A Functional Analytic Approach to SAR Image Reconstruction

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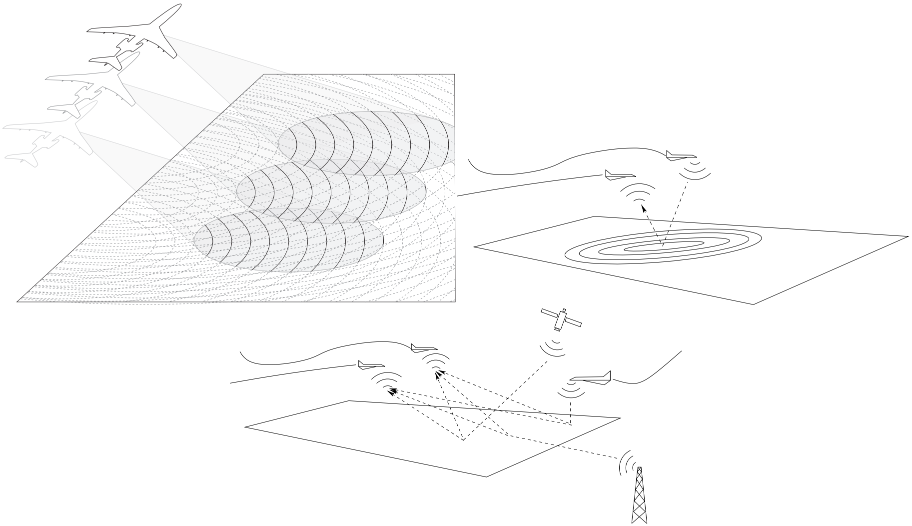
Synthetic-Aperture Radar



SAR Data Model

- Starting from Maxwell's equations and assuming free space we find that the wave equation is a good model for EM wave propagation:

$$(\nabla^2 - c_0^{-2}\frac{\partial^2}{\partial t^2})\mathcal{E}^{in}(t,x) = j(t,x)$$

$$(\nabla^2 - c^{-2}(x)\frac{\partial^2}{\partial t^2})\mathcal{E}^{tot}(t,x) = j(t,x)$$

where

$$c^{-2}(x) = c_0^{-2} - T(x)$$
$$\mathcal{E}^{tot} = \mathcal{E}^{in} + \mathcal{E}^{sc}$$

- $T(x)$ is called the reflectivity function and it models the scene of interest
- We seek data of the form $d = \mathcal{F}[T]$
- Combining the two wave equations and using the Green's function for free space we obtain the Lippmann-Schwinger integral equation

$$\mathcal{E}^{sc}(t,y) = \int\int \frac{\delta(t-\tau-|y-x|/c)}{4\pi|y-x|} T(x)\partial_t^2\mathcal{E}^{tot}(\tau,x)d\tau dx$$

SAR Data Model

- The Born approximated scattered field is given by:

$$\mathcal{E}_B^{sc}(t,y) = \int\int \frac{\delta(t-\tau-|y-x|/c)}{4\pi|y-x|} T(x)\partial_t^2 \mathcal{E}^{in}(\tau,x)d\tau dx$$

- The actual data model is written as

$$d(s,t) = \mathcal{F}[T](s,t) = \int e^{-i\omega(t-\phi(s,\mathbf{x}))} A(x,s,\omega)T(x)dxd\omega$$

  where s is the slow-time which parametrizes the antenna trajectory.

- The phase function takes on a different form depending on which SAR modality you consider:

$$\begin{aligned}
\phi(s,\mathbf{x}) &= r_{s,\mathbf{x}} = 2|\gamma(s)-\mathbf{x}|/c_0 \\
\phi(s,\mathbf{x}) &= r_{s,\mathbf{x}} = |\gamma_T(s)-\mathbf{x}|/c_0 + |\gamma_R(s)-\mathbf{x}|/c_0 \\
\phi(s,\mathbf{x}) &= r_{ij}(s,s',\mathbf{x}) = |\mathbf{x}-\gamma_{R_i}(s)| - |\mathbf{x}-\gamma_{R_j}(s+s')|
\end{aligned}$$

  where $\gamma(s)$ is the antenna position and x is location of a scatterer.

SAR Imaging - Backprojection

- To form an image we aim to invert by applying an imaging operator $\mathcal{K}$

$$
\begin{aligned}
I(z) = \mathcal{K}[d](z) :&= \int e^{i\omega(t-\phi(s,\mathbf{z}))} Q(z,s,\omega) d\omega d(s,t) ds dt \\
&= \int e^{-i\omega\phi(s,\mathbf{z})} Q(z,s,\omega) D(s,\omega) d\omega ds \\
= \mathcal{K}\mathcal{F}[T](z) &= \int e^{i\omega(\phi(s,\mathbf{x})-\phi(s,\mathbf{z}))} Q(z,s,\omega) A(x,s,\omega) d\omega ds T(x) dx
\end{aligned}
$$

- $\mathcal{K}\mathcal{F}$ is called the image-fidelity operator and is a pseudodifferential operator $\Rightarrow$ visible singularities are preserved

Filtered backprojection

- We seek a filtered BP type reconstruction method, i.e. our image is of the form:

$$I(z) = \int e^{-2ikR_{z,s}} Q(z, s, k) D(s, k) dkds$$

  where Q is the backprojection filter and D is the 2D Fourier transform of d.

- Note our image of the form:

$$I(z) = \int K(z, x) T(x) dx$$

- K is called the point-spread function, given below:

$$K(z, x) = \int e^{-2ik(R_{z,s} - R_{x,s})} Q(z, s, k) A(x, s, k) dkds$$

Imaging Continued

- Ideally K would be of the form:

$$\int e^{i(z-x)\cdot\boldsymbol{\xi}} d\boldsymbol{\xi}$$

- We perform the Stolt change of variables $(s,k) \to \boldsymbol{\xi}$ where

$$\boldsymbol{\xi} = \boldsymbol{\Xi}(x,z,s,k) = \int_0^1 \nabla f|_{x+\mu(x-z)} d\mu$$

  where $f(x) = -2kR_{x,s}$.
- After performing symbol calculus we obtain the following expression for K:

$$K \approx \int e^{i(z-x)\cdot\boldsymbol{\xi}} Q(z,\boldsymbol{\xi}) A(z,\boldsymbol{\xi}) \eta(z,\boldsymbol{\xi}) d\boldsymbol{\xi}$$

- Therefore we choose Q as below:

$$Q(z,\boldsymbol{\xi}) = \frac{\chi_\Omega(z,\boldsymbol{\xi})}{A(z,\boldsymbol{\xi})\eta(z,\boldsymbol{\xi})}$$

  where $\chi_\Omega(z,\boldsymbol{\xi})$ is a smooth cut-off function that prevents division by zero and η is the Jacobian resulting from a Stolt change of variables.

BLUE

- We now calculate the best linear unbiased estimate of the reflectivty function from the collected data:
$$D(s,\omega) = \int e^{2ikr_{s,\mathbf{x}}} A(\omega, s, x) T(x) dx + n(s,\omega)$$
where we assume n is zero-mean independently, identically distributed noise in s and ω, i.e. we assume
$$E[n(s,\omega)] = 0$$
$$E[n(s,\omega)\overline{n(s',\omega')}] = \sigma^2 \delta(s-s')\delta(\omega-\omega')$$
where σ^2 is the variance of the noise for a single value of s and ω and δ is the Dirac delta function.

- We aim to estimate $T(x)$ from measurements $D(s,\omega)$ via a linear estimator
$$\widehat{T}(z) = \int Q(z,s,\omega) D(s,\omega) ds d\omega.$$

BLUE continued

- In BLUE we aim to minimize variance while also forcing the estimator to be unbiased.
- We seek to find the weights or filter Q such that the following functional is minimized

$$\mathcal{J}(Q) = E[|\widehat{T}(z) - E[\widehat{T}(z)]|^2] + \lambda(E[\widehat{T}](z) - T(z))$$

- After some calculations and the Stolt change of variables the functional simplifies to

$$\begin{aligned}\mathcal{J}(Q) = &\int |Q(z,\boldsymbol{\xi})|^2 \sigma^2 \eta(z,\boldsymbol{\xi}) d\boldsymbol{\xi} \\ &+ \lambda\Big(\int [Q(z,\boldsymbol{\xi}) e^{ix\cdot\boldsymbol{\xi}} A(z,\boldsymbol{\xi}) \eta(z,\boldsymbol{\xi}) - e^{i(x-z)\cdot\boldsymbol{\xi}}] d\boldsymbol{\xi} \Big).\end{aligned}$$

BLUE continued

- To find the minimizer we seek Q such that the variational derivative of $\mathcal{J}(Q)$ with respect to Q is zero and that the derivative of $\mathcal{J}(Q)$ with respect to λ is also zero

$$\frac{d}{d\epsilon}(\mathcal{J}(Q+\epsilon Q_\epsilon))|_{\epsilon=0} = 2\,\mathrm{Re}\left[\int \sigma^2 Q_\epsilon(z,\boldsymbol{\xi})\overline{Q}(z,\boldsymbol{\xi})\overline{\eta}(z,\boldsymbol{\xi})d\boldsymbol{\xi}\right] + \lambda\int Q_\epsilon(z,\boldsymbol{\xi})A(z,\boldsymbol{\xi})e^{ix\cdot\boldsymbol{\xi}}d\boldsymbol{\xi} = 0.$$

- Taking the derivative of $\mathcal{J}(Q)$ with respect to λ we obtain

$$\frac{d\mathcal{J}}{d\lambda} = \int [Q(z,\boldsymbol{\xi})e^{ix\cdot\boldsymbol{\xi}}A(z,\boldsymbol{\xi})\eta(z,\boldsymbol{\xi}) - e^{i(x-z)\cdot\boldsymbol{\xi}}]d\boldsymbol{\xi} = 0$$

  which implies Q must satisfy

$$Q(z,\boldsymbol{\xi}) = \frac{e^{-iz\cdot\boldsymbol{\xi}}}{A(z,\boldsymbol{\xi})\eta(z,\boldsymbol{\xi})}.$$

BLUE continued

- If we insert the expression found for Q into the definition for $\widehat{T}(z)$ we obtain

$$\widehat{T}(z) = \int \frac{e^{-iz\cdot\boldsymbol{\xi}}}{A(z,\boldsymbol{\xi})\eta(z,\boldsymbol{\xi})} e^{ix\cdot\boldsymbol{\xi}} A(z,\boldsymbol{\xi}) T(x) \eta(z,\boldsymbol{\xi}) d\boldsymbol{\xi} dx + \int Q(z,\boldsymbol{\xi}) n(\boldsymbol{\xi}) d\boldsymbol{\xi}.$$

- Looking at the first term above we see that we obtain precisely the backprojected image from the previous section, i.e.

$$\widehat{T}(z) = \int e^{i(x-z)\cdot\boldsymbol{\xi}} \tilde{Q}(z,\boldsymbol{\xi}) \eta(z,\boldsymbol{\xi}) A(z,\boldsymbol{\xi}) T(x) d\boldsymbol{\xi} dx + \int Q(z,\boldsymbol{\xi}) n(\boldsymbol{\xi}) d\boldsymbol{\xi}$$

where

$$\tilde{Q}(z,\boldsymbol{\xi}) = \frac{1}{A(z,\boldsymbol{\xi})\eta(z,\boldsymbol{\xi})}$$

where in this case we have assumed the data collection manifold is the entire $\boldsymbol{\xi}-$plane.

Observations

- We first conclude that in ‘ideal’ circumstances we can say the backprojected image in SAR is equivalent to the BLUE of the reflectivity function.
- By ‘ideal’ we mean a full data collection manifold and that the imaging plane is $\mathbb{R}^2$.
- In practice these ‘ideal’ conditions are never met and hence using these techniques do not result in a truly unbiased estimator of the reflectivity function (also there is the step when we ignore higher order terms after performing the Stolt change of variables).
- This does lead to interesting questions about how to find an unbiased estimator.

Bias

- If we return to the unbiased constraint from the BLUE calculation, note an unbiased estimator is defined by:

$$E[\widehat{T}(z)] = T(z) = \int K(x,z)T(x)dx$$

- We note this requires that K is a reproducing kernel or evaluator
- The question becomes does T lie in a reproducing kernel Hilbert space and can we find a kernel such that our estimator is unbiased?

RKHS definitions and background

- **Definition.** An evaluation functional over the Hilbert space of functions $\mathcal{H}$ is a linear functional $\mathcal{F}_t : \mathcal{H} \to \mathbb{R}$ that evaluates each function in the space at the point t, or

$$\mathcal{F}_t[f] = f(t), \quad \forall f \in \mathcal{H}.$$

- **Definition.** A Hilbert space $\mathcal{H}$ is a reproducing kernel Hilbert space (RKHS) if the evaluation functionals are bounded, i.e. if for all t there exists some $M > 0$ such that

$$|\mathcal{F}_t[f]| = |f(t)| \leq M||f||_{\mathcal{H}} \quad \forall f \in \mathcal{H}.$$

- **Theorem.** If $\mathcal{H}$ is a RKHS then for each $t \in X$ there exists a function $K_t \in H$ (called the representer of t or reproducing kernel) with the reproducing property

$$\mathcal{F}_t[f] = \langle K_t, f \rangle_{\mathcal{H}} = f(t) \quad \forall f \in \mathcal{H}.$$

- Note a reproducing kernel is symmetric and positive definite. Also an RKHS defines a corresponding RK and a RK defines a unique RKHS.

SAR in an RKHS framework

- We begin by supposing the reflectivity function we wish to reconstruct lies in the Hilbert space $L^2(Y)$ where Y is the imaging plane, for simplicity say it is the rectangle $Y = \{-a \leq x \leq a, -b \leq y \leq b\}$.
- Note the inner product on this space is given by

$$\langle f(x), g(x) \rangle_{L^2(Y)} = \int_Y f(x)\overline{g(x)}dx.$$

- We also note we may express the reflectivity in terms of its Fourier transform:

$$T(x) = \frac{1}{2\pi^2} \int_{\mathbb{R}^2} e^{-ix\cdot\xi} \widehat{T}(\xi) d\xi.$$

SAR in RKHS framework continued

- We now consider the SAR data expression, and observe it is the result of a linear operator $\mathcal{F}$ acting on T:
$$D(k,s) = \mathcal{F}[V(x)] = \int_Y e^{2ikR_{x,s}} A(x,s,k) T(x) dx$$

- The Reisz representation theorem states we may rewrite the data expression as an inner product of V with a unique element of $L^2(Y)$, i.e.
$$D(k,s) = \langle T(x), L(x;s,k) \rangle_{L^2(Y)}$$
where $L(x;s,k) = e^{-2ikR_{x,s}} \overline{A(x,s,k)}$.

SAR in RKHS framework continued

- Now consider the SAR image

$$I(z) = \int_{\Omega} Q(z, s, k) D(s, k) dsdk$$

where Q is to be determined and Ω is the data collection manifold.

- Note we may say $D \in L^2(\Omega)$, a Hilbert space, with the inner product

$$\langle D(s, k), P(s, k) \rangle_{L^2(\Omega)} = \int_{\Omega} D(s, k) \overline{P(s, k)} dsdk$$

- Now I is a linear operator acting on D and again by the Reisz representation theorem we have that

$$I(z) = \langle D(s, k), \overline{Q(z, s, k)} \rangle_{L^2(\Omega)}$$

SAR in RKHS framework continued

- Now inserting the data expression into the image we have

$$\begin{aligned} I(z) &= \langle D(s,k), \overline{Q(z,s,k)} \rangle_{L^2(\Omega)} \\ &= \langle \mathcal{F}[T(x)](s,k), \overline{Q(z,s,k)} \rangle_{L^2(\Omega)} \\ &= \langle T(x), \mathcal{F}^*[\overline{Q(z,s,k)}](x) \rangle_{L^2(Y)} \end{aligned}$$

  where $\mathcal{F}^*$ is the formal adjoint of the operator $\mathcal{F}$.

- We note that the formal adjoint is given by

$$\mathcal{F}^*[Q(z,s,k)](x) = \int_\Omega e^{-2ikR_{x,s}} \overline{A(x,s,k)} Q(z,s,k) ds dk$$

- Ideally we would have

$$I(z) = T(z)$$

$$\Rightarrow I(z) = \langle T(x), \mathcal{F}^*[\overline{Q(z,s,k)}](x) \rangle_{L^2(Y)} = T(z)$$

  which implies $\mathcal{F}^*[\overline{Q(z,s,k)}](x)$ should be the evaluator or reproducing kernel.

Finding the RK for $L^2(Y)$

- Now we seek the element $K_z(x) \in L^2(Y)$ such that

$$\langle T(x), K_z(x) \rangle_{L^2(Y)} = T(z)$$

- Therefore we consider the following integral equation:

$$\int_Y T(x)\overline{K_z(x)}dx = T(z)$$

which is equivalent to

$$\int_Y \left[\int_{\mathbb{R}^2} e^{-ix\cdot\xi}\widehat{T}(\xi)d\xi \right] \overline{K_z(x)}dx = \int_{\mathbb{R}^2} e^{-iz\cdot\xi}\widehat{T}(\xi)d\xi$$

Finding the RK for $L^2(Y)$ continued

- Rearranging the RHS we see that we require the following

$$\overline{\widehat{K}_z(\boldsymbol{\xi})} = e^{-iz\cdot\boldsymbol{\xi}}$$

or

$$K_z(x) = \int_{\mathbb{R}^2} e^{iz\cdot\boldsymbol{\xi}} e^{-ix\cdot\boldsymbol{\xi}} d\boldsymbol{\xi} = \delta(z-x)$$

- We note that $\delta(z-x) \notin L^2(Y)$ as it is not bounded, hence it is not possible to find Q such that $\mathcal{F}^*[\overline{Q}] = \delta(z-x)$.

- Note there are methods to obtain something 'close' to the delta function, i.e. the microlocal technique of Cheney and the Backus-Gilbert method.

A Different Hilbert space

- Let us now suppose that our reflectivity function lies in a different Hilbert space, say

$$H = \{T(x) \in L^2(\mathbb{R}^2) \mid supp(\widehat{T}(\boldsymbol{\xi})) \subseteq Y\}$$

  with the inner product

$$\langle T(x), f(x)\rangle_H = \int_{\mathbb{R}^2} T(x)\overline{f(x)}dx$$

- Note we have

$$T(x) = \frac{1}{2\pi^2}\int_Y e^{-ix\cdot\boldsymbol{\xi}}\,\widehat{T}(\boldsymbol{\xi})d\boldsymbol{\xi}$$

- Now again we look for $K_z(x)$ such that

$$\langle T(x), K_z(x)\rangle_H = T(z)$$

A Different Hilbert space continued

- We now consider the integral equation

$$\int_{\mathbb{R}^2} \left[\int_Y e^{-ix\cdot\xi} \widehat{T}(\xi) d\xi \right] \overline{K_z(x)} dx = \int_Y e^{-iz\cdot\xi} \widehat{T}(\xi) d\xi$$

- We find that

$$\begin{aligned} K_z(x) &= \frac{1}{2\pi^2} \int_Y e^{-iz\cdot\xi} e^{ix\cdot\xi} d\xi \\ &= \frac{ab}{\pi^2} \operatorname{sinc}(a(x_1 - z_1)) \operatorname{sinc}(b(x_2 - z_2)) \end{aligned}$$

supposing Y is a rectangle.

A different Hilbert space continued

- Now considering the SAR image we have

$$\begin{aligned} I(z) &= \int_{\Omega} Q(z,s,k)D(s,k)dsdk \\ &= \langle D(s,k), \overline{Q(z,s,k)} \rangle_{L^2(\Omega)} \\ &= \langle T(x), \mathcal{F}^*[\overline{Q(z,s,k)}](x) \rangle_H \\ &= T(z) \end{aligned}$$

- Therefore we see we require

$$\mathcal{F}^*[\overline{Q(z,s,k)}](x) = \frac{ab}{\pi^2}\mathrm{sinc}(a(x_1 - z_1))\mathrm{sinc}(b(x_2 - z_2))$$

Choosing Q

- We obtain the following integral equation for Q

$$\int_{\mathbb{R}^2} e^{-2ikR_{x,s}} \overline{A(x,s,k)Q(z,s,k)} \chi_\Omega(s,k) ds dk = \frac{1}{2\pi^2} \int_{\mathbb{R}^2} e^{-i(x-z)\cdot\boldsymbol{\xi}} \mathrm{rect}\left(\frac{\xi_1}{a}\right) \mathrm{rect}\left(\frac{\xi_2}{b}\right) d\boldsymbol{\xi}$$

  where χ_Ω is an indicator function that is one on the data collection manifold and zero elsewhere.

- Using a technique similar to that used in backprojection, we perform the Stolt change of variables on the LHS and let $\overline{Q(z,s,k)} = e^{2ikR_{z,s}} \tilde{q}(z,s,k)$

$$\int_{\mathbb{R}^2} e^{-i(x-z)\cdot\boldsymbol{\xi}} \overline{A(x,\boldsymbol{\xi})} \tilde{q}(z,\boldsymbol{\xi}) \chi_\Omega(\boldsymbol{\xi}) \eta(x,z,\boldsymbol{\xi}) d\boldsymbol{\xi}$$
$$= \frac{1}{2\pi^2} \int_{\mathbb{R}^2} e^{-i(x-z)\cdot\boldsymbol{\xi}} \mathrm{rect}\left(\frac{\xi_1}{a}\right) \mathrm{rect}\left(\frac{\xi_2}{b}\right) d\boldsymbol{\xi}$$

Choosing Q continued

- Using symbol calculus we may say the LHS of above is equivalent to the LHS of below plus higher order terms

$$\int e^{-i(x-z)\cdot\boldsymbol{\xi}}\overline{A(z,\boldsymbol{\xi})}\tilde{q}(z,\boldsymbol{\xi})\chi_\Omega(\boldsymbol{\xi})\eta(z,z,\boldsymbol{\xi})d\boldsymbol{\xi}$$
$$= \frac{1}{2\pi^2}\int_{\mathbb{R}^2} e^{-i(x-z)\cdot\boldsymbol{\xi}}\mathrm{rect}\left(\frac{\xi_1}{a}\right)\mathrm{rect}\left(\frac{\xi_2}{b}\right)d\boldsymbol{\xi}$$

- This implies we may choose $\tilde{q}$ to be

$$\tilde{q}(z,\boldsymbol{\xi}) = \frac{\mathrm{rect}\left(\frac{\xi_1}{a}\right)\mathrm{rect}\left(\frac{\xi_2}{b}\right)}{\overline{A(z,\boldsymbol{\xi})}\eta(z,z,\boldsymbol{\xi})\chi_\Omega(\boldsymbol{\xi})}$$

Further questions

- We see that the original filter Q is therefore given by:

$$Q(z, \xi) = e^{-z \cdot \xi} \frac{\operatorname{rect}\left(\frac{\xi_1}{a}\right) \operatorname{rect}\left(\frac{\xi_2}{b}\right)}{\overline{A(z, \xi)} \eta(z, z, \xi) \chi_\Omega(\xi)}$$

- We note this Q is still approximate because of our use of the symbol calculus to find only the first order term

- Also note that typically a, b or the support of the Fourier transform of the reflectivity function are unknown so actually implementing this filter in practice is not possible.

- Question: is there a RKHS that contains $T(x)$ for most scenarios in which we can find Q such that are image is exact, i.e. unbiased?

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