Primal and Mixed Finite Element Methods for Elliptic PDEs in Non–Divergence Form

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Outline

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PDEs in Non–Divergence Form

Model Problem

$$\mathcal{L}u := -A : D^2u = f \qquad \text{in } \Omega \subset \mathbb{R}^n, \tag{1a}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{1b}$$

where

- D^2u denotes the Hessian matrix, e.g.,

$$D^2u = \begin{pmatrix} \dfrac{\partial^2 u}{\partial x_1^2} & \dfrac{\partial^2 u}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 u}{\partial x_1 \partial x_2} & \dfrac{\partial^2 u}{\partial x_2^2} \end{pmatrix} \qquad (n = 2).$$

- $A = A(x) \in \mathbb{R}^{n\times n}$ is SPD, but non–smooth.
- $A : D^2u := \sum_{i,j=1}^n A_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j}$ is the Frobenius inner product.

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Application I

- Linearization of fully nonlinear operators give rise to problems of the form (1).
- For example, the linearization of the Monge–Ampère operator $v \to \det(D^2v)$ at u is

$$\mathrm{cof}(D^2u) : D^2v.$$

- Convergence analysis of numerical schemes for fully nonlinear problems and iterative solvers.

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Application II

- The stationary Hamilton–Jacobi–Bellman problem is given by

$$\sup_{\alpha \in \mathcal{A}} \left(\mathcal{L}_\alpha u - f_\alpha \right) = 0.$$

- $\{\mathcal{L}_\alpha\}_{\alpha \in \mathcal{A}}$ denotes a family of elliptic operators in non-divergence form with non–smooth coefficients.
- The solution of the HJB problem characterizes the supremum of the cost function associated with the optimally controlled stochastic process.

Solution Concepts

- Since A is non–smooth, a variational formulation using integration–by–parts is not viable. Therefore, other solution concepts are needed.
- Classical (Schauder): If $A \in [C^{0,\alpha}(\Omega)]^{n\times n}$ and $\partial\Omega \in C^{2,\alpha}$, there exists a unique $u \in C^{2,\alpha}(\Omega)$ satisfying the PDE.
- Strong (Calderón–Zygmund): If $A \in [C^{0}(\Omega)]^{n\times n}$ and $\partial\Omega \in C^{1,1}$ or if Ω is convex, there exists a unique $u \in H^2(\Omega)$ satisfying the PDE a.e. Moreover,

$$\|v\|_{H^2(\Omega)} \le C\|\mathcal{L}v\|_{L^2(\Omega)} \quad \forall v \in H^2(\Omega) \cap H^1_0(\Omega).$$

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Remark

If $A \in [L^\infty(\Omega)]^{n\times n}$, then uniqueness is generally lost for $n \ge 3$. For example, consider

$$A = I_{n\times n} + \Big(-1 + \frac{n-1}{1-\lambda}\Big)\frac{xx^T}{|x|^2}, \qquad \Omega = B_1(0)$$

with $n > 2(2-\lambda)$. Then $u = (|x|^\lambda - 1) \in H^2(\Omega)\cap H_0^1(\Omega)$ satisfies $-A : D^2u = 0$.

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Primal Finite Element Method: Derivation

The Obvious Difficulty

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- Assume for the moment that A is smooth (e.g., $A \in [W^{1,\infty}(\Omega)]^{n\times n}$).
- Write $f = -A : D^2 u = -\nabla \cdot (A\nabla u) + (\nabla \cdot A) \cdot \nabla u$.
- A finite element method reads

$$\int_\Omega (A\nabla u_h) \cdot \nabla v_h \, dx + \int_\Omega \big((\nabla \cdot A) \cdot \nabla u_h\big) v_h \, dx = \int_\Omega f v_h \, dx \quad \forall v_h \in X_h,$$

where $X_h = \{v_h \in H_0^1(\Omega) : \ v_h|_T \in \mathbb{P}_k(T) \ \forall T \in \mathcal{T}_h\}$ is the Lagrange finite element space.

Primal Finite Element Method: Derivation

$$\int_\Omega (A\nabla u_h)\cdot\nabla v_h\,dx + \int_\Omega \big(({\color{red}(\nabla\cdot A)})\cdot\nabla u_h\big)v_h\,dx = \int_\Omega f v_h\,dx \quad \forall v_h\in X_h.$$

- Integration-by-parts gives

$$\int_\Omega (A\nabla u_h)\cdot\nabla v_h\,dx = -\int_\Omega (A : D_h^2 u_h)v_h\,dx + \sum_{e\mathcal{E}_h^I}\int_e [\![A\nabla u_h]\!]v_h\,ds - \int_\Omega \big((\nabla\cdot A)\cdot\nabla u_h\big)v_h\,dx.$$

where D_h^2 is the piecewise Hessian operator, $\mathcal{E}_h^I$ denotes the set of interior edges/faces in $\mathcal{T}_h$, and $[\![\cdot]\!]$ is the "jump" operator.

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Primal Finite Element Method: Find $u_h \in X_h$ such that for all $v_h \in X_h$

$$\langle \mathcal{L}_h u_h, v_h\rangle := -\int_\Omega (A : D_h^2 u_h)v_h\,dx + \sum_{e\in\mathcal{E}_h^I}\int_e [\![A\nabla u_h]\!]v_h\,dx = \int_\Omega f v_h\,dx. \quad (2)$$

Mixed Finite Element Method: Derivation

- Define the matrix–valued, discontinuous finite element space

$$\Sigma_h := \{\sigma_h \in [L^2(\Omega)]^{n\times n} : \ \sigma_h|_T \in [\mathbb{P}_k(T)]^{n\times n}\ \forall T \in \mathcal{T}_h\}.$$

Finite Element Hessian

Define $\mathbb{H}_h : X_h \to \Sigma_h$ such that

$$\int_\Omega \mathbb{H}_h(v_h) : \mu_h\, dx = -\int_\Omega \nabla v_h \cdot (\nabla_h \cdot \mu_h)\, dx + \sum_{e\in\mathcal{E}_h} \int_e \{\!\{\nabla v_h\}\!\} \cdot [\![\mu_h]\!]\, ds$$

for all $\mu_h \in \Sigma_h$.

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Mixed Finite Element Method: Find $u_h \in X_h$ such that for all $v_h \in X_h$

$$\langle \mathcal{L}_h u_h, v_h \rangle := -\int_\Omega (A : \mathbb{H}_h(u_h)) v_h \, dx = \int_\Omega f v_h \, dx. \tag{3}$$

Remark

Both methods are relatively simple and can be implemented in current finite element software packages (e.g., FEniCS, deal.II, Comsol, Dune, etc.)

Remark

If A is constant, then both methods reduce to

$$\int_\Omega (A\nabla u_h)\cdot\nabla v_h\,dx = \int_\Omega f v_h\,dx \qquad \forall v_h \in X_h.$$

Remark

The method is consistent in the sense that

$$\langle \mathcal{L}_h u, v_h\rangle = \int_\Omega f v_h\,dx + \langle \mathcal{R}_h u, v_h\rangle \quad \forall v_h \in X_h$$

with $\mathcal{R}_h u \to 0$ in X_h' as $h \to 0^+$. For the primal method, $\mathcal{R}_h u = 0$.

Outline

Convergence Analysis

- The methods are consistent, and the problem is linear. Therefore, existence and error estimates all reduce to the stability of the schemes.

Key Idea

Mimic the a priori estimate

$$\|v\|_{H^2(\Omega)} \lesssim \|\mathcal{L}v\|_{L^2(\Omega)} \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{4}$$

- A proof of the elliptic estimate (4) relies on the following observation.

Locally (e.g., in a small ball), the PDE behaves like a PDE with constant coefficients (The discretization inherits the same behavior).

- This observation leads to a Garding inequality

$$\|v\|_{H^2(\Omega)} \lesssim \|\mathcal{L}v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega).$$

Convergence Analysis Based on the Discrete Adjoint Problem

- Define $\mathcal{L}_h^* : V_h \to V_h'$ to be the discrete adjoint operator of $\mathcal{L}_h$, i.e.,

$$\langle \mathcal{L}_h^* w_h, v_h \rangle = \langle \mathcal{L}_h v_h, w_h \rangle.$$

- We obtain stability estimates of $\mathcal{L}_h$ by deriving stability estimates of $\mathcal{L}_h^*$.

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- We obtain stability estimates of $\mathcal{L}_h$ by deriving stability estimates of $\mathcal{L}_h^*$.
- The first step is the following local stability estimate for finite element functions with small support.

Lemma (Local Stability)

There exists $r_\dagger > 0$ independent of h such that for $D \subset \Omega$ with $\mathrm{diam}(D) \le r_\dagger$, there holds

$$\|w_h\|_{L^2(D)} \lesssim \sup_{v_h \in X_h(D_h)\setminus\{0\}} \frac{\langle \mathcal{L}_h^* w_h, v_h \rangle}{\|v_h\|_{H_h^2(D_h)}} \quad \forall w_h \in X_h(D),$$

with $D_h = \{x \in \Omega : \ \mathrm{dist}(x, D) \le 2h\}$.

Convergence Analysis Based on the Discrete Adjoint Problem

- The local stability estimate leads to a Garding–type inequality.

Lemma (Garding Inequality)

There holds

$$\|w_h\|_{L^2(\Omega)} \lesssim \sup_{v_h \in X_h \setminus \{0\}} \frac{\langle \mathcal{L}_h^* w_h, v_h \rangle}{\|v_h\|_{H_h^2(\Omega)}} + \|w_h\|_{H^{-1}(\Omega)} \quad \forall w_h \in X_h.$$

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Lemma (Garding Inequality)

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Lemma (Duality Argument)

Suppose that $k \geq 2$. Then for any $\varepsilon > 0$, there exists $h_\varepsilon > 0$ such that for $h \leq h_\varepsilon$, there holds

$$\|w_h\|_{H^{-1}(\Omega)} \lesssim \sup_{v_h \in X_h \setminus \{0\}} \frac{\langle \mathcal{L}_h^* w_h, v_h \rangle}{\|v_h\|_{H_h^2(\Omega)}} + \varepsilon \|w_h\|_{L^2(\Omega)} \quad \forall w_h \in X_h.$$

Convergence Analysis: Main Results

Theorem

There holds, for h sufficiently small and for $k \geq 2$,

$$\|w_h\|_{L^2(\Omega)} \lesssim \sup_{v_h \in X_h \setminus \{0\}} \frac{\langle \mathcal{L}_h^* w_h, v_h \rangle}{\|v_h\|_{H_h^2(\Omega)}} \quad \forall w_h \in X_h.$$

Consequently,

$$\|v_h\|_{H_h^2(\Omega)} \lesssim \sup_{w_h \in X_h \setminus \{0\}} \frac{\langle \mathcal{L}_h v_h, w_h \rangle}{\|w_h\|_{L^2(\Omega)}} \quad \forall v_h \in X_h.$$

Corollary

There exists a unique solution to the finite element methods. Moreover, the error satisfies

$$\|u - u_h\|_{H_h^2(\Omega)} \leq C h^{s-1} \|u\|_{H^{s+1}(\Omega)}$$

provided $u \in H^{s+1}(\Omega)$ with $1 \leq s \leq k$.

Outline

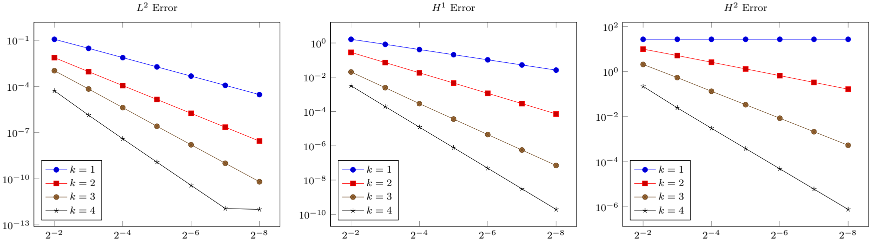
Numerical Test I.

The domain is the unit square $\Omega = (0,1)^2$ and the coefficient matrix is

$$A_{ii} = |\sin(4((x_i - 0.5))\pi))|^{1/5} + 1, \qquad A_{i,j} = \cos(2x_1x_2\pi)\ (i \neq j).$$

The data is chosen such that that exact solution is

$$u = \frac{x_1x_2\sin(2\pi x_1)\sin(3\pi x_2)}{(x_1^2 + x_2^2 + 1)}$$



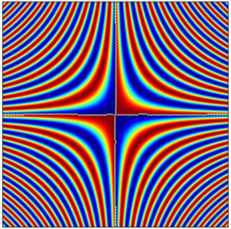
$$\|u - u_h\|_{H_h^2(\Omega)} = \mathcal{O}(h^{k-1}), \qquad \|\nabla(u - u_h)\|_{H^1(\Omega)} = \mathcal{O}(h^k),$$

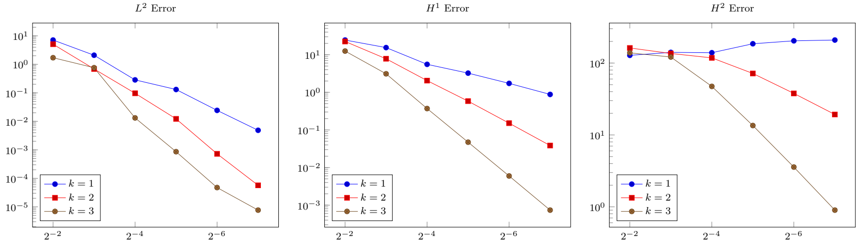
$$\|u - u_h\|_{L^2(\Omega)} = \mathcal{O}(h^{k+\alpha}),\ \exists \alpha \in (0,1).$$

Numerical Test II.

The domain is the square $\Omega = (-1,1)^2$, $A_{ii} = 2$,

$$A_{12} = A_{21} = \sin\left(\pi(20x_1x_2 + 1/2)\right)\frac{x_1x_2}{|x_1||x_2|}.$$





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$$\|u - u_h\|_{L^2(\Omega)} = \mathcal{O}(h^{k+\alpha}),\ \exists \alpha \in (0,1).$$

Numerical Test III.

Consider the Hamilton–Jacobi–Bellman problem

$$\sup_{\alpha\in\mathcal{A}} \left(\mathcal{L}_\alpha u - f_\alpha\right) = 0 \qquad \text{in } \Omega := (0,1)^2,$$

$$u = 0 \qquad \text{on } \partial\Omega$$

with

$$u = x_1 x_2 (1-x_1)(1-x_2)((x_1-0.5)^2 + (x_2-0.5)^2)^{3/4},$$

$$\mathcal{L}_\alpha u := -A_\alpha : D^2 u, \qquad A_\alpha = \alpha^T C \alpha, \quad C = \begin{pmatrix} 20 & 1 \\ 1 & 0.1 \end{pmatrix}$$

and $\mathcal{A}$ is the set of rotation matrices.

h	$\|\nabla(u-u_h)\|_{L^2}$	rate
2^{-1}	$7.51E-02$	
2^{-2}	$4.60E-02$	0.71
2^{-3}	$2.50E-02$	0.88
2^{-4}	$1.31E-02$	0.93
2^{-5}	$6.75E-03$	0.96
2^{-6}	$3.49E-03$	0.95

Outline

Conclusions

- Developed simple primal and mixed finite element methods for elliptic PDEs in non–divergence form.
- Optimal order error estimates in a H^2–type norm.
- The convergence analysis provides local error estimates for free.
- Analysis may be extended to $p \neq 2$ case and other finite element methods (e.g., DG methods).
- Numerical experiments for HJB problem look promising.

Future Work

- Discontinuous coefficients, $k = 1$, error estimates in lower order norms.
- Revisit finite element convergence theory for fully nonlinear problems.
- Convergence theory of HJB problem.