Volume Integral Methods for Waves in Plasmas

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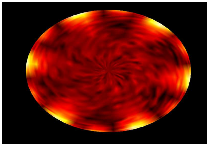


- Linear wave propagation in inhomogeneous media
- Continuous transition from propagating to evanescent medium
- Accurate numerical simulation of this transition

Waves in plasmas

Charged particles

- Various possible models
  - Kinetic or fluid description

Electro-Magnetic fields

- Maxwell's equations
  - Electric and Magnetic fields

Coupling

$\Longmapsto$ Electric currents

$\Longleftarrow$ Lorentz force

Road map to the cold plasma model

- Single fluid model with no thermal velocity
- Linearize the particle model for $|\mathbf{B}_0| >> 1$
- Time-harmonic regime
- Eliminate the current and the magnetic field

The cold plasma mathematical model

Maxwell's time-harmonic equations

$$curl\ curl\ \mathbf{E} - \left(\frac{\omega}{c}\right)^2 \varepsilon \mathbf{E} = 0$$

- $\varepsilon(x) = \begin{pmatrix} 1 - \frac{\omega_p^2(x)}{\omega^2 - \omega_c^2} & i\frac{\omega_c \omega_p^2(x)}{\omega(\omega^2 - \omega_c^2)} & 0 \\ -i\frac{\omega_c \omega_p^2(x)}{\omega(\omega^2 - \omega_c^2)} & 1 - \frac{\omega_p^2(x)}{\omega^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2(x)}{\omega^2} \end{pmatrix}$
- $\omega_p^2(x) = \frac{e^2 n_e(x)}{\varepsilon_0 m_e}$ plasma frequency
- $n_e(x)$ plasma density
- $\omega_c^2 = \frac{-eB_0}{m_e}$ cyclotron frequency

[Stix] **A general analysis of this model is able to provide a surprisingly comprehensive view of plasma waves.**

Dispersion relation : Cutoffs and Resonances

$$\varepsilon(x) = \begin{pmatrix} 1 - \frac{\omega_p^2(x)}{\omega^2 - \omega_c^2} & i\frac{\omega_c \omega_p^2(x)}{\omega(\omega^2-\omega_c^2)} & 0 \\ -i\frac{\omega_c \omega_p^2(x)}{\omega(\omega^2-\omega_c^2)} & 1 - \frac{\omega_p^2(x)}{\omega^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2(x)}{\omega^2} \end{pmatrix}$$

Suppose that the coefficients are slowly varying, and look for plane wave solutions :

$$\mathbf{E}(\mathbf{x}) = \mathbf{p}\ e^{\mathbf{i}(\mathbf{k},\mathbf{x})}, \text{ and } \mathbf{p} \in \mathbb{C}^3.$$

Condition for propagation : $\omega \in \mathbb{R}$ for $\mathbf{k} \in \mathbb{R}^3$.

Definition : Cutoff $\{k = 0\}$.
$\Rightarrow \omega_p^2 = \omega^2 + \eta\omega\omega_c, \qquad \eta = -1, 0, 1.$

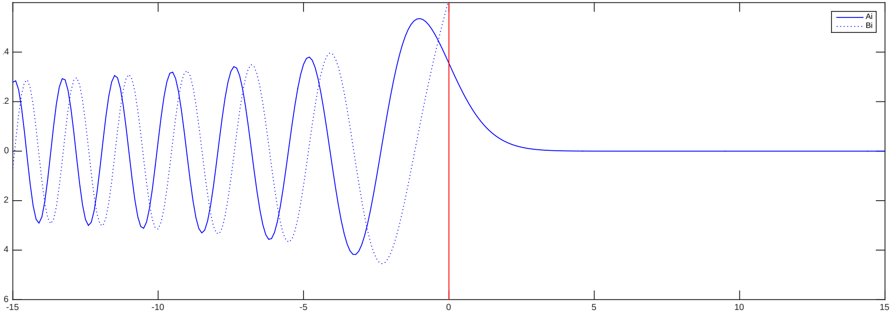
Definition : Resonance $\{k = \infty\}$.
$\Rightarrow \omega_p^2 = \omega^2 - \omega_c^2.$

Perpendicular propagation, parallel polarization
O-mode equation and CutOff

2D Helmholtz equation for the total field

- $-\Delta u - \frac{\omega^2}{c^2}(1 - Cn_e(\mathbf{x}))u = 0$
- smooth variable coefficient : Cutoff $\Leftrightarrow 1 - Cn_e(\mathbf{x}) = 0$

Airy function in 1D : $-u'' + xu = 0$



✓ $n_e < 1/C$ $\Rightarrow$ Propagating waves

▸ $n_e = 1/C$ $\Rightarrow$ Cutoff

✗ $n_e > 1/C$ $\Rightarrow$ Evanescent waves

The scattering problem

2D Helmholtz equation for the total field

- $-\Delta u - \frac{\omega^2}{c^2}(1 - Cn_e(\mathbf{x}))u = 0$
- smooth variable coefficient
  $sign = \pm 1$, Cutoff $\Leftrightarrow 1 - Cn_e(\mathbf{x}) = 0$

Scattering by a penetrable medium

$$\Delta u(\mathbf{x}) + \kappa^2(1 + q(\mathbf{x}))u(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathbb{R}^2$$

$$u = u^{\mathrm{i}} + u^{\mathrm{s}}$$

the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial u^{\mathrm{s}}}{\partial r} - i\kappa u^{\mathrm{s}}\right) = 0 \quad r = |\mathbf{x}|$$

Numerical methods for scattering by a penetrable medium

- Barnett et al. - uniform discretization
  - '15 A spectrally accurate direct solution technique for frequency-domain scattering problems with variable media
- Bruno et al. - large patches
  - '03 Wave scattering by inhomogeneous media : efficient algorithms and applications
  - '04 An efficient, preconditioned, high-order solver for scattering by 2D inhomogeneous media
  - '05 Higher-order Fourier approximation in scattering by two-dimensional, inhomogeneous media
- Fast solvers for dense linear algebra
  - block decomposition of a matrix
  - algebraic/analytic low rank block compression
  - '13 An $O(N \log N)$ fast direct solver for partial hierarchically semi-separable matrices — with application to radial basis function interpolation [Ambikasaran, Darve]

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Novelty

- Adaptive discretization

Challenges

- Integral formulation
- Implementation

Integral formulation : Lippmann-Schwinger Equation II

- Free space Green's function $\Phi(\mathbf{x},\mathbf{y}) = \frac{\imath}{4} H_0^{(1)}(\kappa|\mathbf{x}-\mathbf{y}|)$

$$\Delta\Phi + \kappa^2\Phi = \delta_{\mathbf{x}-\mathbf{y}} \quad + \text{ Sommerfeld condition}$$

- Volume potential representation

$$\begin{aligned}\left(\Delta + \kappa^2(1+q)\right)\int\Phi\rho &= \left(\Delta+\kappa^2\right)\int\Phi\rho + \kappa^2 q\int\Phi\rho \\ &= \rho + \kappa^2 q\int\Phi\rho\end{aligned}$$

$$u^s = \int_{\mathbb{R}^2}\Phi(\mathbf{x},\mathbf{y})\rho(\mathbf{y})\,d\mathbf{y}$$

$$\left(\Delta + \kappa^2(1+q)\right)u^s = -\left(\Delta+\kappa^2\right)u^i - \kappa^2 q u^i$$

$$\Rightarrow \rho(\mathbf{x}) + \kappa^2 q(\mathbf{x})\int_{\mathbb{R}^2}\Phi(\mathbf{x},\mathbf{y})\rho(\mathbf{y})\,d\mathbf{y} = -\kappa^2 q(\mathbf{x})u^i(\mathbf{x})$$

Lippmann-Schwinger Equation
Summary

$$\mathcal{V} : f \mapsto \int_{\mathbb{R}^2} \Phi(\cdot, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y}$$

I Green's theorem

$$(I + \kappa^2 \mathcal{V}(q\cdot))u = u^i$$

- ✗ Integral kernel depending on q

II Volume potential representation

$$\begin{cases} (I + \kappa^2 q\mathcal{V})\rho = -\kappa^2 q u^i \\ u^s = \mathcal{V}\rho \end{cases}$$

- ✓ Integral kernel translation invariant
- ✓ Compact support of the unknown
- ✓ Easy computation of ∇u

Difficulty

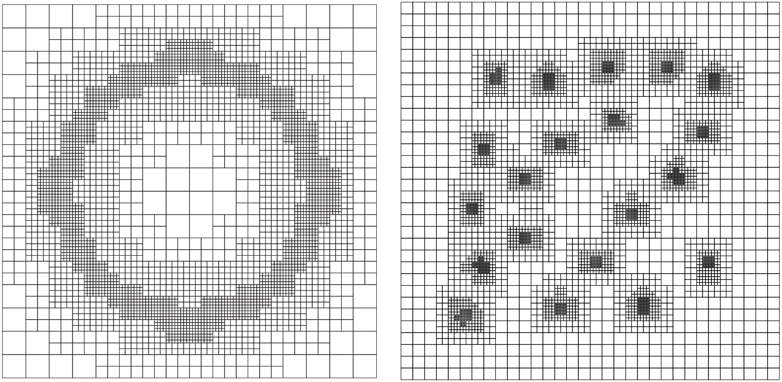
- ➢ Rapid, high order discretization of the volume integral

[Ambikasaran, Borges, IG, Greengard, '16]

Adaptive discretization : Quad-tree structure

Resolve
- the contrast q
- the local wavenumber $\kappa\sqrt{1+q}$
- the background wavenumber κ

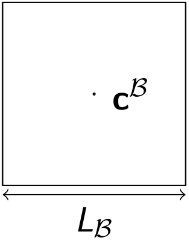
User-specified parameter ε



Integral discretization and collocation method

Leaf node of the tree structure

Notation $\mathcal{B}$



For mth order accuracy

Translation invariance

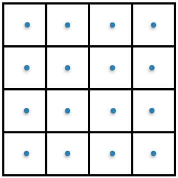
- mth order polynomial approximation

$$\left\{ \left(\frac{\mathbf{x}_1 - \mathbf{c}_{\mathcal{B},1}}{L_{\mathcal{B}}} \right)^j \left(\frac{\mathbf{x}_2 - \mathbf{c}_{\mathcal{B},2}}{L_{\mathcal{B}}} \right)^k, 0 \leq j + k \leq m - 1 \right\}$$

Nystrom discretization

For mth order accuracy

- $m \times m$ grid points on each box



Integral computation

Generic domain of integration

P_ℓ being the ℓth monomial of degree $\leq m-1$

$$\begin{aligned} V_\kappa(\mathbf{x},\ell) &= \int_{\mathcal{B}} H_0^{(1)}(\kappa|\mathbf{x}-\mathbf{y}|)P_\ell\left(\frac{\mathbf{y}-\mathbf{c}_{\mathcal{B}}}{L_{\mathcal{B}}}\right)d\mathbf{y} \\ &= \int_{\mathcal{B}_0} H_0^{(1)}(\kappa L_{\mathcal{B}}|\xi-\eta|)P_\ell(\eta)(L_{\mathcal{B}})^2 d\eta \end{aligned}$$

$$\begin{cases} \xi = \dfrac{\mathbf{x}-\mathbf{c}_{\mathcal{B}}}{L_{\mathcal{B}}} \text{ is called the target} \\ \eta = \dfrac{\mathbf{y}-\mathbf{c}_{\mathcal{B}}}{L_{\mathcal{B}}} \text{ is called the source} \qquad \eta \in \mathcal{B}_0 = [-0.5, 0.5]^2 \end{cases}$$

$\Rightarrow$ Only $L_{\mathcal{B}}$ and ξ depend on $\mathcal{B}$

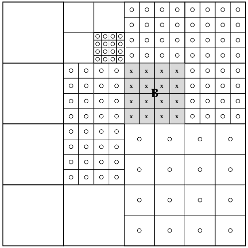
But what about the singularity of $H_0^{(1)}$?

Identify two regimes : near and far field interactions

Local Interactions Level restricted tree

Expansion of the Hankel function

$$H_0^{(1)}(\alpha r) = \sum_{p=0}^{\infty} c_p \alpha^{2p} \left(\frac{r}{2}\right)^{2p} + \sum_{p=0}^{\infty} d_p \alpha^{2p} \left(\frac{r}{2}\right)^{2p} \log\left(\frac{r}{2}\right)$$



Precomputation independent of κ

$$V_\kappa(\mathbf{x}, \ell) \approx \frac{iL_{\mathcal{B}}^2}{4} \sum_{p=0}^{p_{\max}} c_p (\kappa L_{\mathcal{B}})^{2p} \int_{\mathcal{B}_0} \left(\frac{|\xi - \eta|}{2}\right)^{2p} P_\ell(\eta)\, d\eta$$

$$+ \frac{iL_{\mathcal{B}}^2}{4} \sum_{p=0}^{p_{\max}} d_p (\kappa L_{\mathcal{B}})^{2p} \int_{\mathcal{B}_0} \left(\frac{|\xi - \eta|}{2}\right)^{2p} \log\left(\frac{|\xi - \eta|}{2}\right) P_\ell(\eta)\, d\eta$$

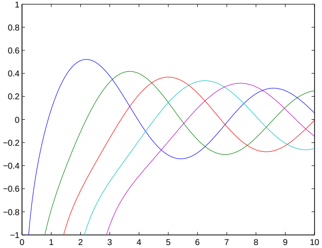
Far-field Interactions

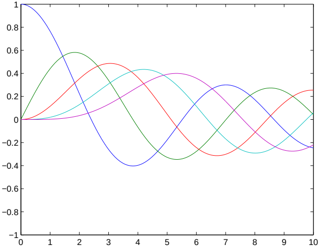
If the target and source points are far away

Multipole expansion of $H_0^{(1)}$ - Graf's addition theorem

$$H_0^{(1)}(\kappa L_{\mathcal{B}}|\xi-\eta|) \;=\; \sum_{k=-\infty}^{\infty} H_k(\kappa L_{\mathcal{B}}|\xi|) J_k(\kappa L_{\mathcal{B}}|\eta|) e^{ik(t_\xi - t_\eta)}, \forall |\xi| > |\eta|$$

$$\begin{cases} \xi = |\xi| \cdot (\cos(t_\xi), \sin(t_\xi)) \\ \eta = |\eta| \cdot (\cos(t_\eta), \sin(t_\eta)) \end{cases}$$





$$V_\kappa(x,\ell) \approx (L_{\mathcal{B}})^2 \sum_{k=-k_m}^{k_m} H_k(\kappa|\xi|) e^{ikt_\xi} \int_{\mathcal{B}_0} J_k(\kappa L_{\mathcal{B}}|\eta|) e^{-ikt_\eta} P_\ell(\eta)\, d\eta$$

Summary

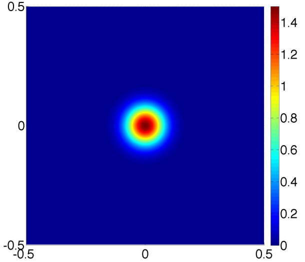
- ➢ Far field
  - $O(N)$ interactions per target point
  - Smooth
- ➢ Near field
  - $O(1)$ interactions per target point but $O(1)$ storage
  - Delicate but precomputed

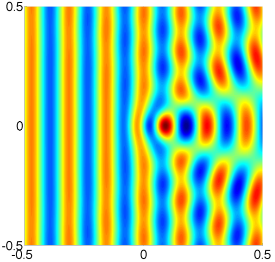
Quadrature characteristics

- ✔ Accuracy $O(h^m) + O(\varepsilon)$
  - Precomputation
- ✔ Speed
  - Precomputation

☞ Very fast access to matrix entries
for the scattering problem
☞ κ as a parameter
☞ q as a parameter

Scattering test case
Gaussian contrast and Total field





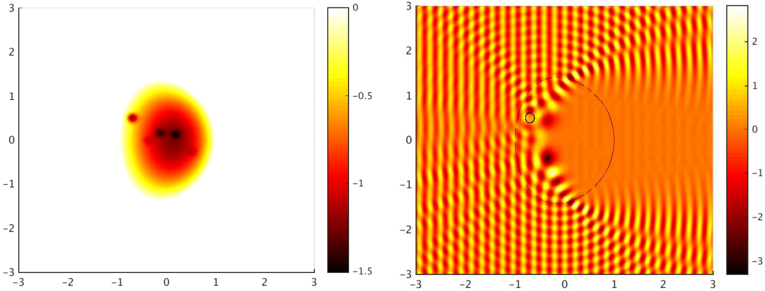
Uniform grid

N	$e(0.5, 0)$	$e(1, 0.5)$
256	$0.8925E+0$	$0.4890E+0$
1024	$0.1809E+0$	$0.1406E+0$
4096	$1.0938E-2$	$9.6485E-3$
16384	$3.1169E-4$	$3.2633E-4$
65536	$1.4300E-5$	$1.2537E-5$
262144	$8.8874E-7$	$6.4485E-7$

Adaptive grid

ϵ	N	$e(0.5, 0)$	$e(1, 0.5)$
$1E-4$	4096	$1.09E-2$	$9.65E-3$
$1E-5$	4864	$7.89E-4$	$1.20E-4$
$1E-6$	6400	$2.99E-4$	$3.33E-4$
$1E-7$	10240	$4.83E-5$	$8.75E-6$
$1E-8$	16384	$1.13E-5$	$5.35E-6$
$1E-9$	34816	$2.30E-6$	$3.86E-7$
$1E-10$	70912	$7.14E-7$	$3.16E-7$
$1E-11$	138688	$4.25E-8$	$1.94E-8$

Scattering by blobs in a plasma
Contrast and Total field



Thank you for your attention

Collaborators : L. Greengard, C. Borges, and S. Ambikasaran.