Westfälische
Wilhelms-Universität
Münster

Flexible frameworks for error control in (localized) Reduced Basis methods

Fully Certified, Adaptive and Localized Reduced Basis Methods

living.knowledge
WWU Münster

Ohlberger, Rave, Schindler

July 14th, 2016

acknowledgments

funding

- German Research Foundation (DFG) [OH 98/4-2]

  Multi-scale analysis of two-phase flow in porous media with complex heterogeneities
- German Federal Ministry of Education and Research (BMBF) [05M13PMA]

  MULTIBAT: Multiskalenmodelle und Modellreduktionsverfahren zur Vorhersage der Lebensdauer von Lithium-Ionen-Batterien
- University of Münster (WWU)
- Center for Nonlinear Science (WWU) CeNoS

collaboration

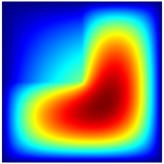
- B. Haasdonk, S. Kaulmann (University of Stuttgart)
- R. Milk, M. Ohlberger, S. Rave (WWU)

parametric problems

$$IO : \mathcal{P} \subset \mathbb{R}^4 \to H_0^1(\Omega)$$
$$\mu \mapsto p(\mu), \text{ s. t. } -\nabla \cdot \big(\lambda(\mu) \nabla p(\mu)\big) = f$$

- *many-query context* (optimization, UQ, ...)
- *real-time context* (embedded devices, ...)

parametric problems



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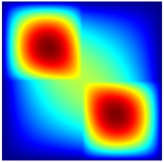

$\mu = (0.1, 1, 1, 0.1)$

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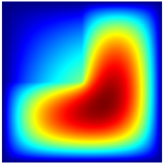
Ω

μ_1	μ_2
μ_3	μ_4

parametric problems



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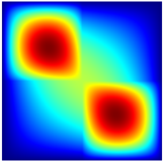

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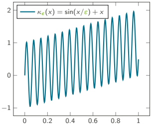
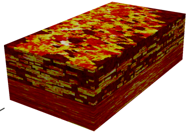
$\mu = (0.1, 1, 1, 0.1)$

& multi-scale problems



given $0 < \varepsilon \ll |\Omega|$, find $p_\varepsilon \in H^1_0(\Omega)$, s. t. $-\nabla\cdot\big(\kappa_\varepsilon \nabla p_\varepsilon\big) = f$

- κ_ε exhibits *strong oscillations* or *high contrast*
- grids have to resolve multi-scale features: $h < \varepsilon$



$\kappa_\varepsilon \in [7 \cdot 10^{-4}; 4 \cdot 10^4]$ (log)
spe.org/web/csp/datasets/set02.htm

elliptic parametric (multi-scale) problems

Let $\mu \in \mathcal{P}$ (a bounded set of admissible parameters $\mathcal{P} \subset \mathbb{R}^{p \in \mathbb{N}}$),

$$\text{find } p(\mu) \in H_0^1(\Omega): \qquad \underbrace{b(p(\mu), q; \mu)}_{\int_\Omega (\lambda(\mu)\kappa_\varepsilon \nabla p(\mu)) \cdot \nabla q \, \mathrm{d}x} = \overbrace{l(q)}^{\int_\Omega f q \, \mathrm{d}x} \qquad \forall q \in H_0^1(\Omega);$$

elliptic parametric (multi-scale) problems

Let $\mu \in \mathcal{P}$ (a bounded set of admissible parameters $\mathcal{P} \subset \mathbb{R}^{p \in \mathbb{N}}$),

$$\text{find } p(\mu) \in H_0^1(\Omega): \qquad b(p(\mu), q; \mu) = l(q) \qquad \forall q \in H_0^1(\Omega);$$

to be more precise: compute approximations

$$\tilde{p}(\mu) \in \tilde{Q}(\tau_h): \qquad b(\tilde{p}(\mu), \tilde{q}; \mu) = l(\tilde{q}) \qquad \forall \tilde{q} \in \tilde{Q}(\tau_h), \tag{1}$$

accurately, such that (for a prescribed tolerance $\Delta > 0$)

$$\|p(\mu) - \tilde{p}(\mu)\| < \Delta,$$

- in *real-time contexts*: for some $\mu \in \mathcal{P}$ as fast as possible ($\Rightarrow$ **online efficient**) or
- in *multi-query contexts*: for all parameters of interest $\mu \in \mathcal{P}_{\text{int}}$ as “cheap” as possible ($\Rightarrow$ **overall efficient**).

outline

introdcution

elliptic parametric (multi-scale) problems
- model reduction with reduced basis methods
- the localized reduced basis (multi-scale) method
- error control based on diffusive flux reconstruction
- adaptive online enrichment
- numerical experiments

parabolic parametric problems
- error control based on elliptic reconstruction
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summary

model reduction with reduced basis methods

[Fox, Miura, 1971]
[Maday, Patera, Turinici, 2002]

Idea: given a *multi-purpose high-dimensional* approximation space $Q_h(\tau_h)$,

[Kolmogoroff, 1936]
[Pinkus, 1985]

if $IO_h(\mathcal{P}) := \big\{ p_h(\mu) \text{ solution of (1)} \,\big|\, \mu \in \mathcal{P} \big\} \subset Q_h(\tau_h)$ "low-dimensional"

- find a **reduced space** $Q_{\text{red}} \subset Q_h(\tau_h)$ using a discrete weak **greedy** algorithm

model reduction with reduced basis methods

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[VEROY, PRUD'HOMME, PATERA, 2003]

given: model reduction error estimate $\|p_h(\mu) - p_{\text{red}}(\mu)\| \leq \eta_{\text{red}}(\mu)$, training parameters $\mathcal{P}_{\text{train}} \subset \mathcal{P}$

- start with $\phi_{\text{red}}^{(0)} := \emptyset, \quad Q_{\text{red}}^{(0)} := \text{span}(\phi_{\text{red}}^{(0)}), \quad n \leftarrow 0$
- find worst approximated parameter: $\mu_* \leftarrow \text{argmax}_{\mu \in \mathcal{P}_{\text{train}}} \eta_{\text{red}}\big(Q_{\text{red}}^{(n)}, \mu\big)$
- extend reduced basis: $\phi_{\text{red}}^{(n+1)} := \texttt{gram\_schmidt}(\phi_{\text{red}}^{(n)} \cup p_h(\mu_*)), \quad n \leftarrow n+1$

until $\max_{\mu \in \mathcal{P}_{\text{train}}} \eta_{\text{red}}(\mu) < \Delta_{\text{red}}$

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  [Veroy, Prud'homme, Patera, 2003]

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  [Binev, Cohen, Dahmen, DeVore, Petrova, Wojtaszczyk, 2011]
  [Cohen, DeVore, 2014]

- *low-dimensional* reduced space $Q_{red} \approx IO_h(\mathcal{P})$
  - e.g., $100 = n := \dim Q_{red} \ll \dim Q_h(\tau_h) =: N = 10^6$

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- *efficiency* by Galerkin projection and precomputation (if $b(p, q; \mu) = \sum \theta_\xi(\mu)\, b_{h,\xi}(p, q)$)

find $p_h(\mu) \in Q_h(\tau_h)$: $\quad b(p_h(\mu), q_h; \mu) = l_h(q_h) \quad \forall q_h \in Q_h(\tau_h)$ $\qquad \underline{b(\mu)} \in \mathbb{R}^{N \times N}$

$\underline{\phi_{\text{red}}} \cdot \underline{b_{h,\xi}} \cdot \underline{\phi_{\text{red}}}^\perp$

find $p_{\text{red}}(\mu) \in Q_{\text{red}}$: $\quad b(p_{\text{red}}(\mu), q_{\text{red}}; \mu) = l_h(q_{\text{red}}) \quad \forall q_{\text{red}} \in Q_{\text{red}}$ $\qquad \underline{b_{\text{red}}(\mu)} \in \mathbb{R}^{n \times n}$

accuracy vs. efficiency of RB methods

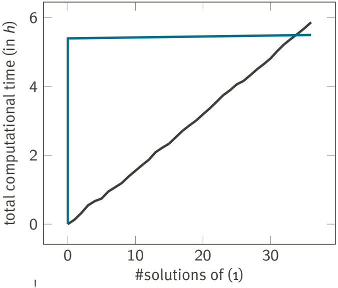
[A., HAASDONK, KAULMANN, OHLBERGER, 2012]

SWIPDG $N = 4.44 \cdot 10^6$

- offline: -
- online (avg. for one μ): $10min$

RB $n = 23$

- offline: $5.4h$
- online (avg. for one μ): $0.3ms$



$$\|p(\mu) - p_{\text{red}}(\mu)\| \leq \|p(\mu) - p_h(\mu)\| + \|p_h(\mu) - p_{\text{red}}(\mu)\| \overset{!}{<} \Delta$$

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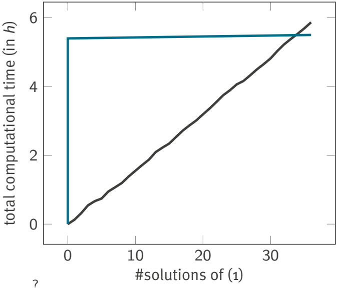

$$\|p(\mu) - p_{\text{red}}(\mu)\| \leq \underbrace{\|p(\mu) - p_h(\mu)\|}_{\leq\ ?} + \underbrace{\|p_h(\mu) - p_{\text{red}}(\mu)\|}_{\leq \eta_{\text{red}}(\mu)} \overset{?}{<} \Delta$$

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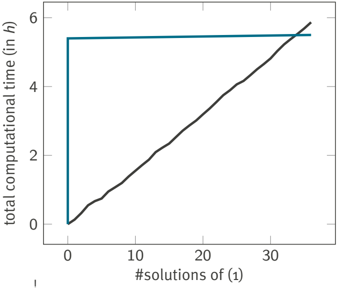

$$\|p(\mu) - p_{\text{red}}(\mu)\| \leq \underbrace{\|p(\mu) - p_h(\mu)\|}_{\leq \eta_h(\mu)} + \|p_h(\mu) - p_{\text{red}}(\mu)\| \overset{!}{<} \Delta$$

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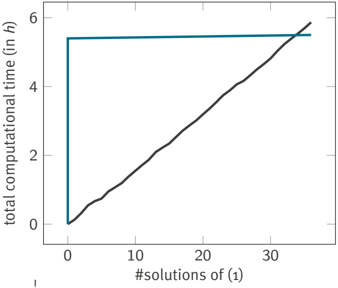


$$\|p(\mu) - p_{\text{red}}(\mu)\| \leq \|p(\mu) - p_h(\mu)\| + \|p_h(\mu) - p_{\text{red}}(\mu)\| \stackrel{!}{<} \Delta$$

challenges

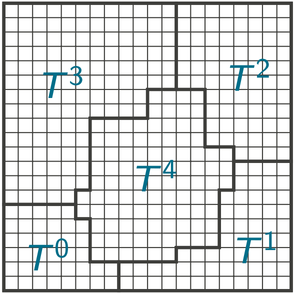
- offline: *"no way out of Q_h"* $\Longrightarrow$ [Ali, Steih, Urban, 2014], [Yano, 2015]
- online: *"no way out of Q_{red}"* $\Longrightarrow$ [Carlberg, 2015], [Ohlberger, S., 2015]
- offline computational complexity $\approx \varepsilon^{-l}$ (for multi-scale problems, with $l \in \mathbb{N}$)

outline

the localized reduced basis (multi-scale) method

Idea of the **LRBMS:** given a *multi-purpose highly-resolved* grid τ_h

- decompose approximation space into *local* spaces $Q_h(\tau_h) = \oplus_{T \in \mathcal{T}_H} Q_h^T$
- associated with *arbitrary* (connected) subdomains $T \in \mathcal{T}_H$



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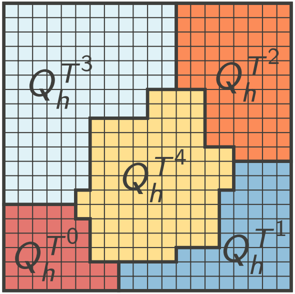
  independent local discretizations and approximation spaces (CG or DG)



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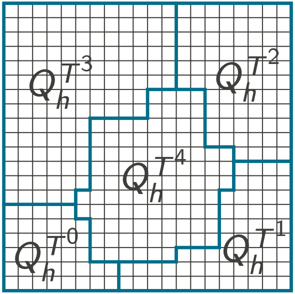

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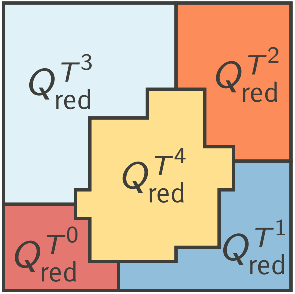
  independent local discretizations and approximation spaces (CG or DG) and global SWIPDG coupling [Ern, Stephansen, Zunino, 2009]
- build local reduced spaces $Q_{\text{red}}^T \subset Q_h^T$
- reduced *broken* space $Q_{\text{red}} = \bigoplus_{T \in \mathcal{T}_H} Q_{\text{red}}^T$



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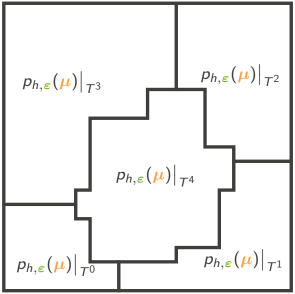
notes

- standard theory applicable
  (But: residual-based estimates are expensive!)
- greedy basis generation applicable
  (by localizing snapshots)



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summary

error control based on diffusive flux reconstruction

locally computable a-posteriori error estimate [Ohlberger, S., 2014]

[Karakashian, Pascal, 2003]

- nonconformity estimator: Oswald interpolation operator $I_{\mathrm{OS}}[\cdot] \in H_0^1(\Omega)$
  $\eta_{nc}^T(\tilde{p}(\mu); \overline{\mu}) := |||\tilde{p}(\mu) - I_{\mathrm{OS}}[\tilde{p}(\mu)]|||_{\overline{\mu}, T}$

[Ern, Stephansen, Vohralík, 2010]

- residual estimator: diffusive flux reconstruction $-\lambda(\mu)\kappa_\varepsilon \nabla_h \cdot \approx R_h[\cdot] \in H_{\mathrm{div}}(\Omega)$
  $\eta_r^T(\tilde{p}(\mu)) \quad := (C_P^T / c_\kappa^T)^{1/2} h_T \| f - \nabla \cdot R_h[\tilde{p}(\mu); \mu] \|_{L^2, T}$
- diffusive flux estimator:
  $\eta_{df}^T(\tilde{p}(\mu); \hat{\mu}) := \left\| (\lambda(\hat{\mu})\kappa_\varepsilon)^{-1/2} (\lambda(\mu)\kappa_\varepsilon \nabla_h \tilde{p}(\mu) + R_h[\tilde{p}(\mu); \mu]) \right\|_{L^2, T}$

error control based on diffusive flux reconstruction

locally computable a-posteriori error estimate [OHLBERGER, S., 2014]

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$$\Rightarrow \quad \|p(\mu) - \tilde{p}(\mu)\|_{\overline{\mu}} \leq \eta(\tilde{p}(\mu); \overline{\mu}, \hat{\mu}) := \frac{1}{\sqrt{\alpha(\mu, \overline{\mu})}} \Bigg[\sqrt{\gamma(\mu, \overline{\mu})} \Big[\sum_{T \in \mathcal{T}_H} \eta_{nc}^T(\tilde{p}(\mu))^2 \Big]^{1/2} + \Big[\sum_{T \in \mathcal{T}_H} \eta_r^T(\tilde{p}(\mu))^2 \Big]^{1/2} + \frac{1}{\sqrt{\alpha(\mu, \hat{\mu})}} \Big[\sum_{T \in \mathcal{T}_H} \eta_{df}^T(\tilde{p}(\mu); \hat{\mu})^2 \Big]^{1/2} \Bigg]$$

error control based on diffusive flux reconstruction

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- provides an estimate on
  - discretization error: $\tilde{p}(\mu) = p_h(\mu)$
  - full error: $\tilde{p}(\mu) = p_{\mathrm{red}}(\mu)$
- given $1 \in \tilde{Q}^T$
- $\alpha(\mu, \overline{\mu}) = \min \theta_\xi(\mu) \theta_\xi(\overline{\mu})^{-1}$

outline

adaptive online enrichment

challenges

- $|\tau_h| \approx \varepsilon^{-l}$ $\Rightarrow$ snapshots extremely costly
- limited computing power $\Rightarrow$ insufficient reduced space Q_{red}

LRBMS: offline

- initialize Q_{red}^T with DG basis of order up to $k \in \mathbb{N}$ for all $T \in \mathcal{T}_H$
- optional: greedy basis generation (using localized global snapshots), given available resources

$\rightsquigarrow$ proceed as in standard RB methods ...

adaptive online enrichment

LRBMS: online — for some $\mu \in \mathcal{P}_{\text{int}}$

- compute reduced solution $p_{\text{red}}(\mu)$
- estimate error $\eta_{h,\text{red}}(\mu)$
- if $\eta_{h,\text{red}}(\mu) > \Delta$, start intermediate local enrichment phase:

adaptive online enrichment

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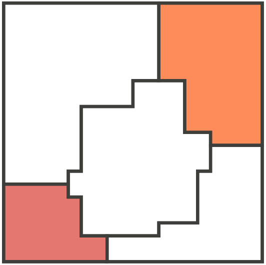
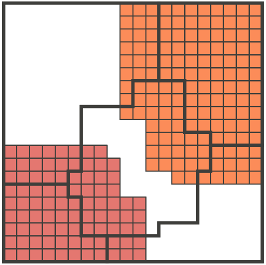
  - compute local error indicators
  - mark subdomains for enrichment: $\tilde{\mathcal{T}}_H = \mathtt{mark}(\mathcal{T}_H)$ (e.g., Dörfler and age)



adaptive online enrichment

LRBMS: online — for some $\mu \in \mathcal{P}_{\text{int}}$



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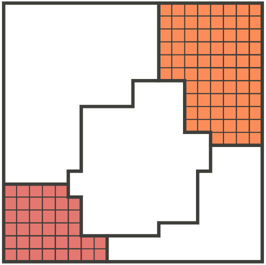
  - solve corrector problem on overlapping subdomain $T^\delta \supset T$ for all $T \in \tilde{\mathcal{T}}_H$:

$$
\begin{aligned}
b(\varphi_h(\mu), q_h; \mu) &= l_h(q_h) && \text{in } T^\delta \\
\varphi_h(\mu) &= p_{\text{red}}(\mu) && \text{on } \partial T^\delta
\end{aligned}
$$

adaptive online enrichment

LRBMS: online — for some $\mu \in \mathcal{P}_{\text{int}}$



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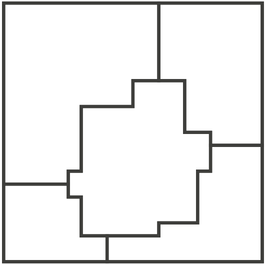
  - extend local reduced basis for all $T \in \tilde{\mathcal{T}}_H$:

$$Q_{\text{red}}^T := \mathtt{gram\_schmidt}(\{Q_{\text{red}}^T \cup \varphi_h(\mu)|_T\})$$

adaptive online enrichment

LRBMS: online — for some $\mu \in \mathcal{P}_{\text{int}}$



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$$Q_{\text{red}}^T := \texttt{gram\_schmidt}(\{Q_{\text{red}}^T \cup \varphi_h(\mu)|_T\})$$

  - update reduced quantities
  - compute updated reduced solution $p_{\text{red}}(\mu)$ and $\eta_{h,\text{red}}(\mu)$

adaptive online enrichment

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  - extend local reduced basis for all $T \in \tilde{\mathcal{T}}_H$:

$$Q_{\text{red}}^T := \mathtt{gram\_schmidt}(\{Q_{\text{red}}^T \cup \varphi_h(\mu)|_T\})$$

  - update reduced quantities

  $\mathcal{O}(\dim Q_h(T^\delta)^l)$ (for the three steps above)

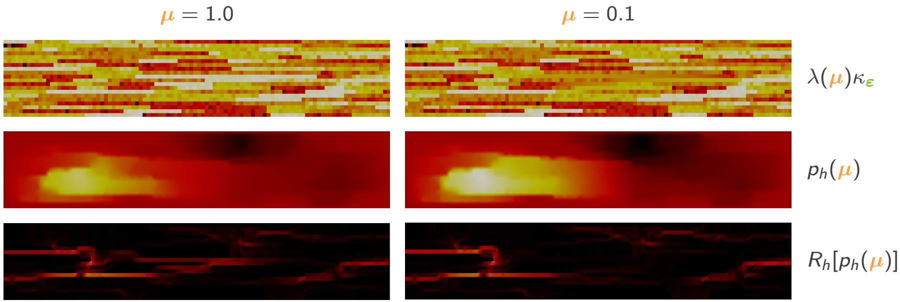
  - compute updated reduced solution $p_{\text{red}}(\mu)$ and $\eta_{h,\text{red}}(\mu)$
- iterate until $\eta_{h,\text{red}}(p_{\text{red}}(\mu)) \leq \Delta$, continue with next parameter

outline

experiment: spe10 model1

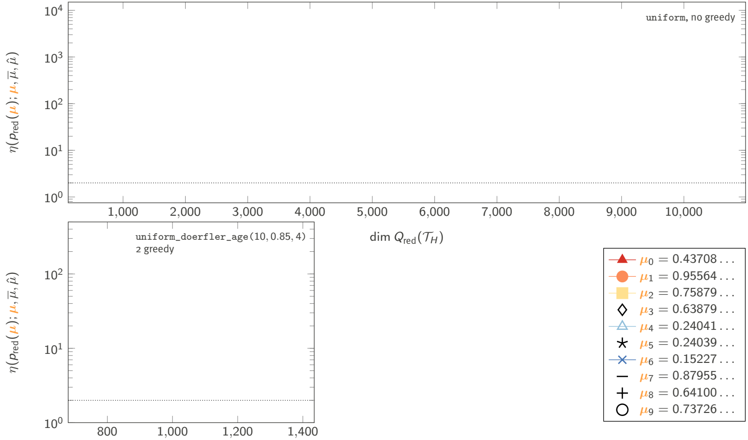
[OHLBERGER, S., 2015]

SPE10: $|\tau_h| = 10^6$, $|\mathcal{T}_H| = 25 \times 5$, $k = 1$



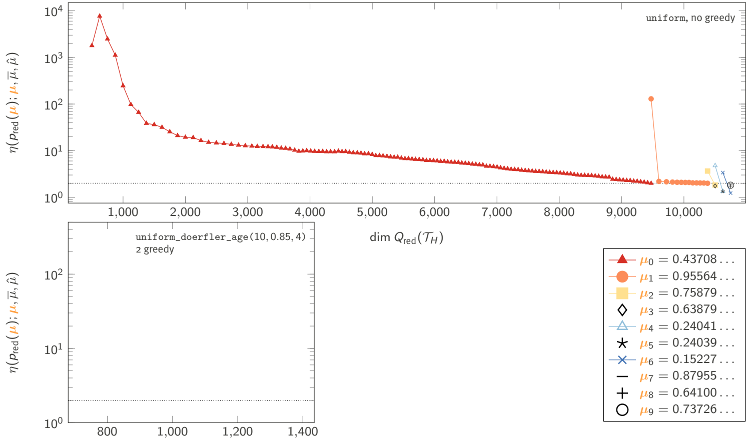
experiment: spe10 model1

[Ohlberger, S., 2015]



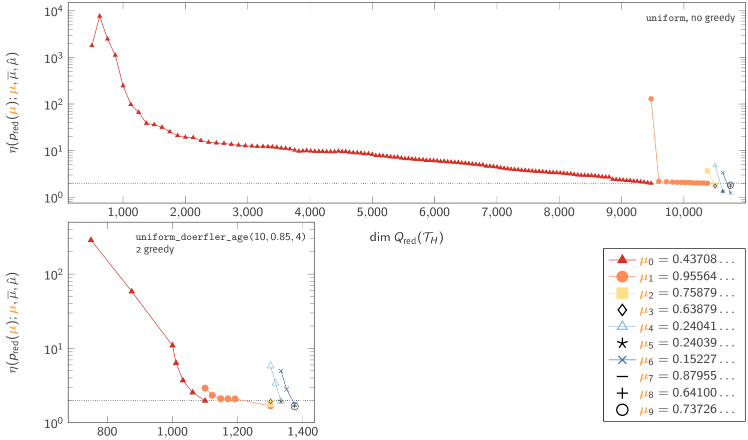
experiment: spe10 model1

[Ohlberger, S., 2015]



experiment: spe10 model1

[Ohlberger, S., 2015]

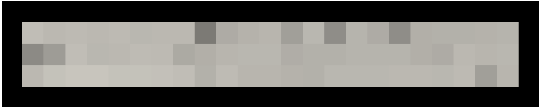


experiment: spe10 model1

[OHLBERGER, S., 2015]

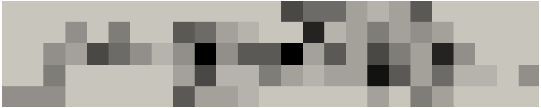
final local basis sizes

`uniform`, no greedy



[24, 148]

`uniform_doerfler_age`(10, 0.85, 4), 2 greedy



[9, 20]

outline

introdcution

elliptic parametric (multi-scale) problems
- model reduction with reduced basis methods
- the localized reduced basis (multi-scale) method
- error control based on diffusive flux reconstruction
- adaptive online enrichment
- numerical experiments

parabolic parametric problems
- error control based on elliptic reconstruction
- numerical experiments

summary

parabolic parametric problems

For a Gelfand triple $Q \subset H \subset Q'$, an end time $T > 0$ and $\mu \in \mathcal{P}$, find $p(\mu, \cdot) \in L^2(0, T; Q)$ with $\partial_t p(\mu, \cdot) \in L^2(0, T; Q')$, $p(\mu, 0) = p_0(\mu) \in Q$, s.t.

$$< \partial_t p(\mu, t), q > \; + b(p(\mu, t), q; \mu) = l(q) \qquad \forall q \in Q, t \in [0, T].$$

Consider $\tilde{Q} \subset H$, find an approximation $\tilde{p}(\mu, \cdot) \in L^2(0, T; \tilde{Q})$, $\partial_t \tilde{p}(\mu, \cdot) \in L^2(0, T; \tilde{Q})$, s.t.

$$\big(\partial_t \tilde{p}(\mu, t), \tilde{q}\big)_H + b(\tilde{p}(\mu, t), \tilde{q}; \mu) = l(\tilde{q}) \qquad \forall \tilde{q} \in \tilde{Q}.$$

parabolic parametric problems

For a Gelfand triple $Q \subset H \subset Q'$, an end time $T > 0$ and $\mu \in \mathcal{P}$, find $p(\mu, \cdot) \in L^2(0, T; Q)$ with $\partial_t p(\mu, \cdot) \in L^2(0, T; Q')$, $p(\mu, 0) = p_0(\mu) \in Q$, s.t.

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FEM/DG	RB	LRBMS
$Q = H^1_0(\Omega) \subset L^2(\Omega) = H$	$Q = H = Q_h(\tau_h)$	$Q = H^1_0(\Omega) \subset L^2(\Omega) = H$
$\tilde{Q} = Q_h(\tau_h) \subset L^2(\Omega)$	$\tilde{Q} = Q_{\mathrm{red}} \subset Q_h(\tau_h)$	$\tilde{Q} = Q_{\mathrm{red}} \subset Q_h(\tau_h)$

outline

elliptic reconstruction

[Georgoulis, Lakkis, Virtanen, 2011]
[Makridakis, Nochetto, 2003]

Given $\tilde{p} \in \tilde{Q}$, define the *elliptic reconstruction* $\mathcal{E}(\tilde{p}) \in Q$, as the solution of

Riesz-representative: $(\tilde{\omega}(\tilde{p}), \tilde{q})_H = b(\tilde{p}, \tilde{q}) \quad \forall \tilde{q} \in \tilde{Q}$

$$b(\mathcal{E}(\tilde{p}), q) = \big(\tilde{\omega}(\tilde{p}) + f - \tilde{\Pi}(f), q\big)_H \qquad \text{for all } q \in Q. \tag{2}$$

L^2-orthogonal projection onto $\tilde{Q}$

elliptic reconstruction

[Georgoulis, Lakkis, Virtanen, 2011]
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L^2-orthogonal projection onto $\tilde{Q}$

Proposition

$\tilde{p} \in \tilde{Q}$ is the $\tilde{Q}$-Galerkin approximation of the solution $\mathcal{E}(\tilde{p}) \in Q$ of (2).

$\Rightarrow$ We can estimate $\|\mathcal{E}(\tilde{p}) - \tilde{p}\|$ by *any* a-posteriori estimate on the elliptic problem (2)!

abstract estimate

[Ohlberger, Rave, S., 2016]

Let $\tilde{p} \in \tilde{Q}$ be arbitrary, let $C := (3|||b||| + 2)^{1/2}$ and C_Q, s.t. $\big\|\tilde{\Pi}(q)\big\|_H \leq C_Q |||q||| \quad \forall q \in Q$ and let $\mathcal{R}_T(\tilde{p}) \in \tilde{Q}$ denote the Riesz-representative of the time-stepping residual. Then

$$\begin{aligned}
\|p - \tilde{p}\|_{L^2(0,T,|||\cdot|||)} \leq & \quad \big\|p(0) - \tilde{p}^c(0)\big\|_H \\
& + \quad C \,\big\|\tilde{p}^d\big\|_{L^2(0,T;|||\cdot|||)} + 2\,\big\|\partial_t \tilde{p}^d\big\|_{L^2(0,T;|||\cdot|||_{-1})} \\
& + (C+1)\,\big\|\mathcal{E}(\tilde{p}) - \tilde{p}\big\|_{L^2(0,T;|||\cdot|||)} \\
& + \quad 2C_Q\,\big\|\mathcal{R}_T(\tilde{p})\big\|_{L^2(0,T;H)}
\end{aligned}$$

recall: $\begin{array}{cccc} \tilde{p} & = \tilde{p}^c & + \tilde{p}^d \\ \Cap & \Cap & \Cap \\ \tilde{Q} & Q & \tilde{Q} \end{array}$

abstract estimate

[Ohlberger, Rave, S., 2016]

Let $\tilde{p} \in \tilde{Q}$ be arbitrary, let $C := (3|||b||| + 2)^{1/2}$ and C_Q, s.t. $\|\tilde{\Pi}(q)\|_H \leq C_Q |||q||| \quad \forall q \in Q$ and let $\mathcal{R}_T(\tilde{p}) \in \tilde{Q}$ denote the Riesz-representative of the time-stepping residual. Then

$$\begin{aligned} \|p - \tilde{p}\|_{L^2(0,T,|||\cdot|||)} \leq \;& \|p(0) - \tilde{p}^c(0)\|_H \\ &+ C\,\|\tilde{p}^d\|_{L^2(0,T;|||\cdot|||)} + 2\,\|\partial_t \tilde{p}^d\|_{L^2(0,T;|||\cdot|||_{-1})} \\ &+ (C+1)\,\|\mathcal{E}(\tilde{p}) - \tilde{p}\|_{L^2(0,T;|||\cdot|||)} \\ &+ 2C_Q\,\|\mathcal{R}_T(\tilde{p})\|_{L^2(0,T;H)} \end{aligned}$$

recall: $\tilde{p} = \tilde{p}^c + \tilde{p}^d$, with $\tilde{p} \in \tilde{Q}$, $\tilde{p}^c \in Q$, $\tilde{p}^d \in \tilde{Q}$

- $\Rightarrow$ bound $|||\mathcal{E}(\tilde{p}(t)) - \tilde{p}(t)||| \leq \eta_{\text{ellip.}}(\tilde{p})$ by an elliptic estimate on (2)
- $\Rightarrow$ bound norms and constants, depending on spaces and time stepping
- $\Rightarrow$ localizable, if $\eta_{\text{ellip.}}$ is
- $\Rightarrow$ offline/online decomposable, if $\eta_{\text{ellip.}}$ is

[OHLBERGER, RAVE, S., 2016]

estimate on the discretization error DG

$\|p(\mu) - p_h(\mu)\| \leq \dots$

- [MAKRIDAKIS, NOCHETTO, 2003]
- [GEORGOULIS, LAKKIS, VIRTANEN, 2011]

estimate on the model reduction error RB

$\|p_h(\mu) - p_{\text{red}}(\mu)\|_{L^2(0,T;H^1(\Omega))} \leq \dots$

- use standard residual-based estimate on the elliptic reconstruction error

estimate on the full approximation error LRBMS, implicit euler

$$\begin{aligned}
\|p(\mu) - p_{\text{red}}(\mu)\|_{L^2(0,T,|||\cdot|||_{\overline{\mu}})} \leq \alpha(\mu,\overline{\mu})^{-1}\Big\{ & \left\|p(0) - p_{\text{red}}^c(0)\right\|_{L^2(\Omega)} \\
+ & \sqrt{5}\left\|p_{\text{red}}^d(\mu)\right\|_{L^2(0,T;|||\cdot|||_{\mu})} \\
+ & 2\,\alpha(\mu,\hat{\mu})^{-1} C_Q(\hat{\mu})\left\|\partial_t p_{\text{red}}(\mu)^d\right\|_{L^2(0,T;L^2(\Omega))} \\
+ & (\sqrt{5}+1)\left(4/3\,\Delta t \textstyle\sum_{n=0}^{nT}\eta(p_{\text{red}}(\mu,n\Delta t),\tilde{\mu})\right)^{1/2} \\
+ & 2\,\alpha(\mu,\hat{\mu})^{-1} C_Q(\hat{\mu})\left\|\mathcal{R}_T(p_{\text{red}}(\mu))\right\|_{L^2(0,T;L^2(\Omega))}\Big\}
\end{aligned}$$

[OHLBERGER, RAVE, S., 2016]

estimate on the discretization error DG

$\|p(\mu) - p_h(\mu)\| \leq \dots$

- [MAKRIDAKIS, NOCHETTO, 2003]
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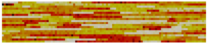
$$\|p(\mu) - p_{\mathrm{red}}(\mu)\|_{L^2(0,T,|||\cdot|||_{\overline{\mu}})} \leq \alpha(\mu,\overline{\mu})^{-1}\Big\{ \big\|p(0) - p^c_{\mathrm{red}}(0)\big\|_{L^2(\Omega)}$$
$$+ \sqrt{5}\,\big\|p^d_{\mathrm{red}}(\mu)\big\|_{L^2(0,T;|||\cdot|||_{\mu})}$$
$$+ 2\,\alpha(\mu,\hat{\mu})^{-1} C_Q(\hat{\mu})\,\big\|\partial_t p_{\mathrm{red}}(\mu)^d\big\|_{L^2(0,T;L^2(\Omega))}$$
$$+ \;(\sqrt{5}+1)\,\Big(4/3\,\Delta t \sum_{n=0}^{nT} \eta(p_{\mathrm{red}}(\mu, n\Delta t), \tilde{\mu})\Big)^{1/2}$$
$$+ 2\,\alpha(\mu,\hat{\mu})^{-1} C_Q(\hat{\mu})\,\big\|\mathcal{R}_T(p_{\mathrm{red}}(\mu))\big\|_{L^2(0,T;L^2(\Omega))}\Big\}$$

- $C_Q(\hat{\mu})$ = Poincaré constant/smalles EV
- time-residual can be exactly computed
- localizable, offline/online decomposable
- $\hat{\mu}, \overline{\mu}, \tilde{\mu} \in \mathcal{P}$ arbitrary

outline

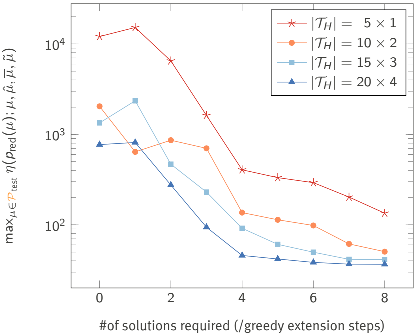
experiment: spe10 model1

[Ohlberger, Rave, S., 2016]



- LRBMS: $|\tau_h| = 8 \cdot 10^3$, SWIPDG in each subdomain
- implicit Euler: $T = 0.05$, $n_T = 10$, $p_0 = 0$
- $\hat{\mu} = \overline{\mu} = \tilde{\mu} = 0.1$
- initial local basis: $\varphi^T_{\text{red}} := \texttt{gram\_schmidt}(\{1, f|_T\})\ \forall T \in \mathcal{T}_H$
- POD-greedy [Haasdonk, Ohlberger, 2008] with localized trajectories
- $|\mathcal{P}_{\text{train}}| = 10$ (uniformly), $|\mathcal{P}_{\text{test}}| = 10$ (randomly)

error evolution during POD-greedy



outline

summary

LRBMS for the model reduction of parametric multi-scale problems

- flexible formulation (links to DG, RB, DD)
- potential for increased *overall* efficiency (compared to standard RB)
- efficient local error control of the *full* approximation error
- online adaptation of Q_{red} to cope with limited offline computing power
- p-adaptive RB

software http://pymor.org/

pyMOR [Rave, Milk, S.] (BSD-2-Clause)

- 21k LOC (since 2012), contributions: A. Buhr, M. Laier, F. Meyer, P. Mlinaric, M. Schäfer
- generic algorithms, based on abstract `VectorArray`, `Operator`, `Discretization` interfaces
- bindings to BEST, `deal.II`, `dune-gdt`, `fenics`, `ngsolve`