CAGD Methods for Invariant Manifold Computations

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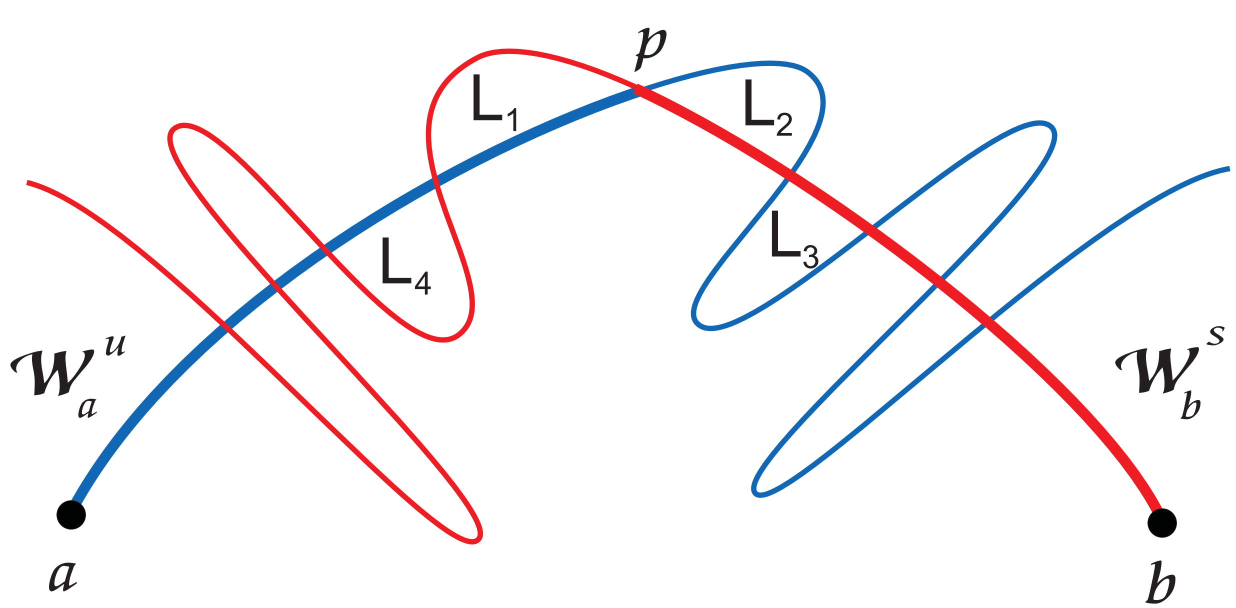
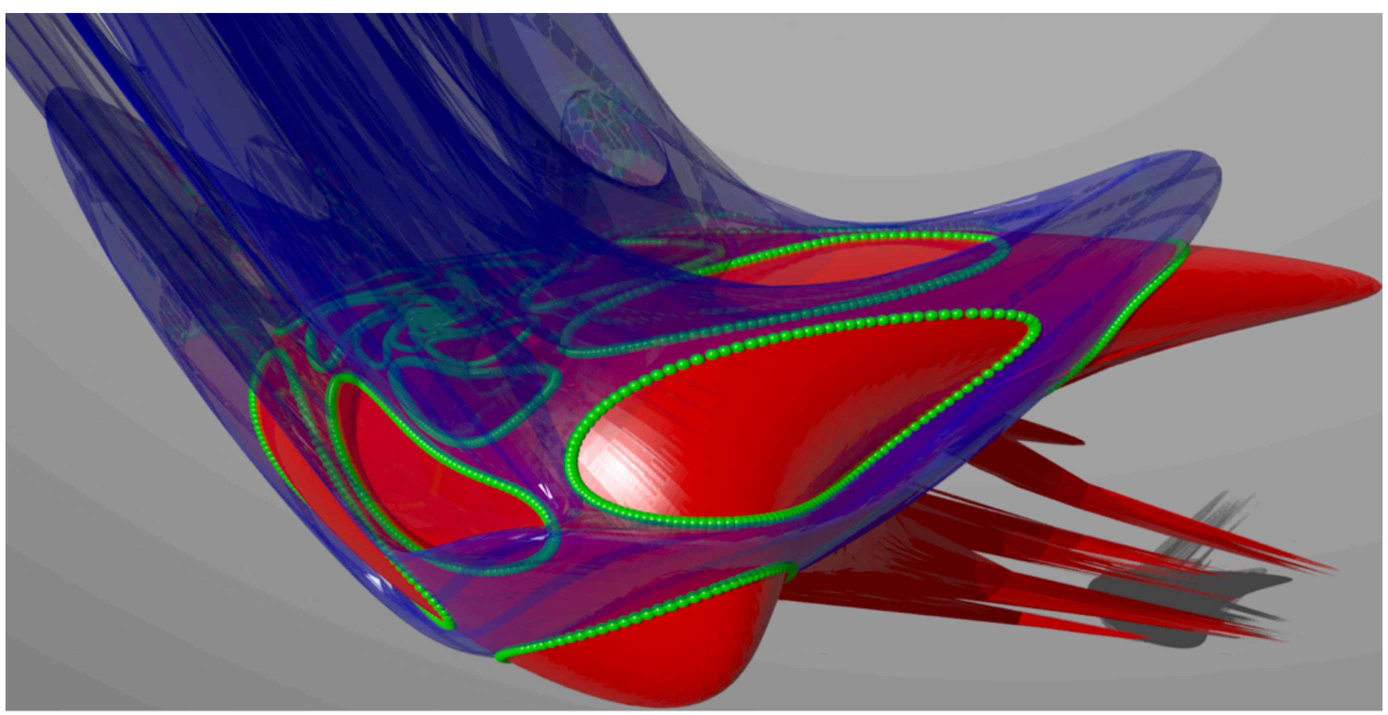
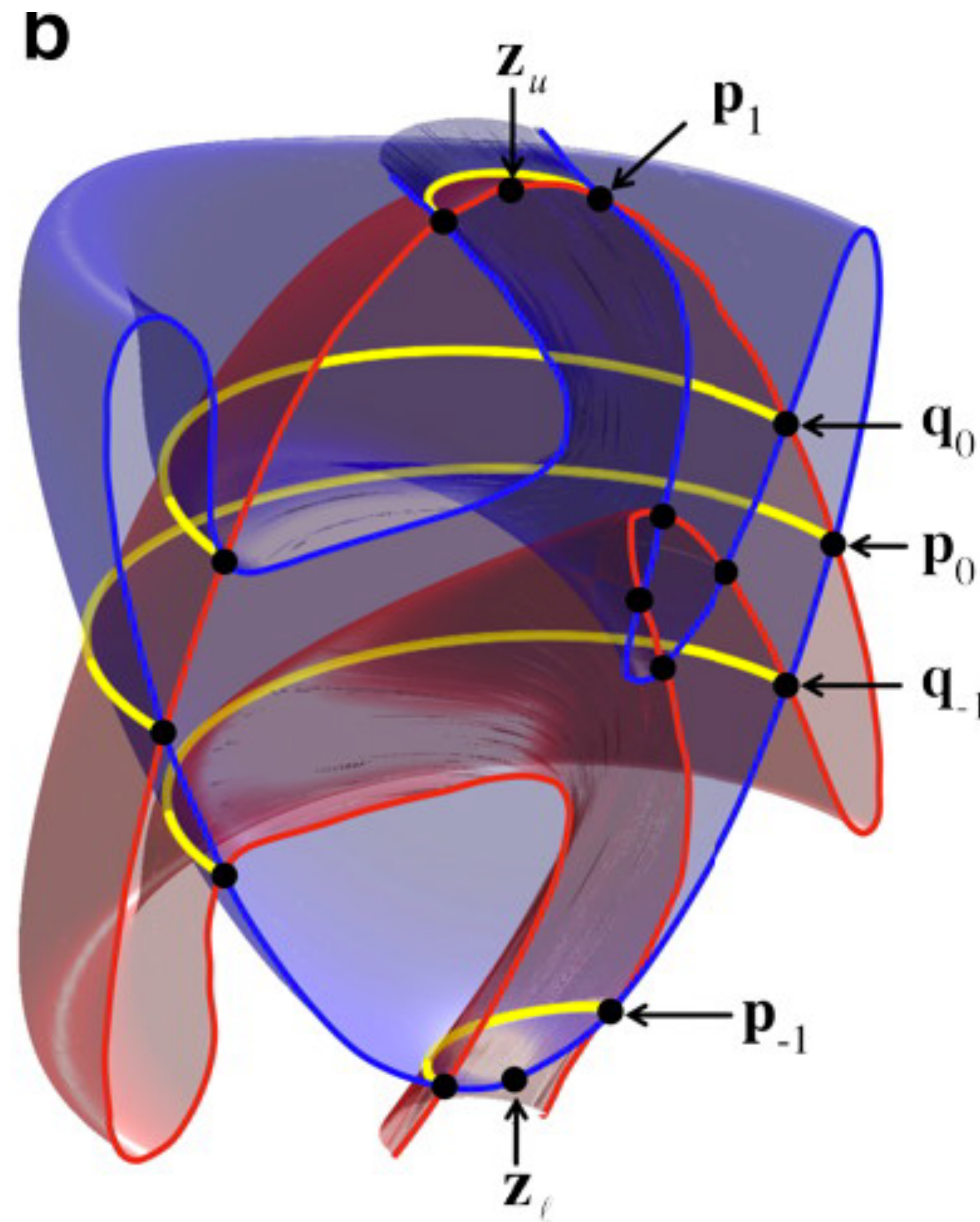
Invariant Manifolds are important in understanding dynamics





Mireles James/Capinski 2017



Smith et al 2017

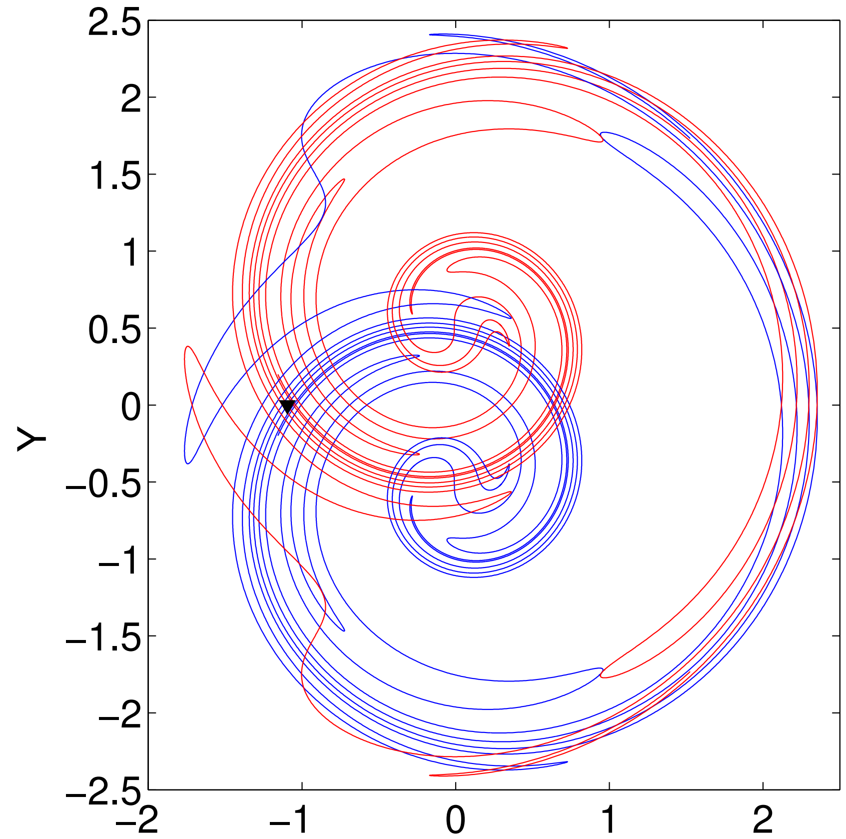
Setup

Consider a diffeomorphic mapping $\mathbf{x}' = f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^2$
with a hyperbolic fixed point $\mathbf{x}^*$
i.e. such that $Df(\mathbf{x}^*)$ has eigenvalues $0 < |\lambda_s| < 1 < |\lambda_u|$

Define the stable and unstable manifolds

$$W^{\mathrm{s}}(\mathbf{x}^*) = \{\mathbf{x} \in \mathbb{R}^2 : f^k(\mathbf{x}) \to \mathbf{x}^* \text{ as } k \to \infty\}$$

$$W^{\mathrm{u}}(\mathbf{x}^*) = \{\mathbf{x} \in \mathbb{R}^2 : f^{-k}(\mathbf{x}) \to \mathbf{x}^* \text{ as } k \to \infty\},$$

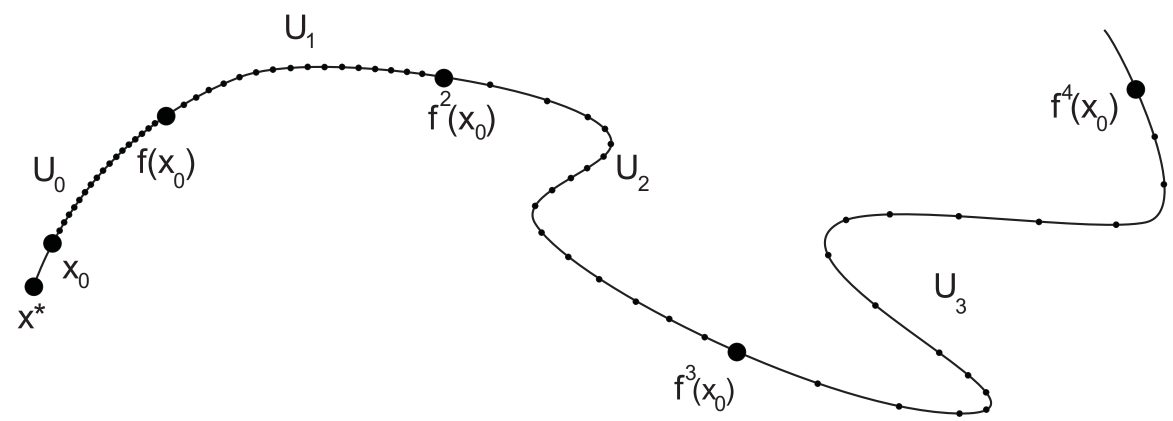


Fundamental segments

Pick a point x_0 near x^* and inductively define $x_{n+1}=f(x_n)$. Then the n^{th} fundamental segment is defined as

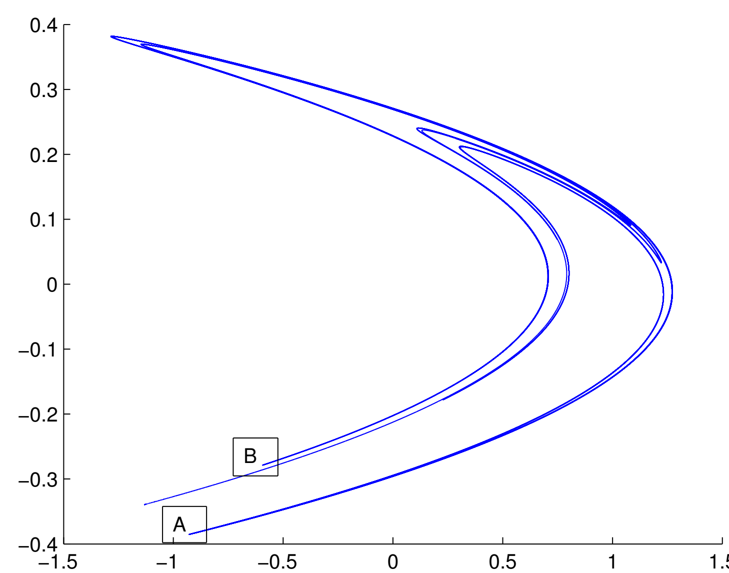
$$U_n = W^{\mathrm{u}}[x_n, x_{n+1}]$$

and $W^{\mathrm{u}} = \bigcup_{n=-\infty}^{\infty} U_n$



Issues:

Not many points required to resolve U_0 but dynamic stretching and folding will require more points to resolve later segments.



Curvature can vary by orders of magnitude after just a few iterates.

Existing Methods of Computation

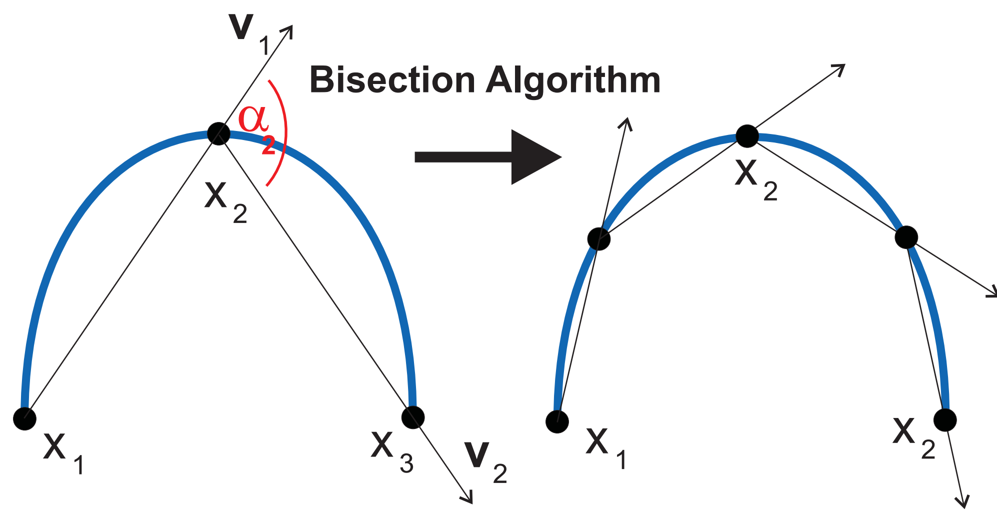
- Iteration of fundamental segments
- Marching methods: given a manifold computed up to a given point, extending manifold by a given amount, see esp. Krauskopf & Osinga
- Parameterization methods: seek functional representation for W^u (generally only useful locally)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} t^k$$

(and many others)

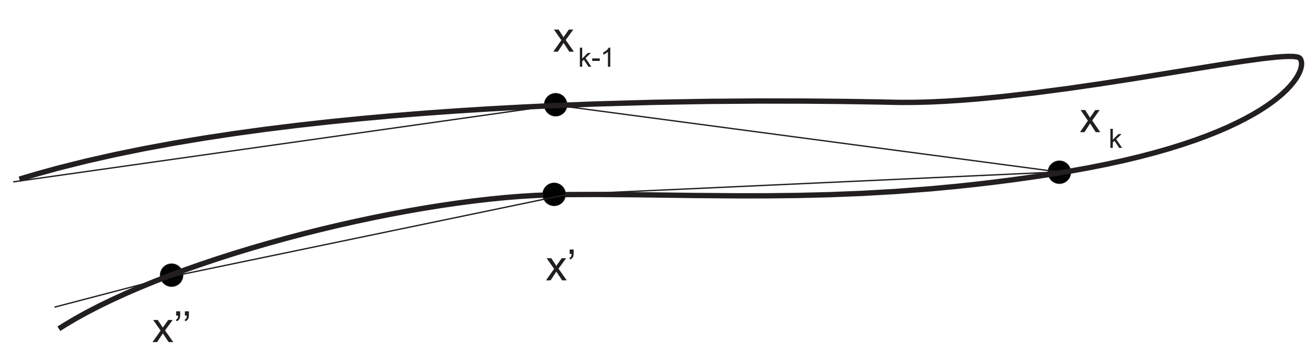
Existing Methods: Linear Interpolation

Consider adaptively drawing a parametric curve. Basis for Carter's adaptive (bisection) algorithm. Bisect if



$$\alpha_k > \mathrm{tol}_1 \text{ or } l_k \alpha_k > \mathrm{tol}_2$$

Marching algorithm (Hobson): Similar. Add one point to previously computed curve. Requires finding the right pre-image at each point, throws away many points. Occasionally cuts corners

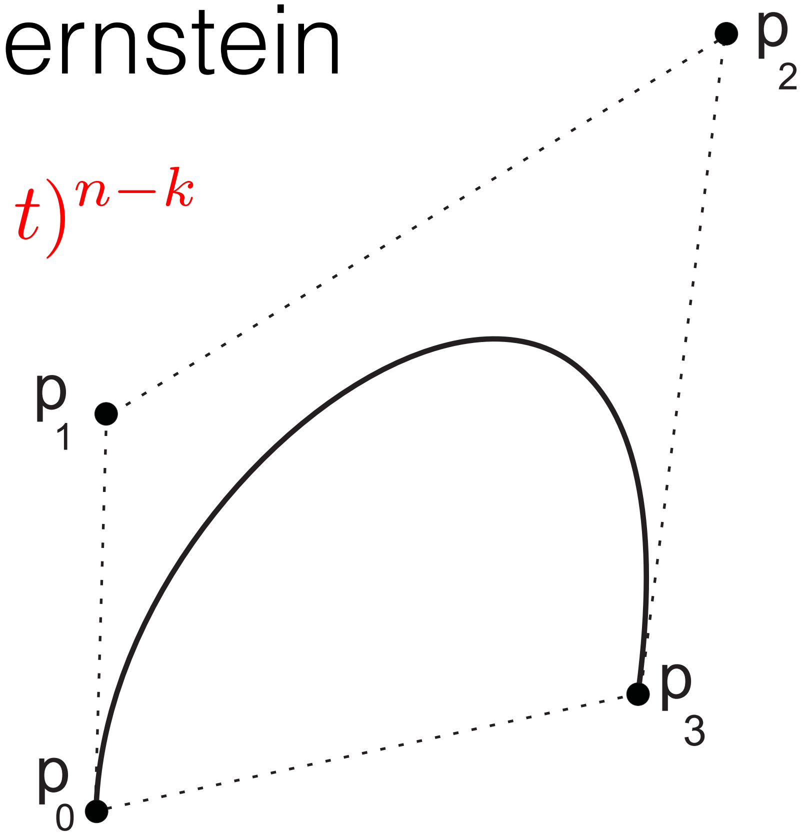


Tools from CAGD

Bézier curves: Form a convex combination of (n+1) *control points* using Bernstein Polynomials

$$\beta(t) = \sum_{k=0}^{n} B_k^n(t)\mathbf{p}_k$$

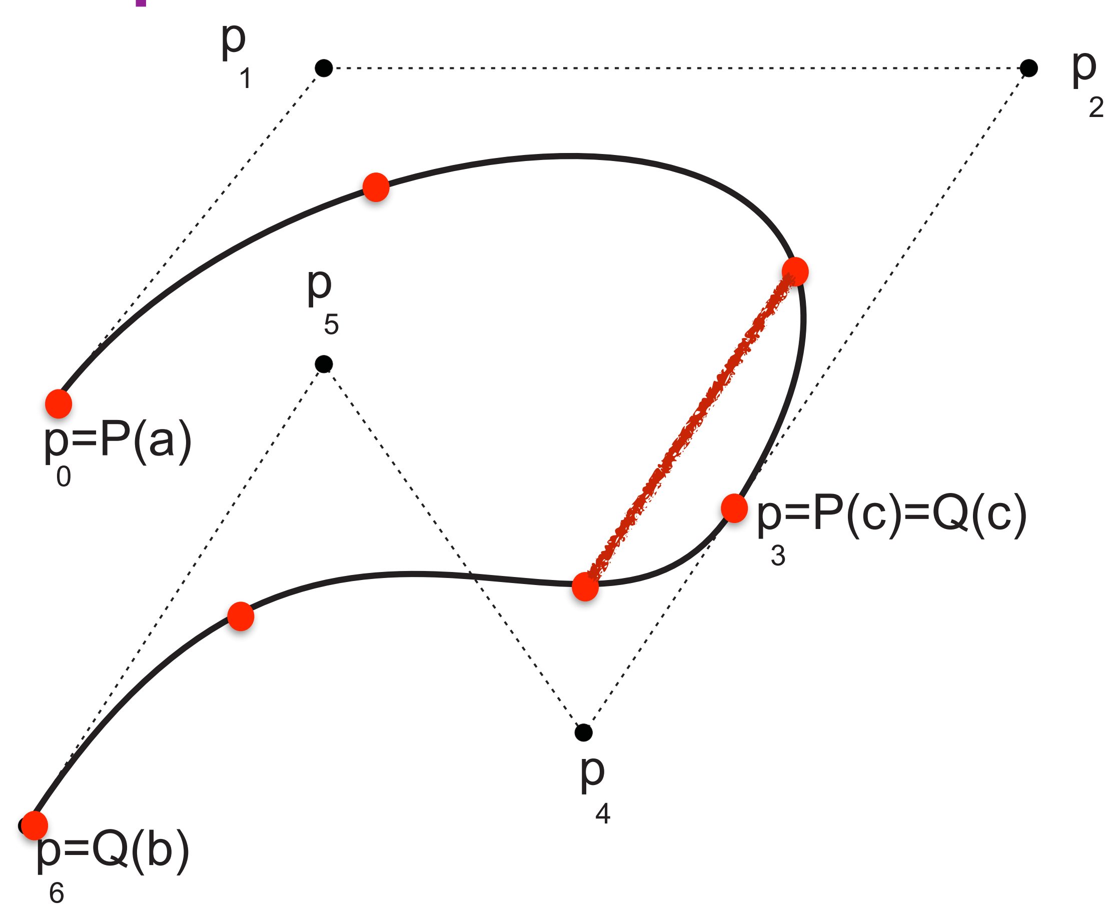
$$B_k^n(t) = \binom{n}{k} t^k (1-t)^{n-k}$$



Cubic Hermite polynomials: construct Bezier curve interpolating points x_0 and x_1 with tangent vectors v_0 and v_1 by letting

$$\mathbf{p}_0 = \mathbf{x}_1, \mathbf{p}_1 = \mathbf{x}_1 + \vec{\mathbf{v}}_1/3, \mathbf{p}_2 = \mathbf{x}_2 - \vec{\mathbf{v}}_2/3, \mathbf{p}_3 = \mathbf{x}_2,$$

Catmull-Rom splines



Partial interpolation using composite cubic Bézier functions.

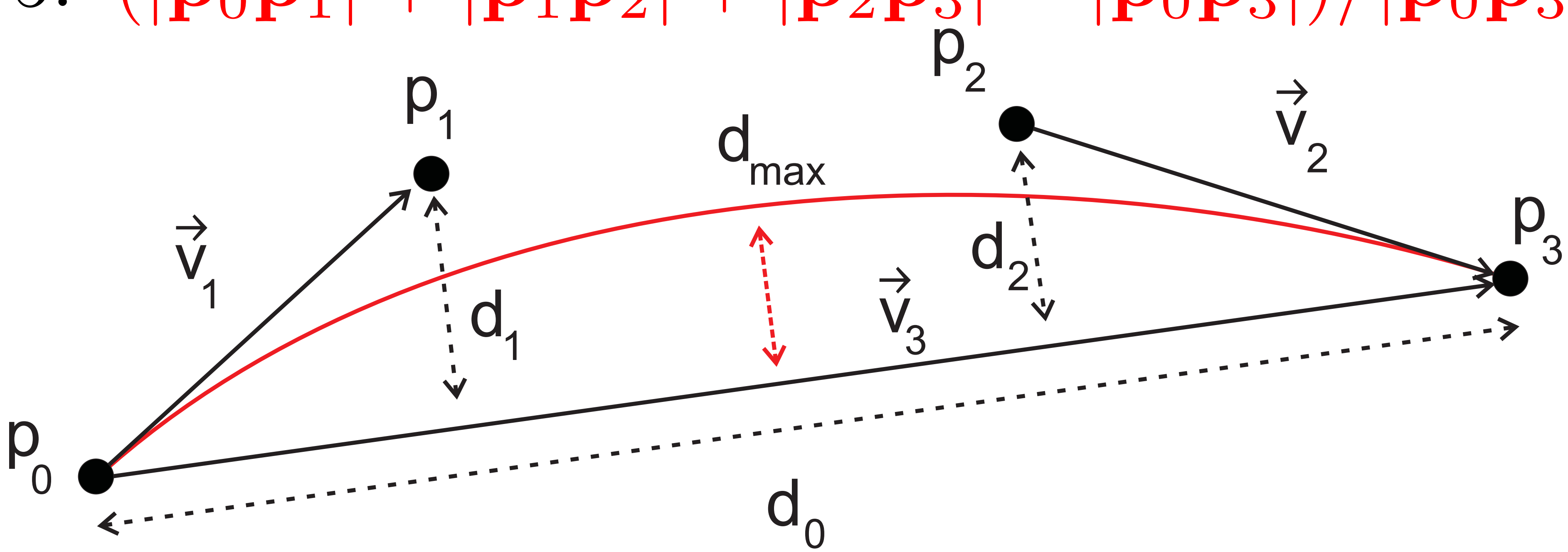
First and last control point on each segment chosen to interpolate data.

Second and third control points on each segment chosen to approximate tangent vector at endpoints using centered differences.

Allows for local refinement (as opposed to more familiar cubic splines).

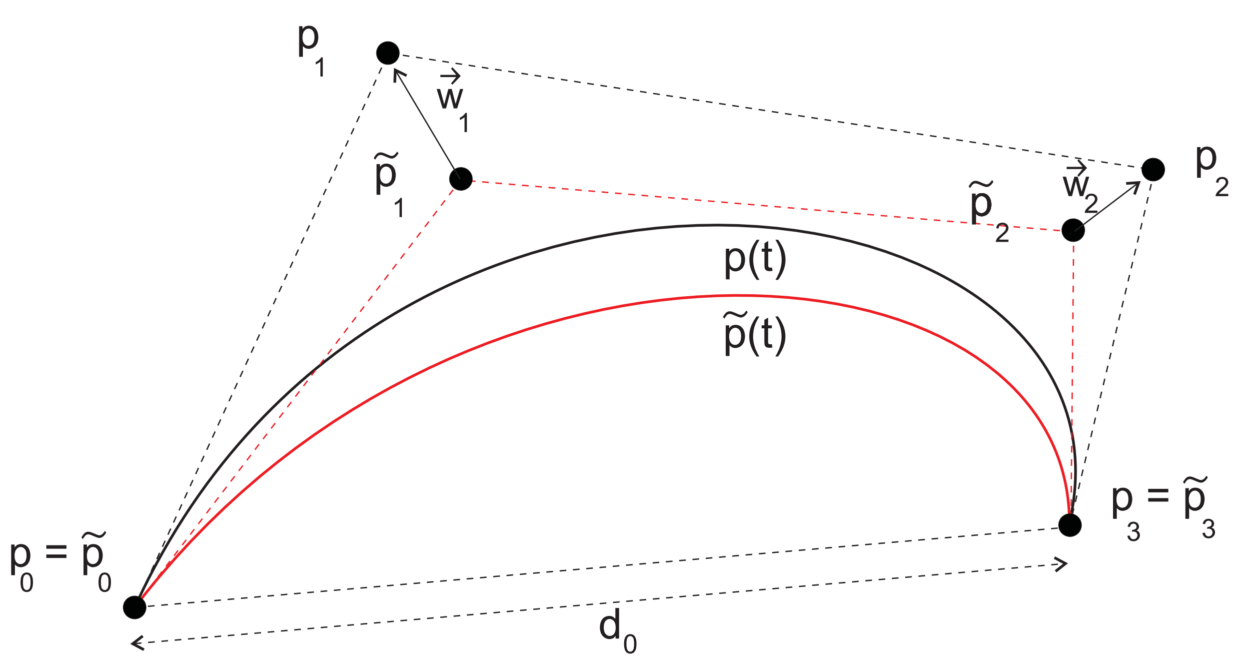
Adaptive Catmull-Rom 1: Flatness conditions

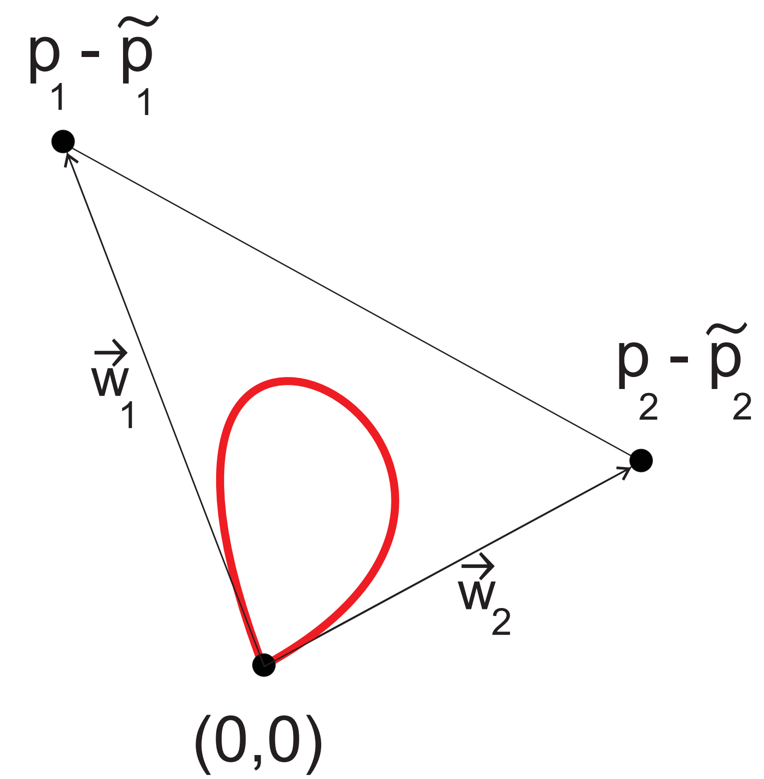
1. $\max\{d_1, d_2\}$
2. $(|\mathbf{p}_0\mathbf{p}_1| + |\mathbf{p}_1\mathbf{p}_2| + |\mathbf{p}_2\mathbf{p}_3| - |\mathbf{p}_0\mathbf{p}_3|)$,
3. $\max\{d_1/d_0, d_2/d_0\}$,
4. the angle between $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$,
5. $(|\mathbf{p}_0\mathbf{p}_1| + |\mathbf{p}_1\mathbf{p}_2| + |\mathbf{p}_2\mathbf{p}_3| - |\mathbf{p}_0\mathbf{p}_3|)/|\mathbf{p}_0\mathbf{p}_3|$.



Adaptive Catmull-Rom 2: Error refinement conditions

Compute two approximations, subtract and obtain the error polygon

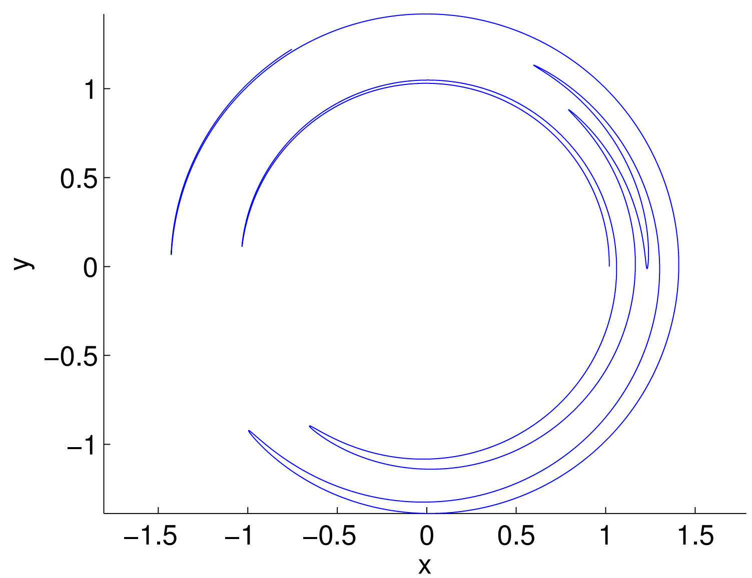


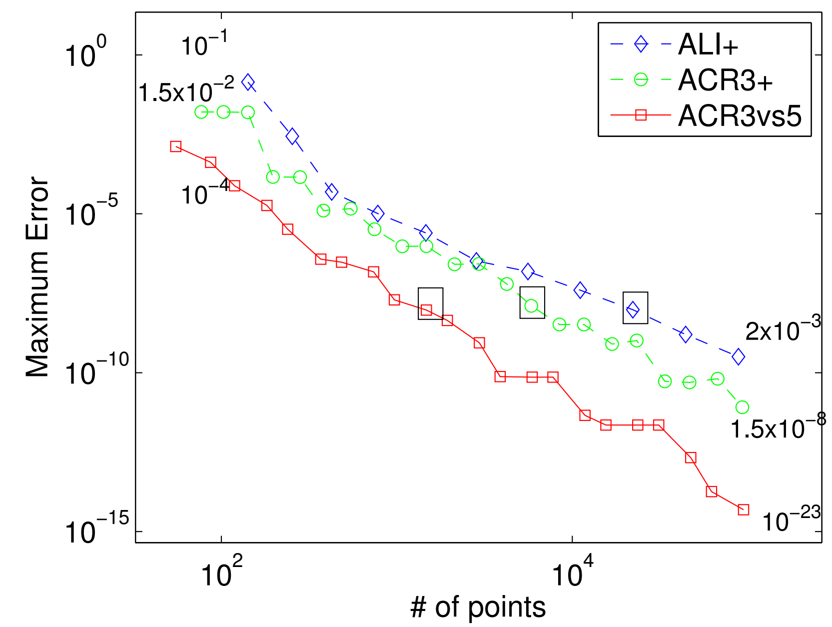


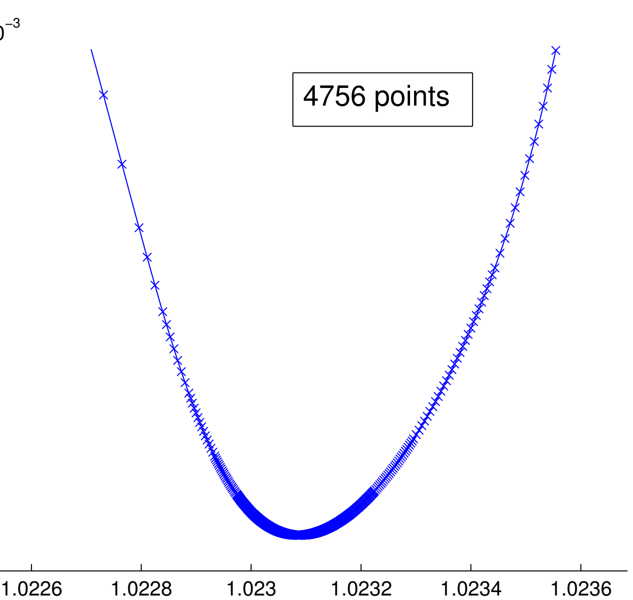
$$1.\quad \max\{|\vec{\mathbf{w}}_1|, |\vec{\mathbf{w}}_2|\},$$

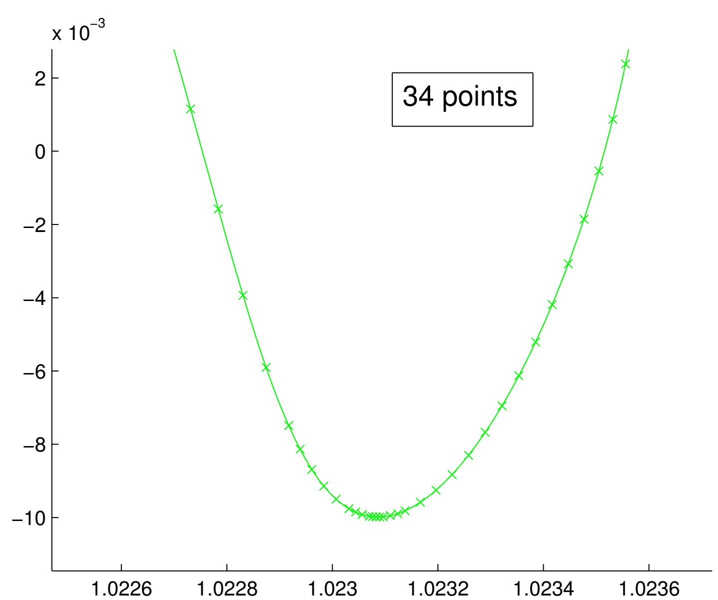
$$2.\quad \max\left\{\frac{|\vec{\mathbf{w}}_1|}{d_0}, \frac{|\vec{\mathbf{w}}_2|}{d_0|}\right\}.$$

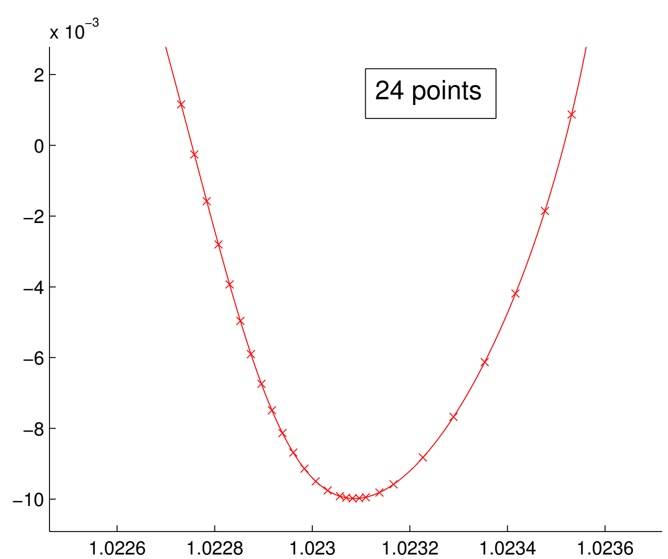
Numerical Results: model problem











Numerical Results: Real Maps

Hénon Map

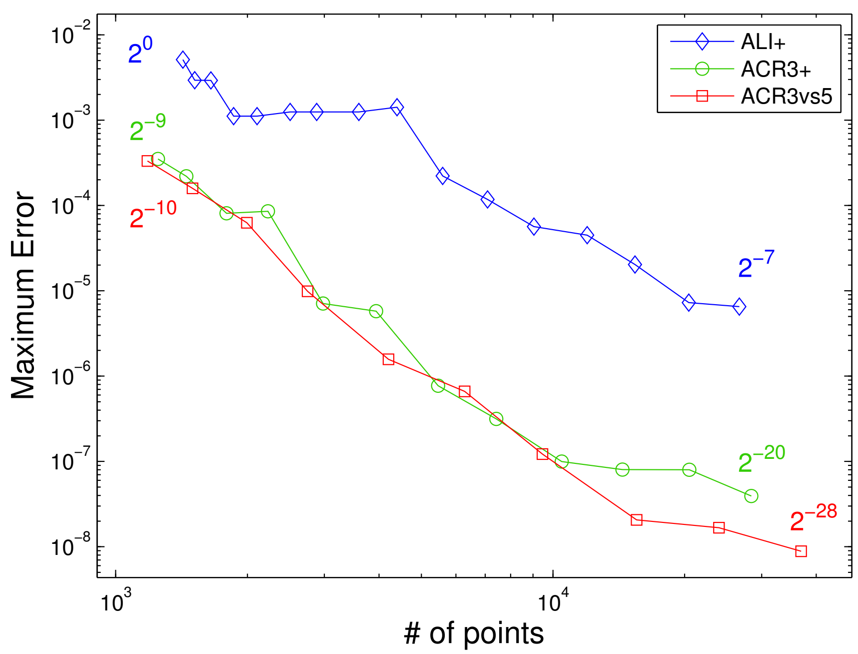
$$x_{n+1} = 1 + y_n - a x_n^2$$

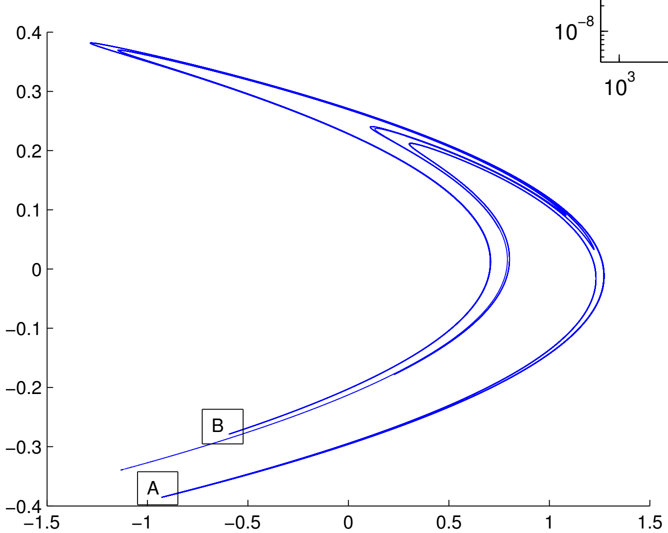
$$y_{n+1} = b x_n$$

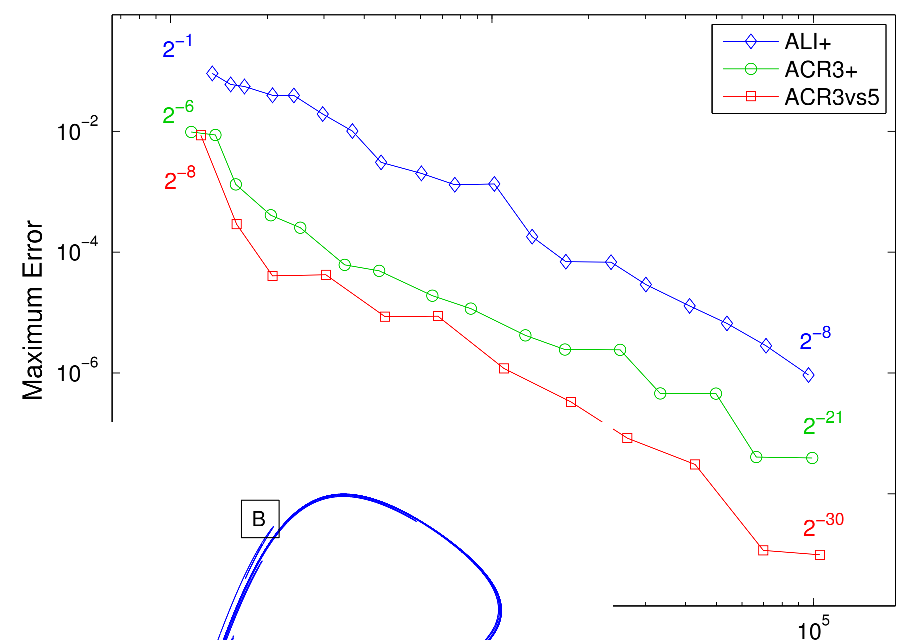
McMillan Map

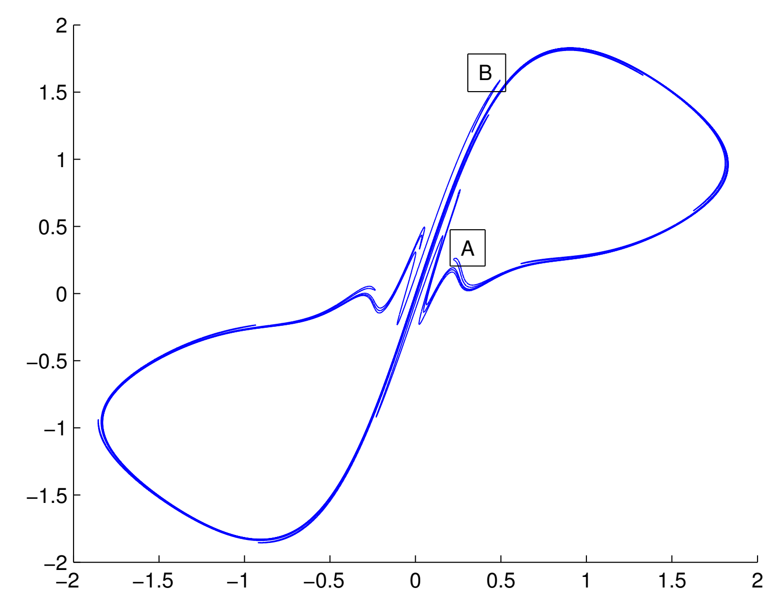
$$x_{n+1} = y_n,$$

$$y_{n+1} = -x_n + 2y_n \left(\frac{\mu}{1 + y_n^2} + \varepsilon \right)$$

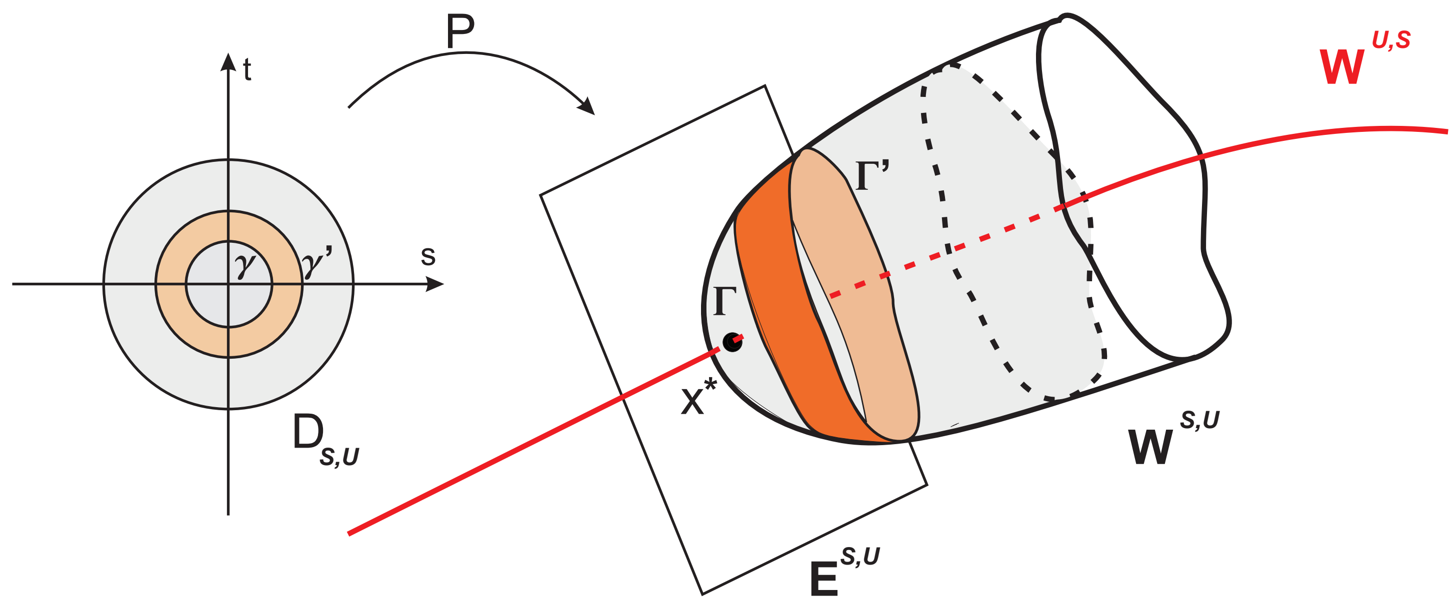








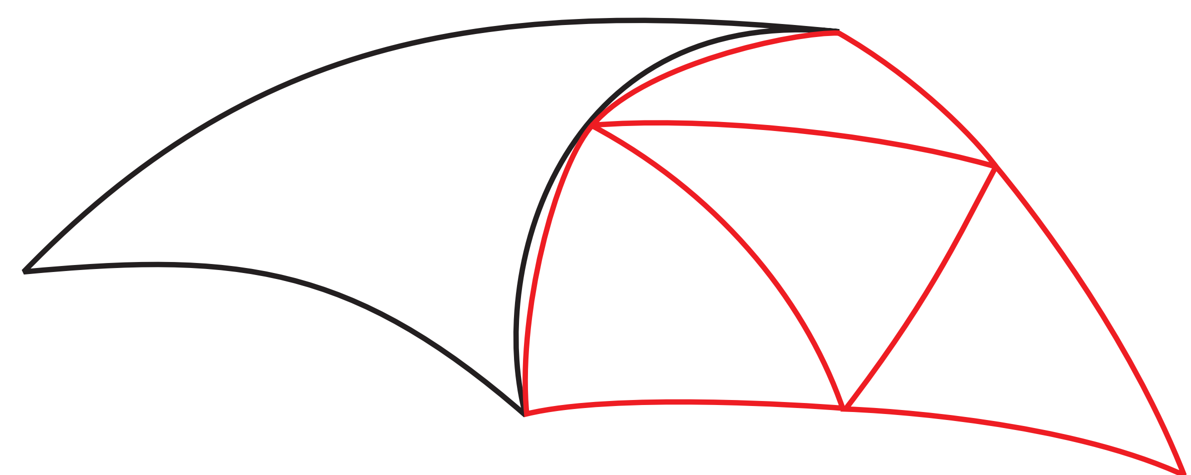
2D Unstable Manifolds for 3D Maps



Significant new challenges

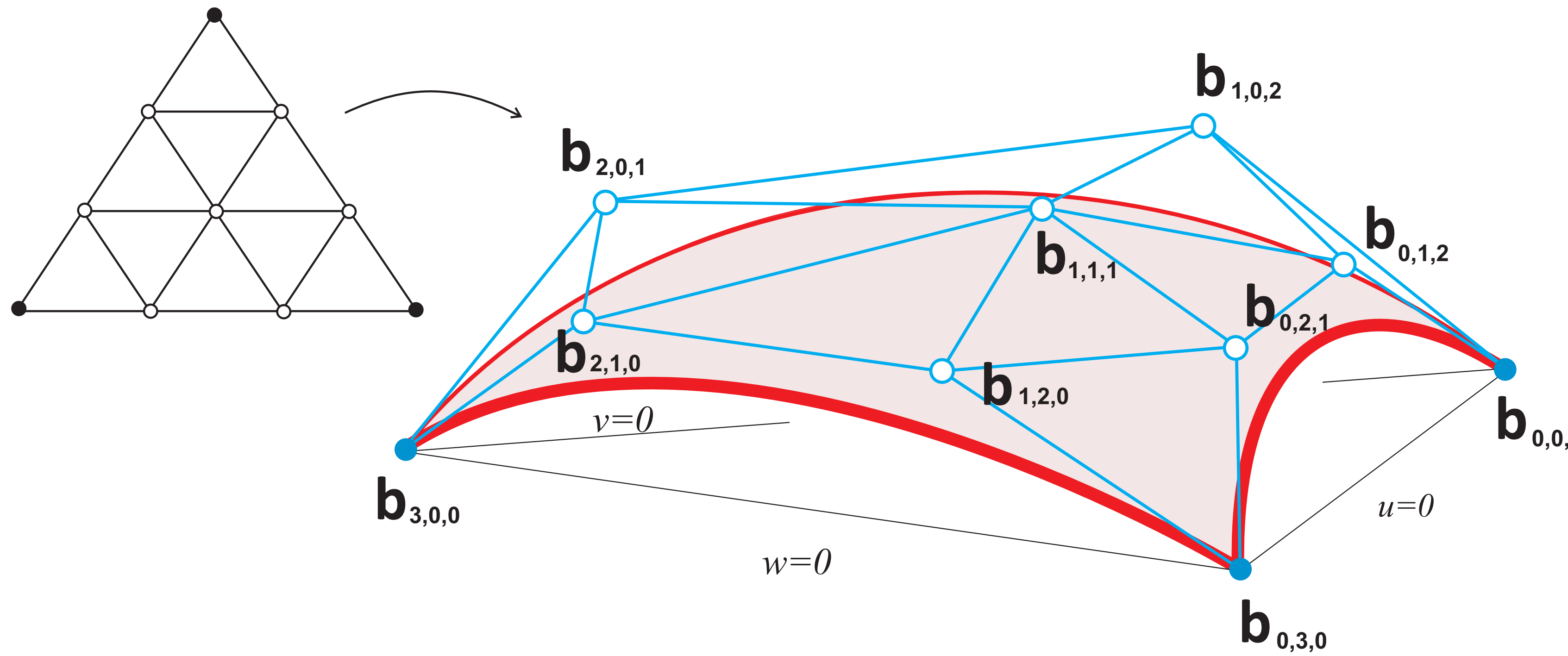
Dynamics: Exponentially anisotropic Growth

Computational/Geometric

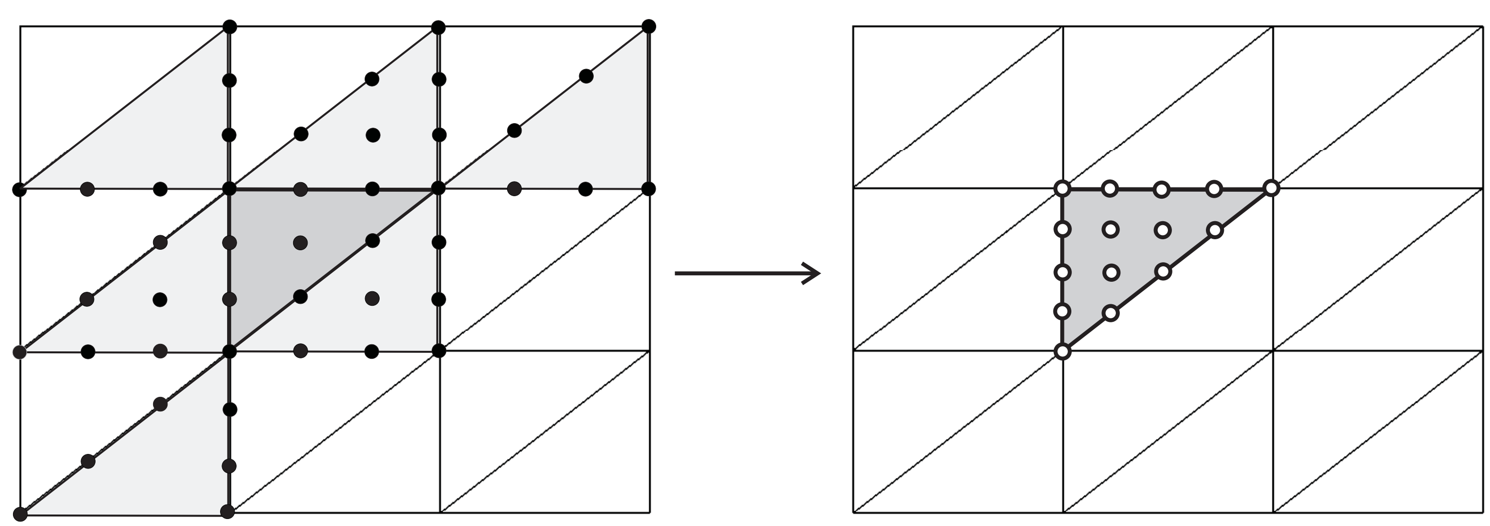


CAGD Toolbox

Bézier triangular patches



Quasi-interpolation using quartic 2D Bernstein polynomials: 10 data points/triangle, +46 points from neighboring triangles, giving 15 control points/patch



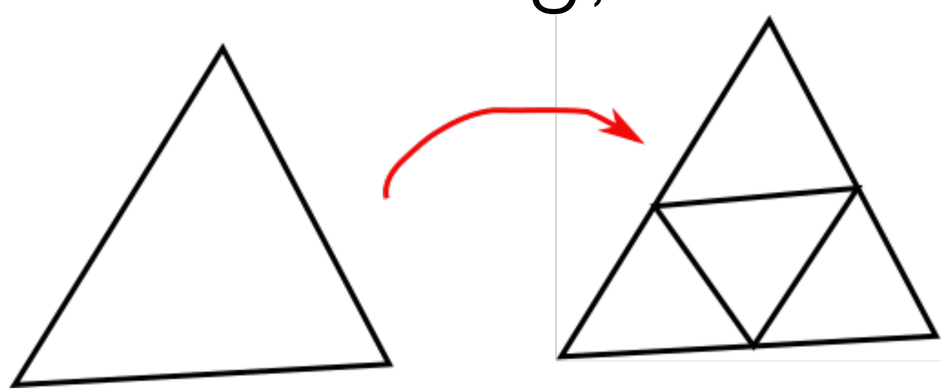
Quasi-interpolation: choose coeffs such that polynomials of given degree represented exactly

Adaptive quasi-interpolating quartic Bezier patches

Sorokina/Zeilfelder 2008, Hering-Bertram et al 2009

A Band-aid approach

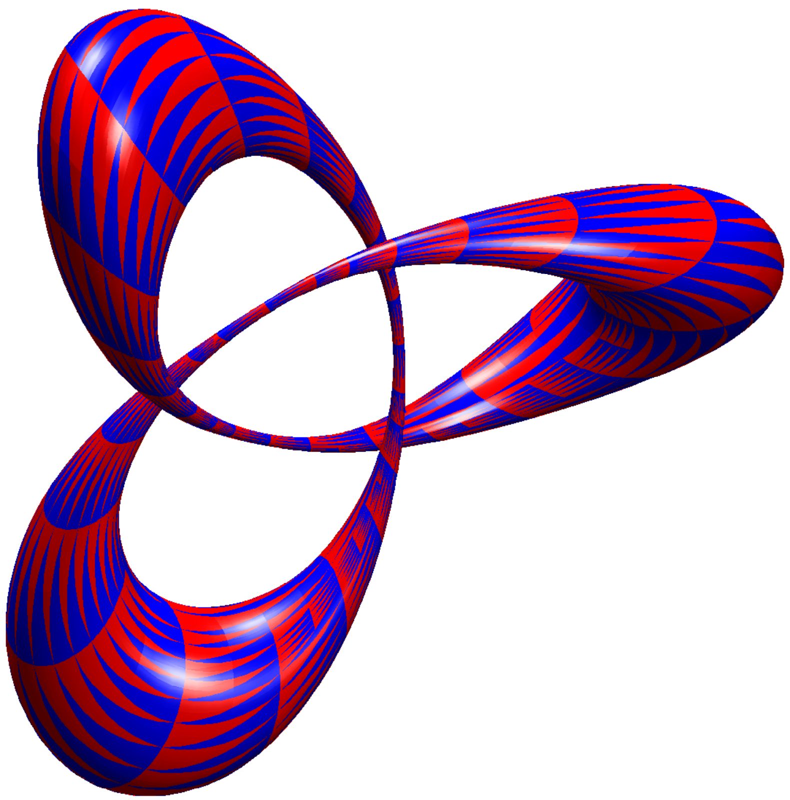
Identify triangles on which need refining, bisect into four half-length triangles

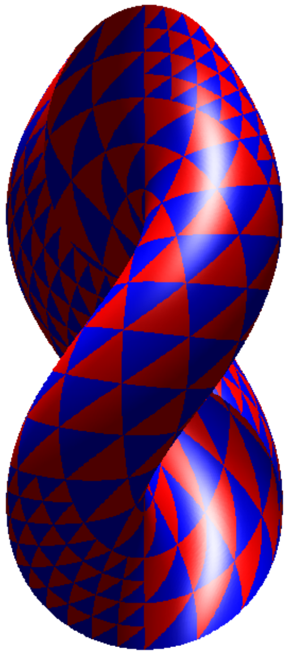


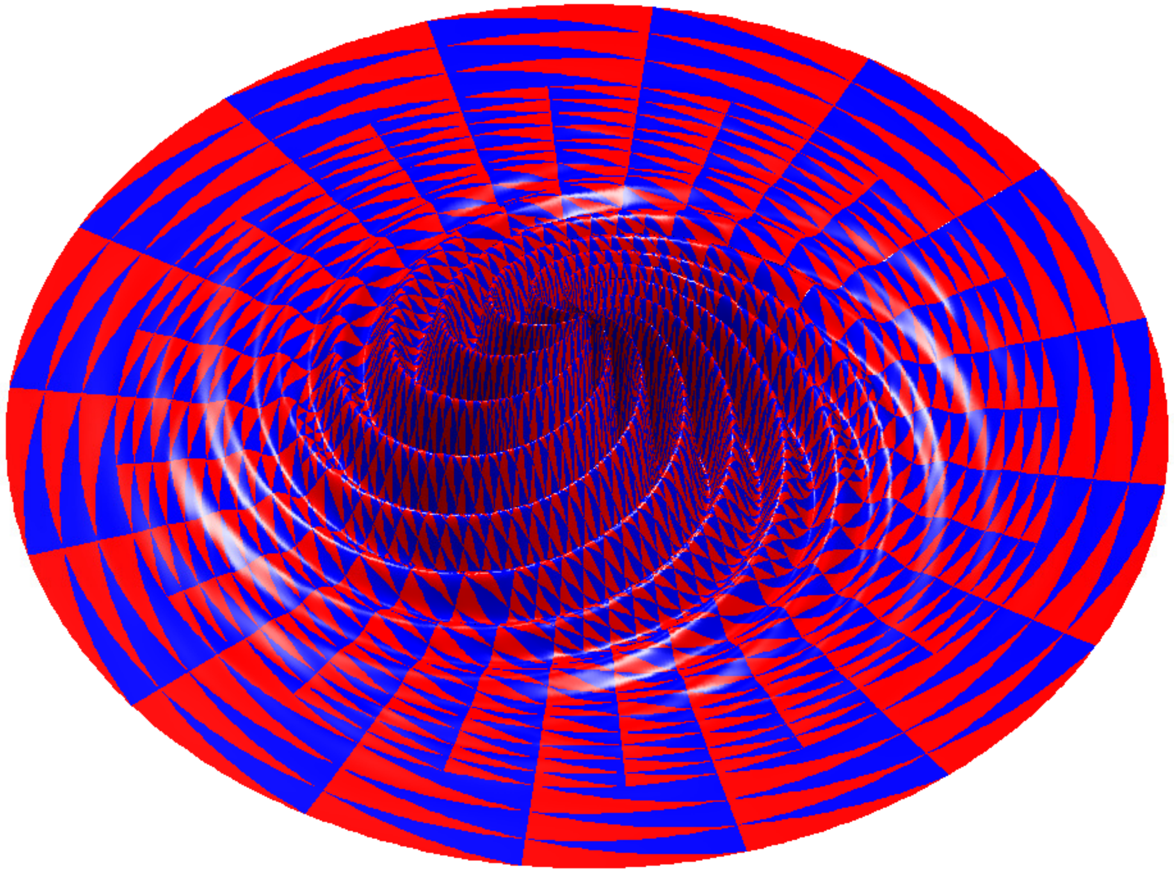
Interpolate the difference between the data and the current interpolant on the refined grid.

Overlay the higher resolution patch on the previous computation

Applied to model problems







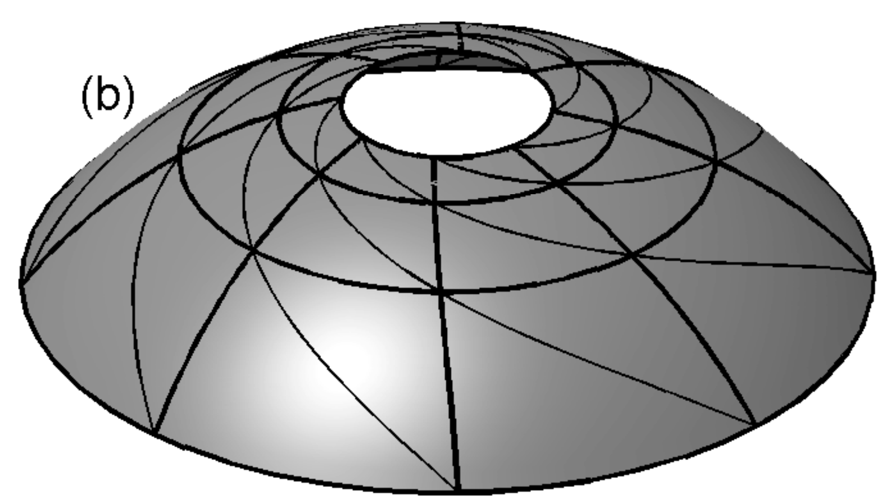
Application to Unstable Manifolds

Fundamental Segments→Fundamental Annuli

Proper loop: A simple closed curve on W^u with the fixed point on its interior and which does not intersect its image under the map.

Fundamental annulus: Region of W^u contained between a proper loop γ and its image $F(\gamma)$.

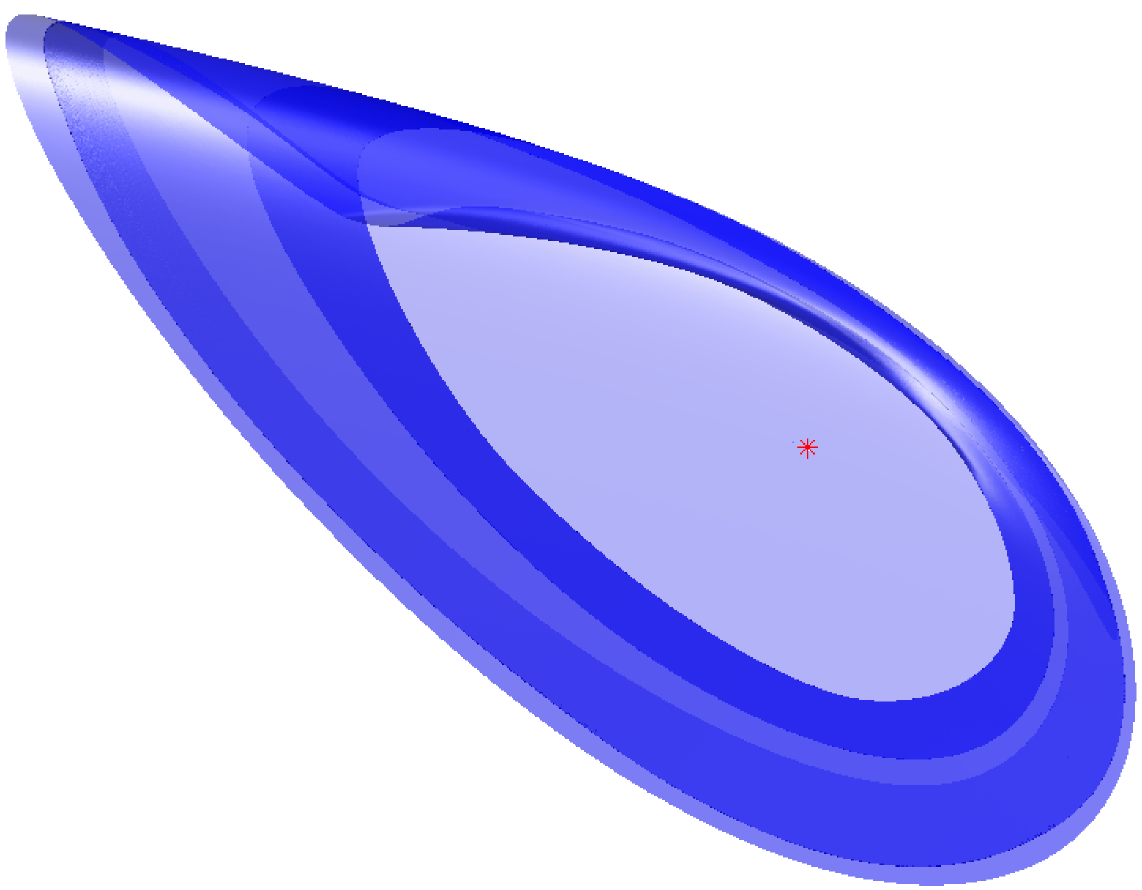
Parameterization method used for initial annulus

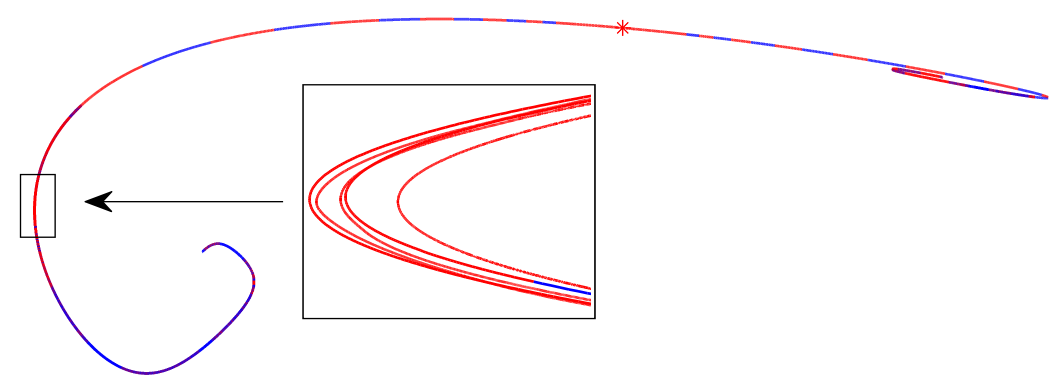


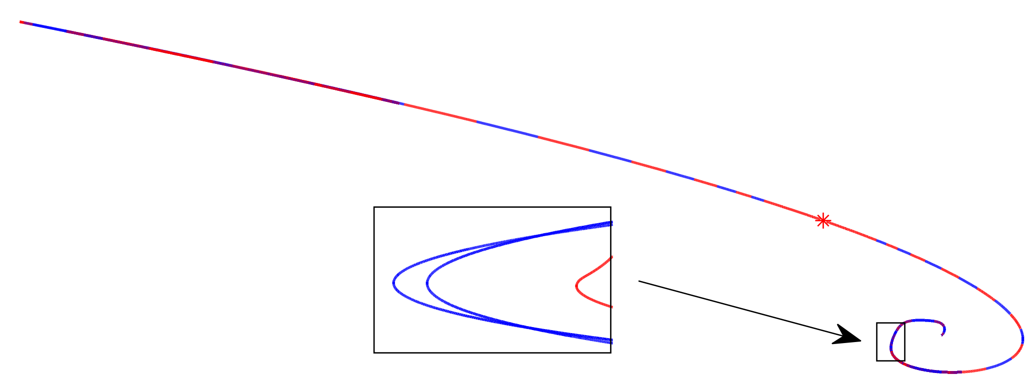
Arneodo–Coullet–Tresser map

A discrete map whose attractor resembles the Rössler system's attractor

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} a x_n - \omega b (y_n - z_n) \\ \frac{b}{\omega} x_n + a (y_n - z_n) \\ c x_n - d x_n^k + e z_n \end{pmatrix}$$

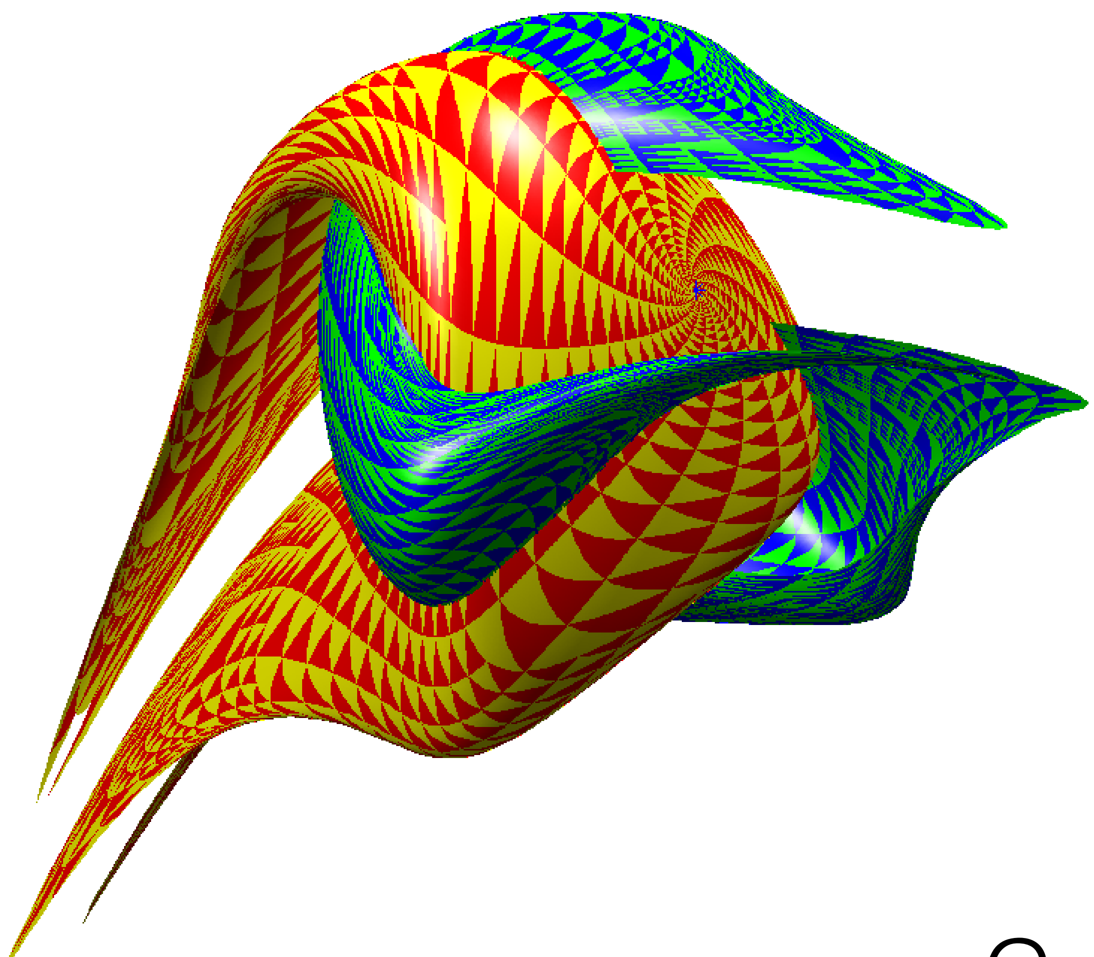




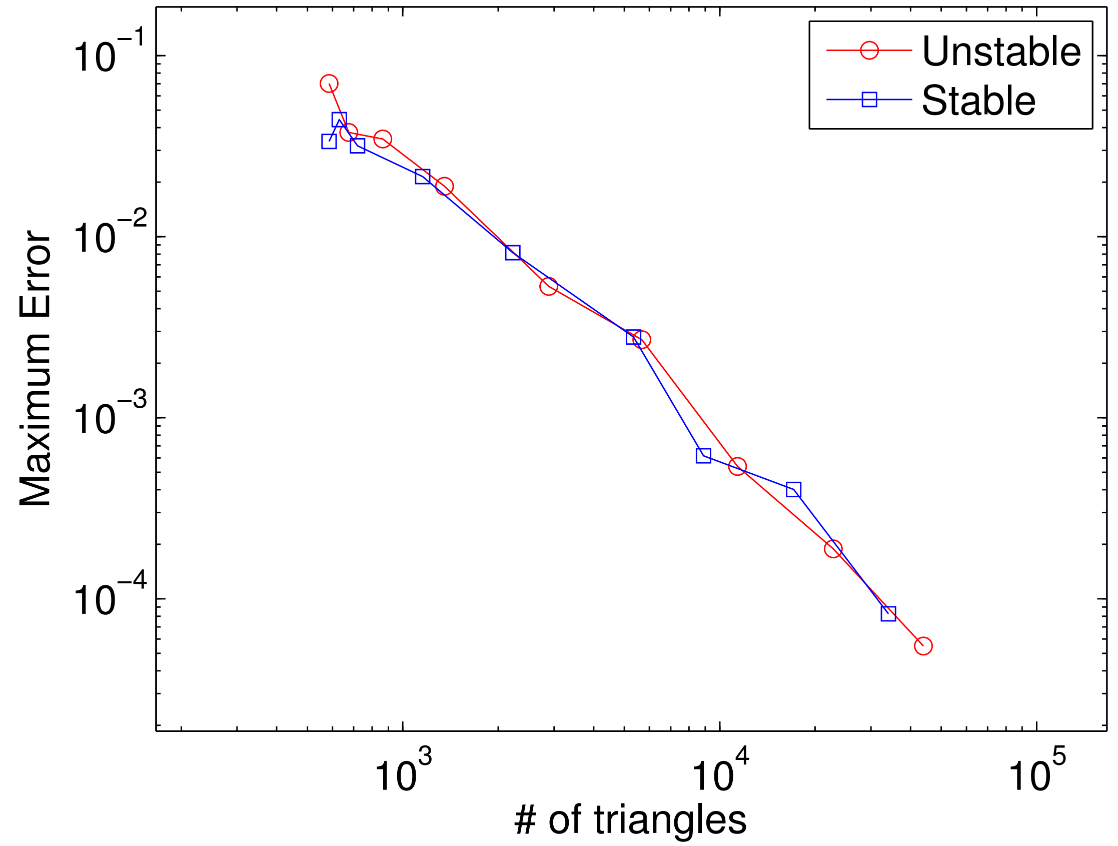


Volume-preserving Hénon Map

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha + \tau x_n + z_n + a x_n^2 + b x_n y_n + c y_n^2 \\ x_n \\ y_n \end{pmatrix}$$



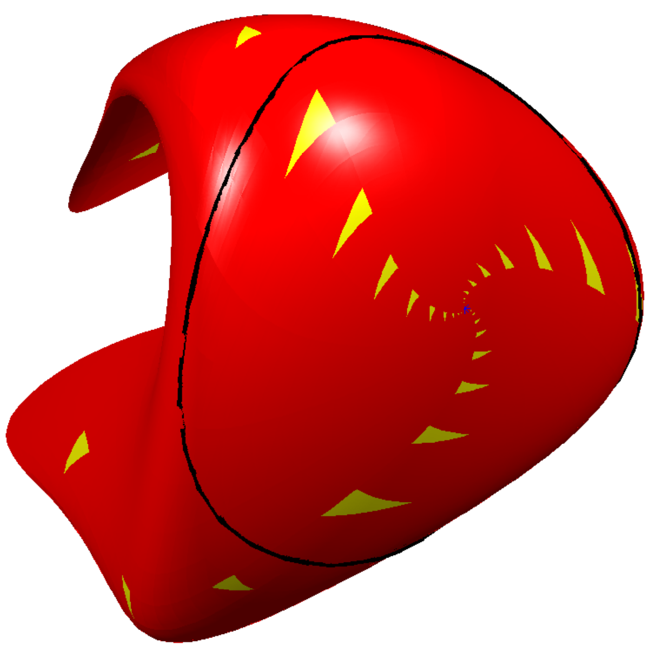
Anisotropic growth



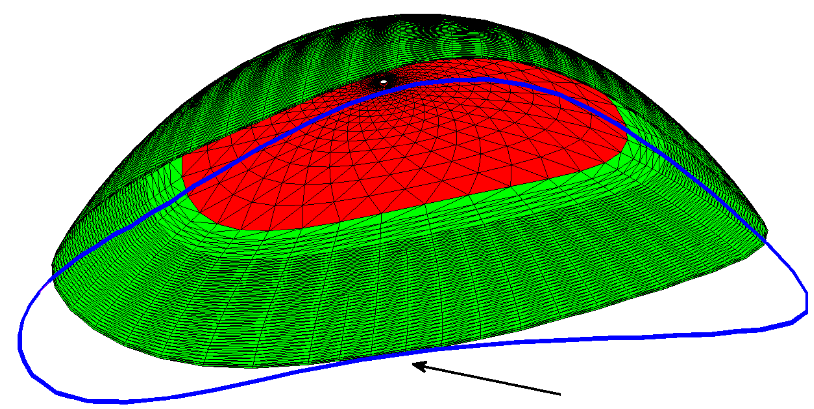
Converges, perhaps 2nd order

Can Geodesic-Level-Set-Method save the day?

Emphatically, no.



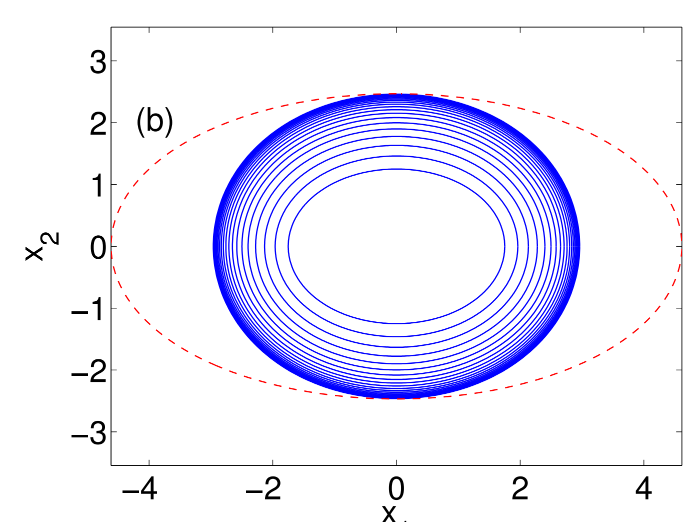
If computational boundary not a proper loop, can't find pre-image of needed next points



Level sets converge

Simplified model problem

$$\mathbf{x}_{n+1} = \begin{pmatrix} 0 & -AB & 0 \\ \frac{A}{B} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \mathbf{x}_n, \quad B > A > 1$$



That leaves parameterization

Let's look at it in 2 space dimensions:
Goal: Construct functions $\mathbf{p}(t)$ and $\Lambda(t)$ such that

$$\mathbf{p}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

parameterizes the unstable manifold, requiring:

- Invariance under the map $\mathbf{f}\left(\mathbf{p}(\mathbf{t})\right) = \mathbf{p}\left(\Lambda(t)\right)$
- Curve passes through fixed point $\mathbf{p}(0) = \mathbf{x}^*$
- Curve tangent to unstable subspace at fixed point $$\left.\frac{d\mathbf{p}(\mathbf{t})}{dt}\right|_{t=0} = c\mathbf{v}_{\mathrm{unstable}}$$

How parameterization works

In practice, defined by a power series $\mathbf{p}(t) = \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} t^k$ and $\Lambda(t) = \lambda_{\text{unstable}} t$

Conditions $\mathbf{p}(0) = \mathbf{x}^*$ and $\left.\dfrac{d\mathbf{p}(\mathbf{t})}{dt}\right|_{t=0} = c\mathbf{v}_{\text{unstable}}$

used to determine leading-order terms.

Invariance condition $\mathbf{f}\left(\mathbf{p}(\mathbf{t})\right) = \mathbf{p}\left(\Lambda(t)\right)$ satisfied by expanding both sides in power series and matching coefficients

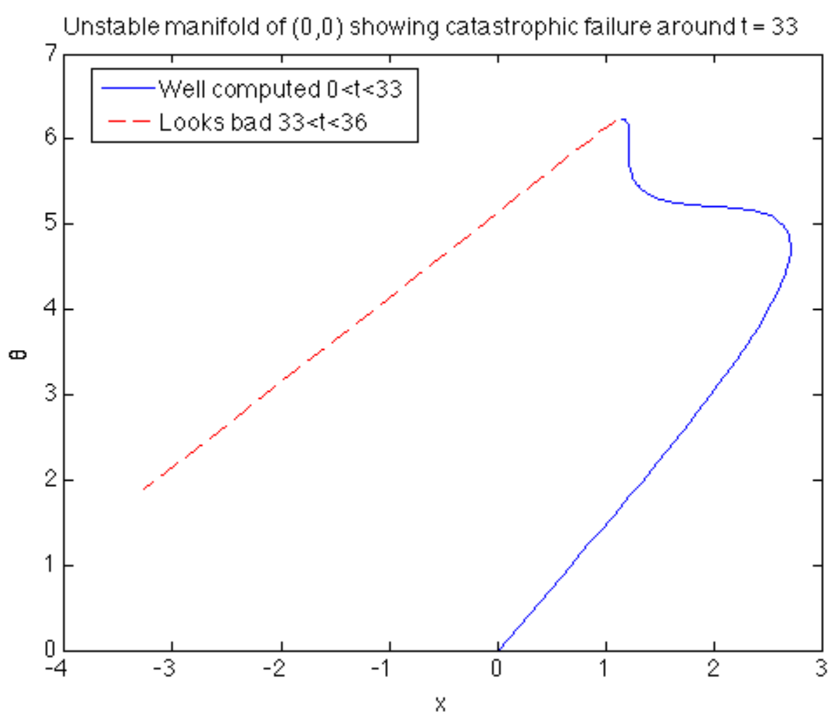
Methods based on automatic differentiation for expanding functions of power series on LHS

In practice only useful for generating small portion of manifolds

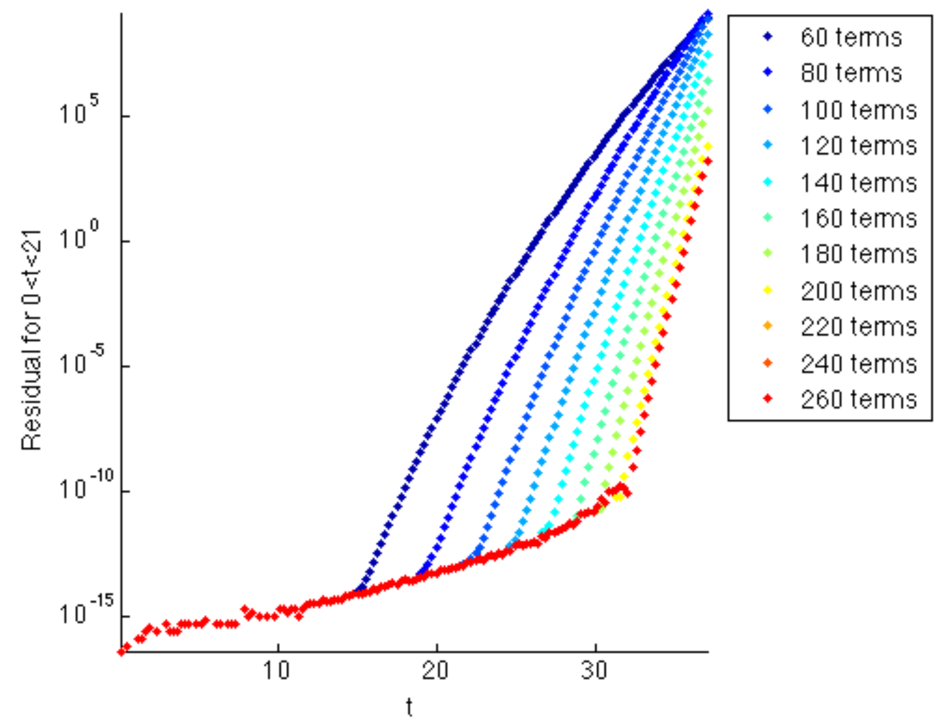
Standard Map

$$x_{n+1} = x_n + \epsilon \sin \theta_n$$

$$\theta_{n+1} = x_n + \theta_n + \epsilon \sin \theta_n$$



Residual $|\mathbf{f}\left(\mathbf{p}(\mathbf{t})\right) - \mathbf{p}\left(\Lambda(t)\right)|$



How can we generalize the method to give more global information?