lee (2014)
    - Stability with partial data: Montalto& T. (2015)
- Anisotropic case:
  - Bal, Guo & Monard (2014, 2015) unique determination (many currents)
  - Hoel, Moradifam &Nachman (2014, within conformal class)

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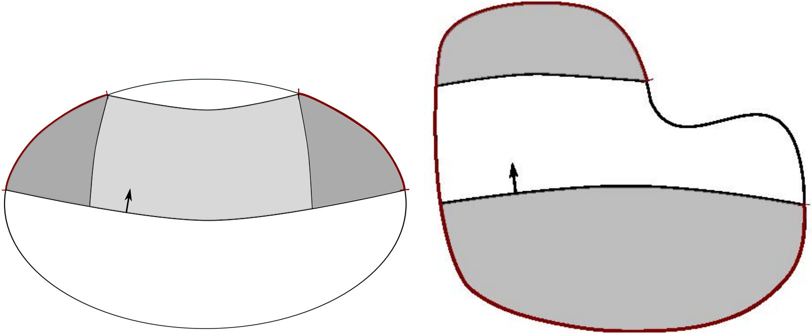

Figure : Left: Stability region $\subsetneq$ Injectivity and Right: Stability region=Injectivity region.

Stability from partial interior and boundary data

Domain: $\Omega \subset \mathcal{R}^n$ is a $C^{2,\alpha}$
Conductivities: $\sigma, \tilde{\sigma} \in C^{1,\alpha}(\overline{\Omega})$
Let $M := \max\{\|\sigma\|_{C^{1,\alpha}}, \|\tilde{\sigma}\|_{C^{1,\alpha}}\}$.
Corrsponding voltages $u, \tilde{u} \in C^{2,\alpha}(\overline{\Omega})$ with $u|_\Gamma = \tilde{u}|_\Gamma$,
where $\Gamma \subseteq \partial\Omega$ open and $\Gamma' \Subset \Gamma$.
Assume that

$$\sigma|_\Gamma = \tilde{\sigma}|_\Gamma, \qquad |\nabla(u+\tilde{u})| \geq \delta > 0, \quad \text{in} \quad \overline{\mathcal{I}(\Gamma', u+\tilde{u})}$$

Then $\exists C = C(\delta, M) > 0$ such that

$$\|\sigma - \tilde{\sigma}\|_{L^2(\mathcal{S}(\Gamma', u+\tilde{u}))} \leq C \left|\left|\nabla \cdot (\Pi_{\nabla(u+\tilde{u})}(\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma})))\right|\right|_{L^2(\mathcal{I}(\Gamma', u+\tilde{u}))}^{\frac{\alpha}{2+\alpha}}$$

$$\|\sigma - \tilde{\sigma}\|_{L^2(\mathcal{S}(\Gamma', u+\tilde{u}))} \leq C \left|\left|\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma})\right|\right|_{H^1(\mathcal{I}(\Gamma', u+\tilde{u}))}^{\frac{\alpha}{2+\alpha}}.$$

Corollary: Local Stability

Domain $\Omega \subset \mathcal{R}^n$ is $C^{2,\alpha}$ and conductivity $\sigma \in C^{1,\alpha}(\overline{\Omega})$
Let

$$\mathcal{U}_\epsilon := \{\tilde{\sigma} \in C^{1,\alpha}(\overline{\Omega}) : \ \|\sigma - \tilde{\sigma}\|_{C^{1,\alpha}} < \epsilon, \ \tilde{\sigma}|_{\partial\Omega} = \sigma|_{\partial\Omega}\}$$

Then $\exists \epsilon > 0$ and $C = C(\epsilon) > 0$, such that $\forall \tilde{\sigma} \in \mathcal{U}_\epsilon$,

$$\|\sigma - \tilde{\sigma}\|_{L^2(\Omega)} \leq C \, ||\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma})||_{H^1(\Omega)}^{\frac{\alpha}{2+\alpha}},$$

where $\mathbf{J}(\tilde{\sigma})$ is the current density corresponding to $\tilde{\sigma}$ induced by imposing **the same** boundary voltage as for generating $\mathbf{J}(\sigma)$.

Note: This recovers the results of Montalto& Stefanov '13, and Kim& Lee '15.

Key idea

Let $u, \tilde{u}$ be σ-(respectively $\tilde{\sigma}$)-harmonic with $\nabla(u+\tilde{u}) \neq 0$ in $\overline{O}$. Then

- $$2\nabla \cdot \Pi_{\nabla(u+\tilde{u})}(\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma})) = \mathcal{L}(u - \tilde{u}) \quad \text{in } O,$$
  $$\mathcal{L}v := -\nabla\cdot(\sigma+\tilde{\sigma})\nabla v + \nabla\cdot\left((\sigma+\tilde{\sigma})\frac{\nabla(u+\tilde{u})\cdot\nabla v}{|\nabla(u+\tilde{u})|^2}\nabla(u+\tilde{u})\right).$$
- $\mathcal{L}$ is an $(n-1)$-dimensional Laplace-Beltrami operator on each level surface of $u+\tilde{u}$:
  In local coordinates (y', y_n) in neighborhood of a level set (parametrized by y' :with $y_n = u + \tilde{u}$:

  $$\mathcal{L} := -\sum_{\alpha,\beta=1}^{n-1} \frac{1}{\sqrt{det(g)}}\frac{\partial}{\partial y_\beta}(\sigma+\tilde{\sigma})g^{\alpha\beta}\sqrt{det(g)}\frac{\partial}{\partial y^\beta}.$$

  where $g = (g_{\alpha\beta})$ is the induced metric on the level set.

The question of Range when only $|\mathbf{J}|$ is available.

- **So far:** Both $\mathbf{J}$ and $\tilde{\mathbf{J}}$ come as currents corresponding to some conductivities $\sigma, \tilde{\sigma}$
- **Is $|\mathbf{J}| + \epsilon$ the magnitude of a current corrsponding to some $\tilde{\sigma}$, induced by same boundary voltage?**
- In general not known
- New technics are required

Stability in a regularized minimization scheme

$(|\mathbf{J}|, f) \in C^{\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\partial\Omega)$ and $|\mathbf{J}| \geq \delta > 0$
$\Delta u_0 = 0$ in Ω with $u_0|_{\partial\Omega} = f$.
Let $a_n \to |\mathbf{J}|$ in $L^2(\Omega)$ and consider

$$h_n = \operatorname{argmin}_{h \in H_0^1(\Omega)} \int_{\Omega} a_n \max\{|\nabla(u_0 + h)|, \delta\} dx + \epsilon_n \int_{\Omega} |\nabla h|^2 dx,$$

where $\epsilon_n = \sqrt{\| \, |\mathbf{J}| - a_n \|_{L^2}}$. Then

- $\exists h^* \in L^q(\Omega)$, $1 \leq q < \frac{d}{d-1}$: $h_n \to h^*,$ in $L^q(\Omega)$.
- $h^* \in C_0^{1,\alpha}(\overline{\Omega})$,
- $|\nabla(u_0 + h^*)| > 0$
- $\sigma = \frac{|\mathbf{J}|}{|\nabla(u_0 + h^*)|}$

Warning: We don't know $\frac{|\mathbf{J}|}{|\nabla(u_0 + h_n)|} \to \sigma$!

We have a better (but not complete) understanding why we get good reconstructions

$$1S/m \leq \sigma \leq 1.8S/m,\ -I_0 = I_1 = 3mA,\ z_0 = z_1 = 8.3m\Omega \cdot m^2$$

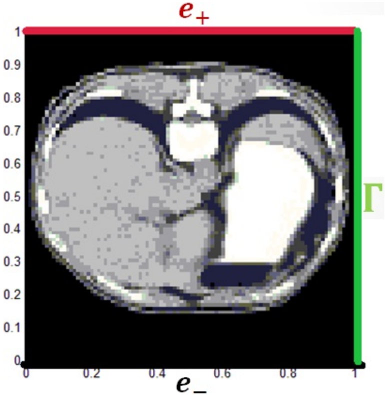

Figure : Exact conductivity (left) vs. reconstructed conductivity (right)

Thank you!