symbolic dynamics, itinerary map, top semi-conjugacy
- a standard proof that the logistic map is chaotic begins by establishin
shift on 2 symbols

(labeled) Conley index theory to prove surjectivity (first, surjectivity onto
continuity and compactness to argue surj. onto symbolic system)

compute cycles, grow isolating nbhd from union of cycles, gluing of reg
different periodic orbits [interplay between grid/resolution and spatial se
gives rise to index pairs with complicated topology (multiple generators

isolation -- S subset of grid, oS one-box neighborhood, check that coll
(invariant set) doesn't change -- Inv(S)=Inv(oS)

need automation to process complicated indices (mult. generators/regi

Combinatorial Approx
Discrete-Time Dy

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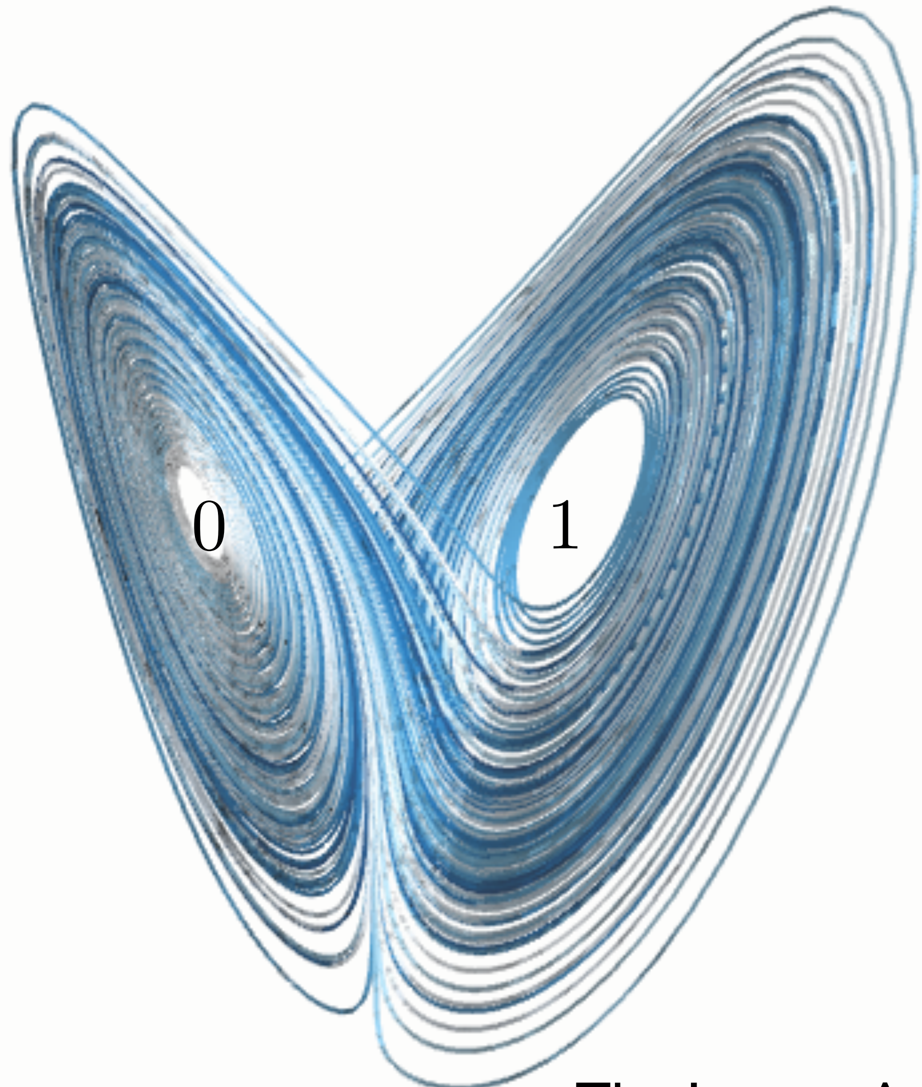
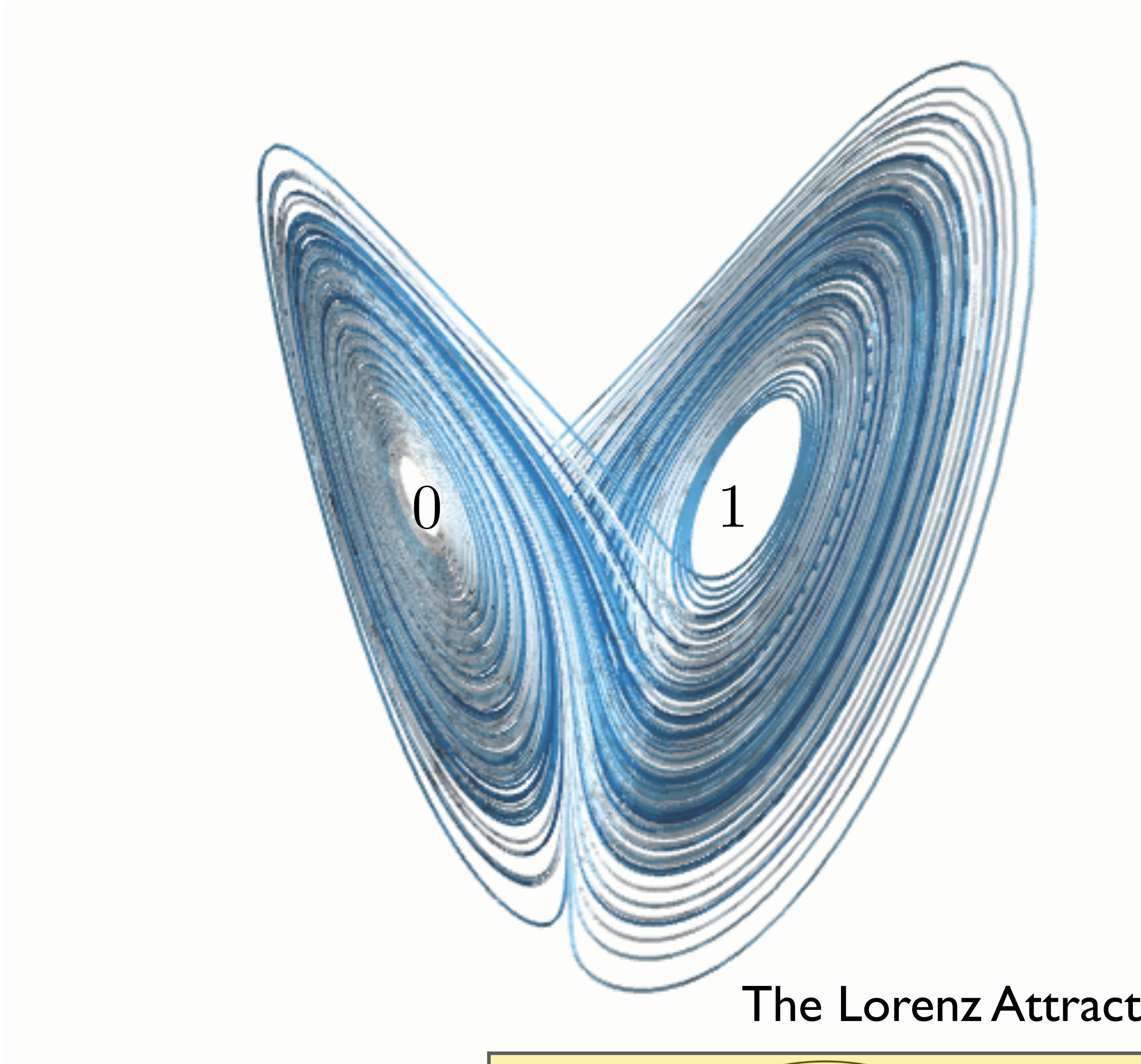
May 23, 2017

Joint work with Rafael Frongillo



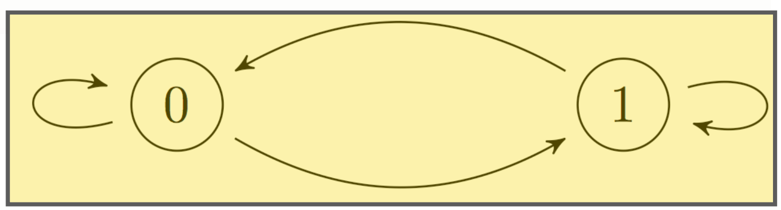
The Lorenz Attractor (Wikipedia)

an itinerary: $\rho(x) = 01100011101011100111\ldots$



The Lorenz Attractor (Wikipedia)

all itineraries:



See also Mischaikow, Mrozek 1995, Tucker 1999

Symbolic Dynamics:

Example 1: the full shift on two symbols $\sigma : \Sigma_2 \to \Sigma_2$

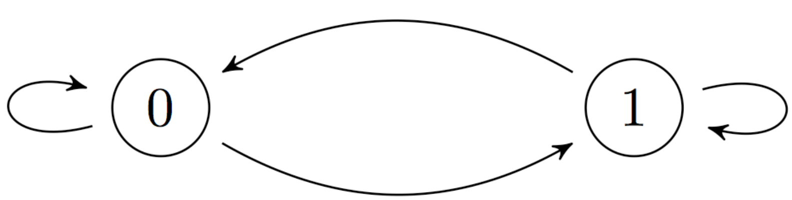
Phase space: $\Sigma_2 = \{\mathbf{s} = s_0 s_1 s_2 \ldots | s_i \in \{0,1\}\}$

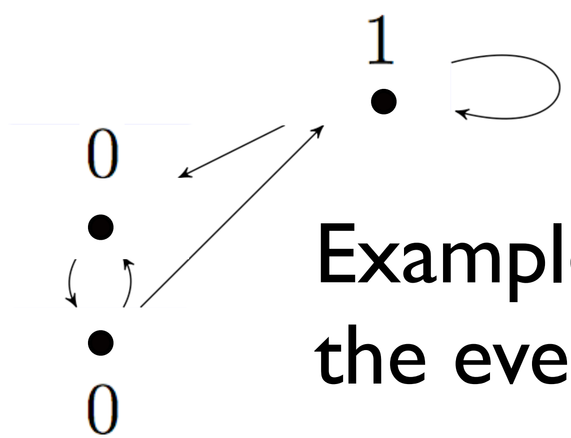
$$010101\ldots = \overline{01}$$

Map: $\sigma(s_0 s_1 s_2 \ldots) = s_1 s_2 s_3 \ldots$

$$\sigma(\overline{01}) = \sigma(010101\ldots) = 101010\ldots = \overline{10}$$

Vertex Shift Presentation:





Example 2:
the even shift

Sofic subshifts serve as catalogues of dynamics.

periodic orbits ~ cycles

connecting orbits ~ connecting paths

recurrent dynamics ~ SCC

topological entropy ~ $\log(sp(A))$

Problem:

Many systems don't come to us as sofic subshifts.

Use computational topology (and ...) to build a symbolic description.

$$\begin{array}{ccc} S & \xrightarrow{f} & S \\ \downarrow \rho & & \downarrow \rho \\ \Sigma_T & \xrightarrow{\sigma} & \Sigma_T \end{array}$$

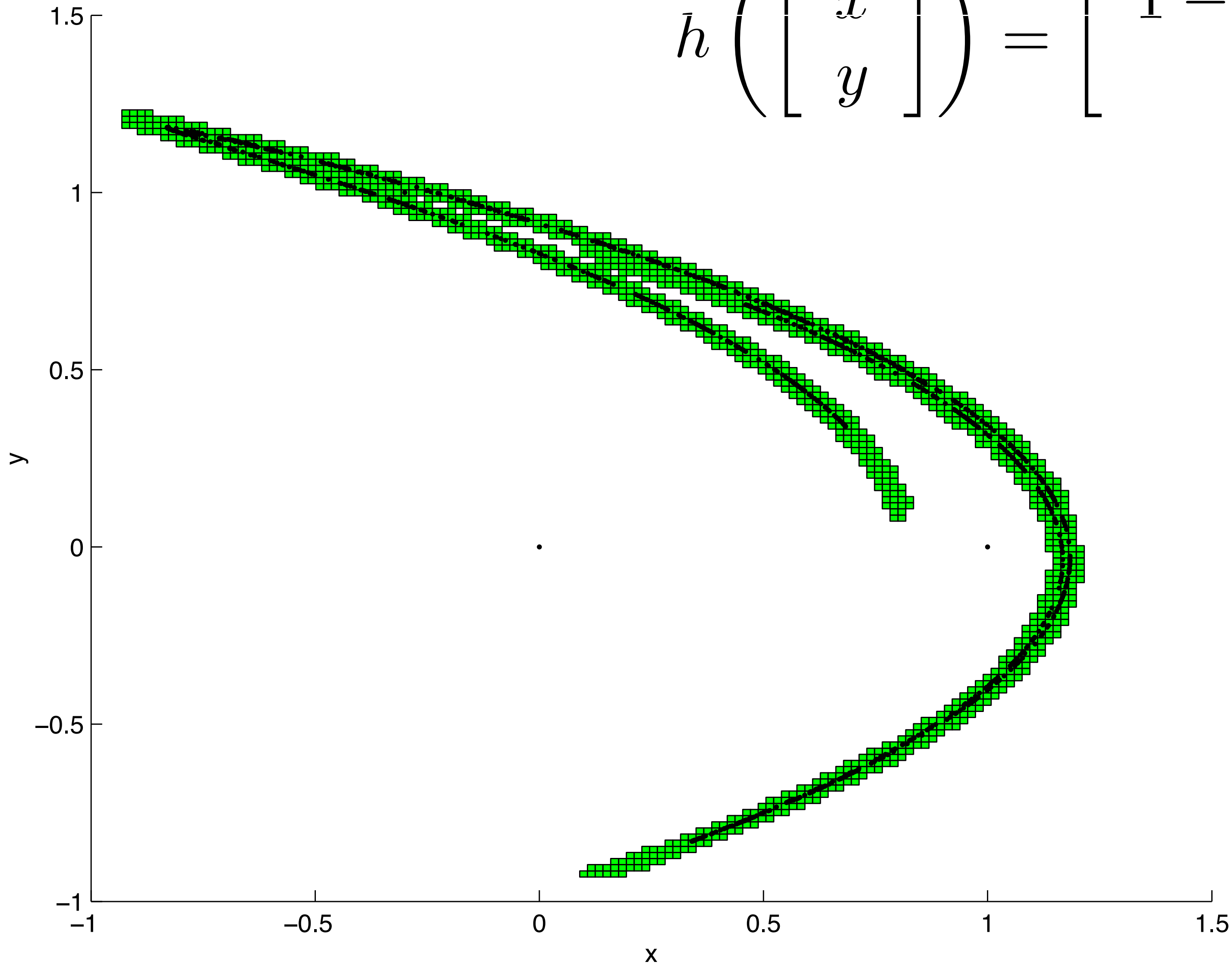
If $\rho \circ f = \sigma \circ \rho$ and ρ is a homeomorphism, then f and σ are *topologically conjugate*.

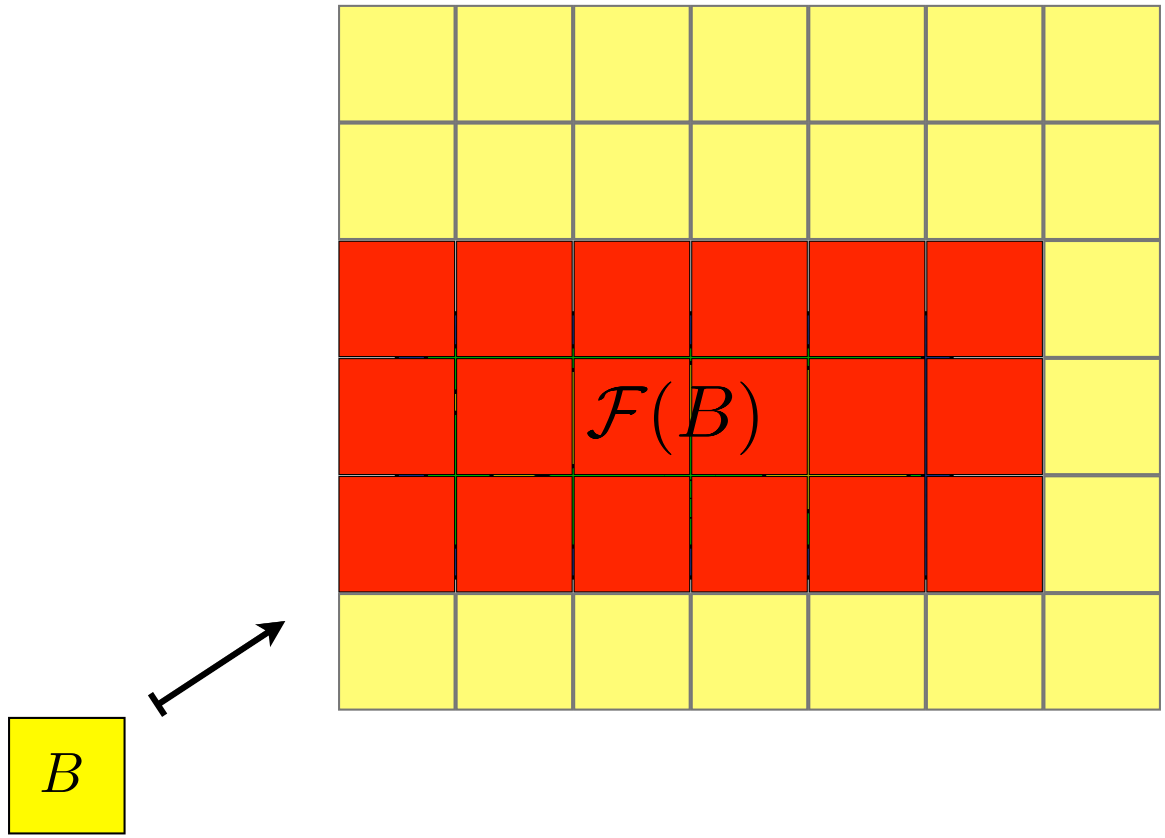
If the diagram commutes but we relax the condition that ρ is 1-to-1, then f is *topologically semi-conjugate* to σ.

Use Conley Index Theory to prove surjectivity.

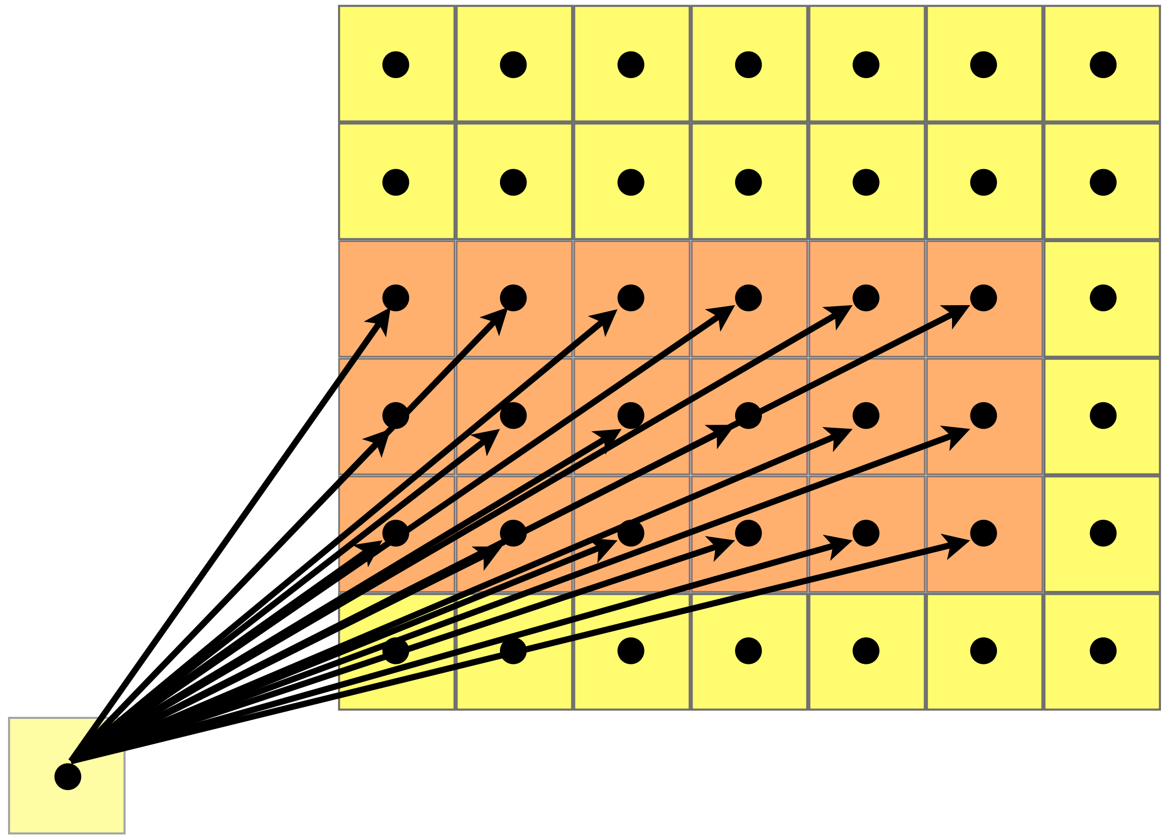
Example: the Henon map

$$h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 - 1.3x^2 + y/5 \\ x \end{bmatrix}$$



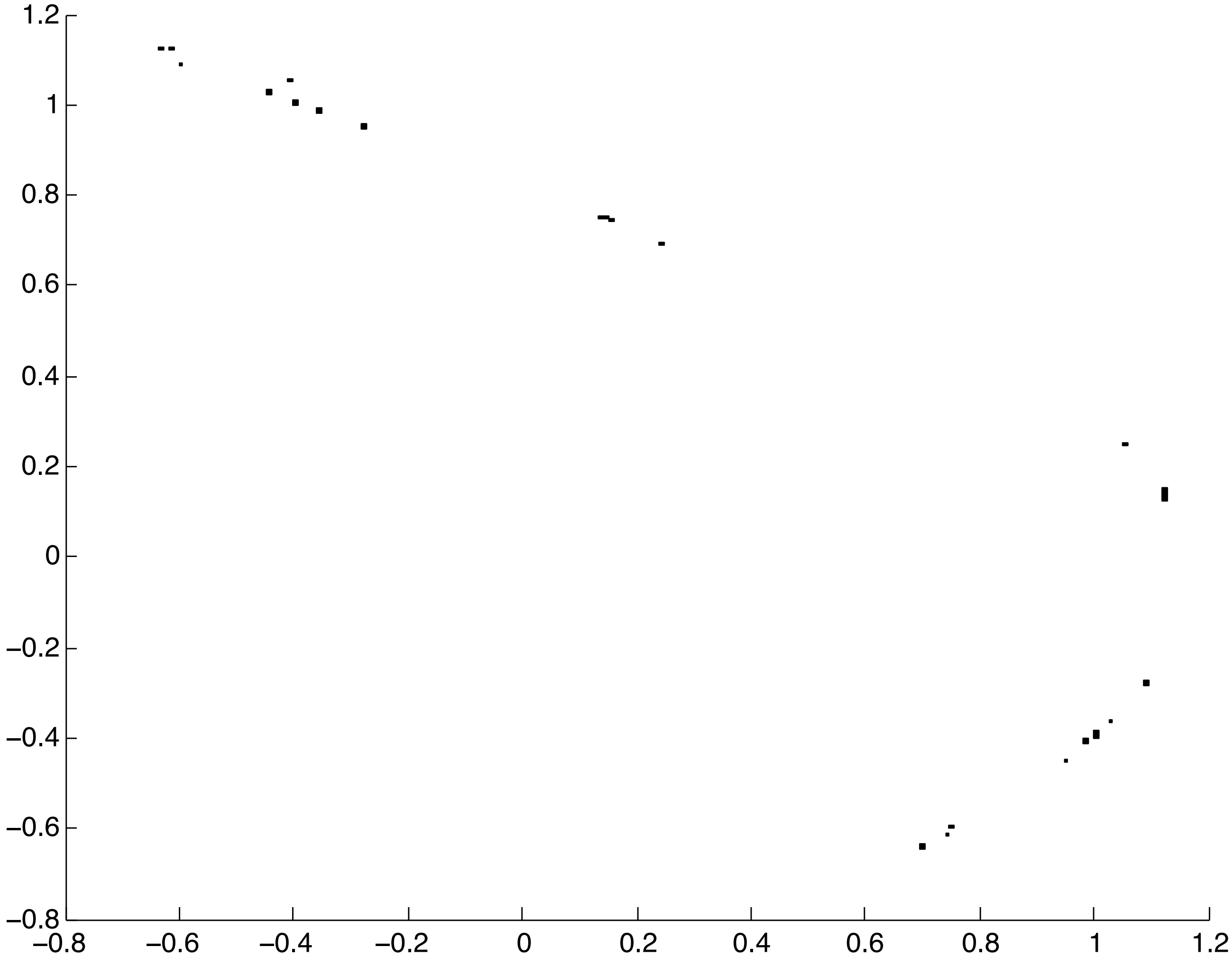


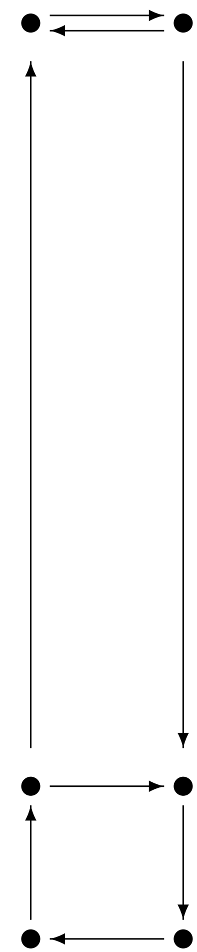
Outer Approximation



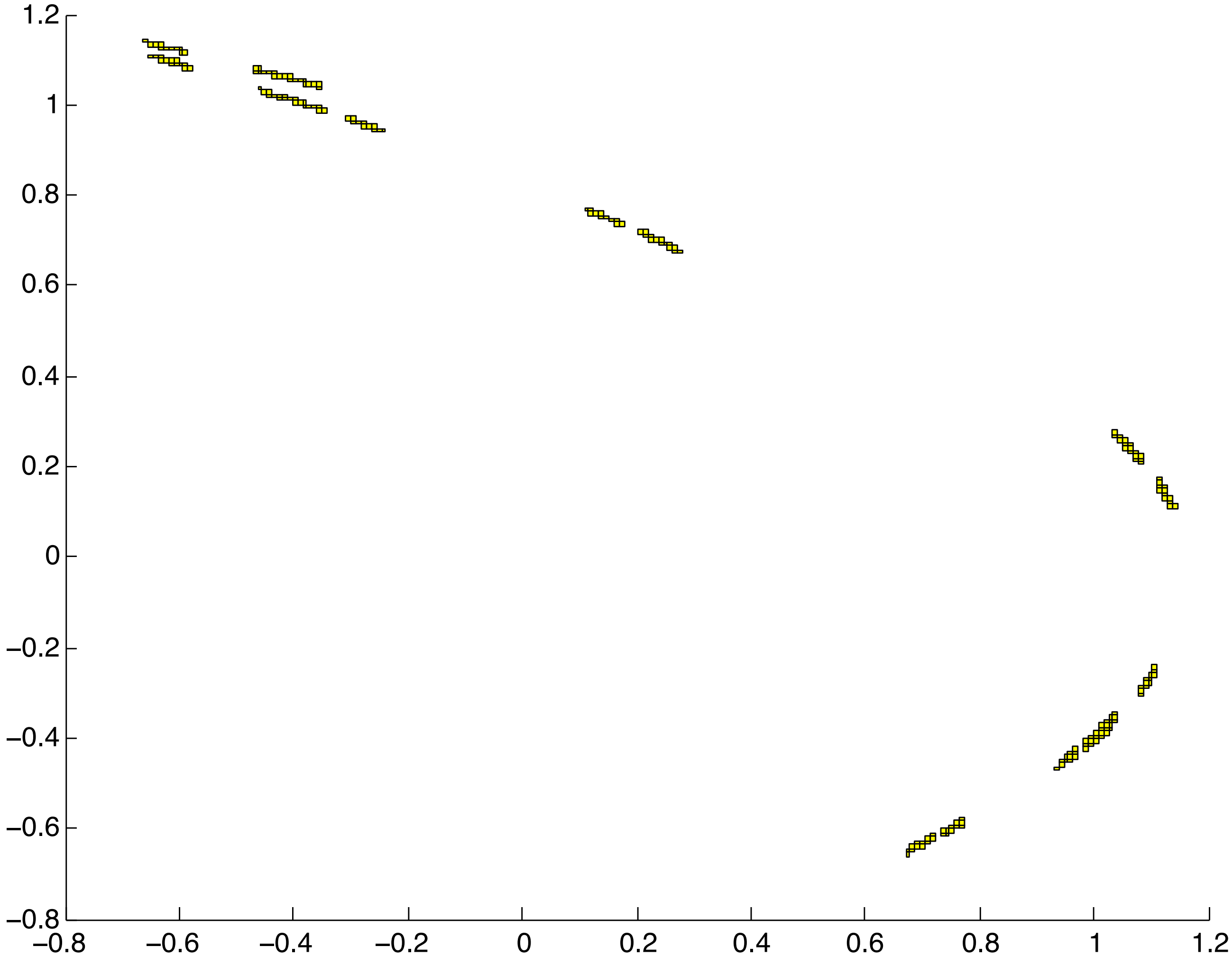
Outer Approximation

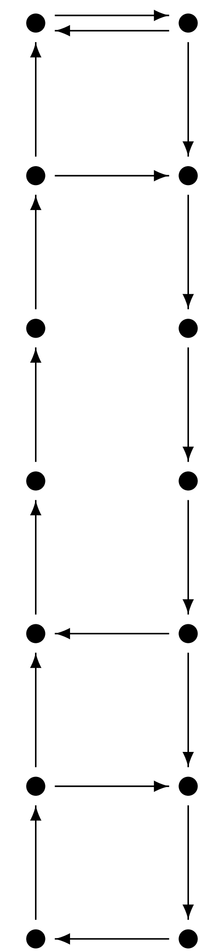
interesting structure in the outer approximation



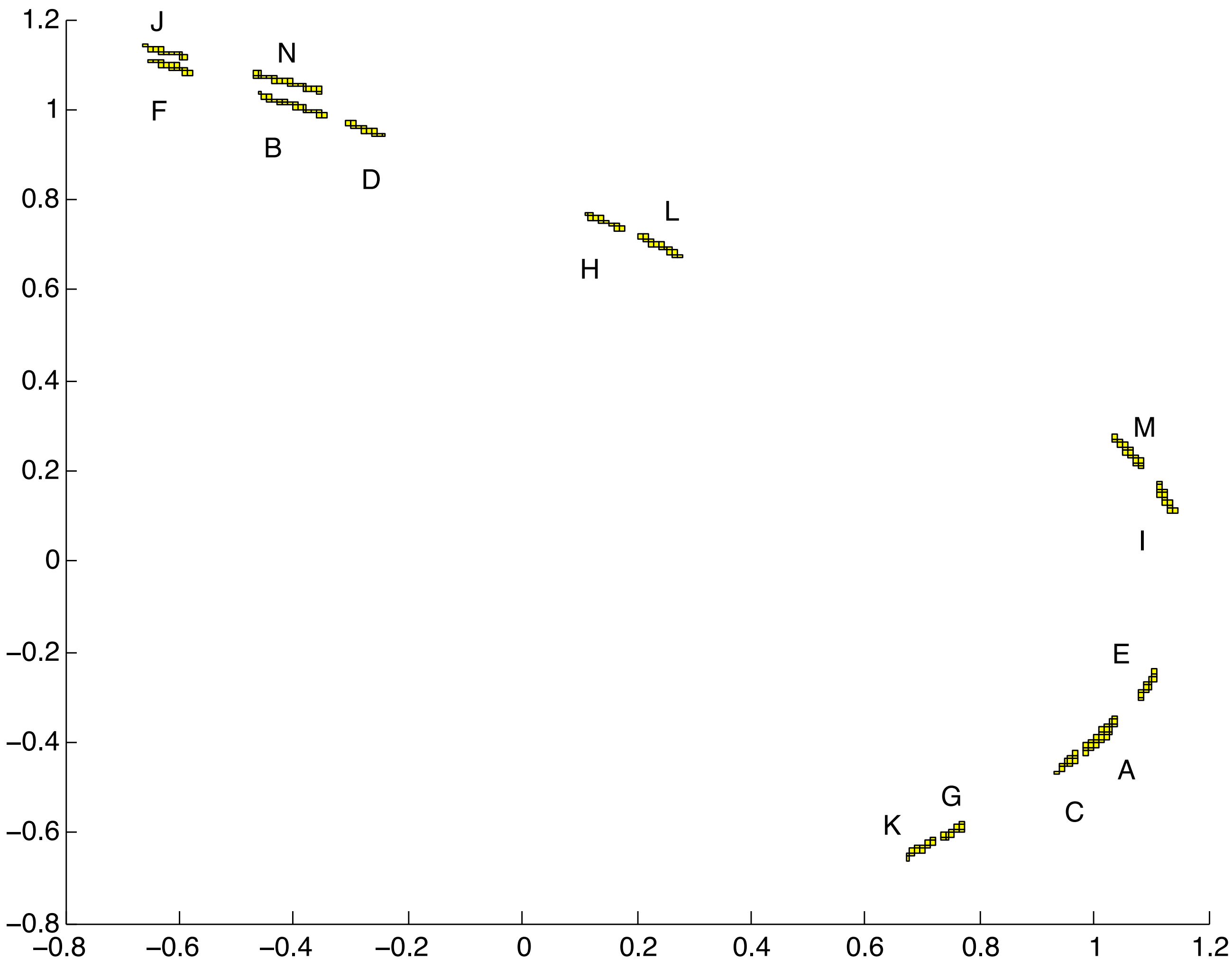


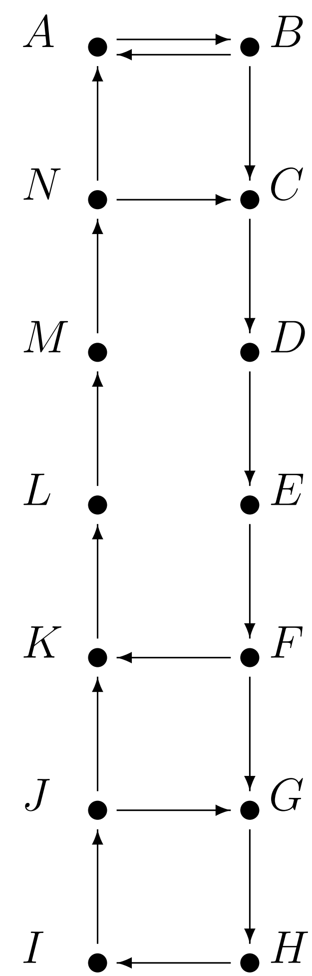
growing an isolating neighborhood



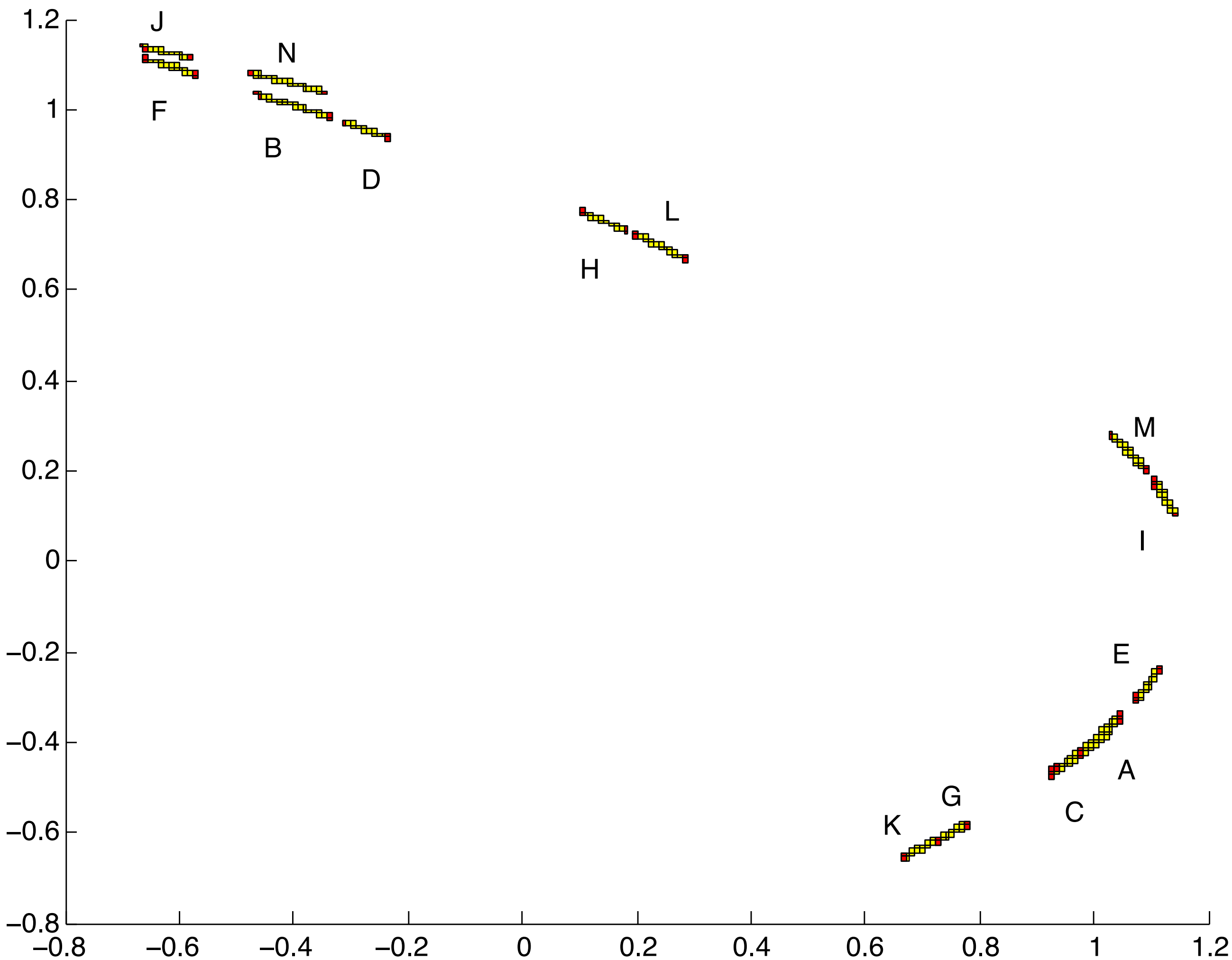


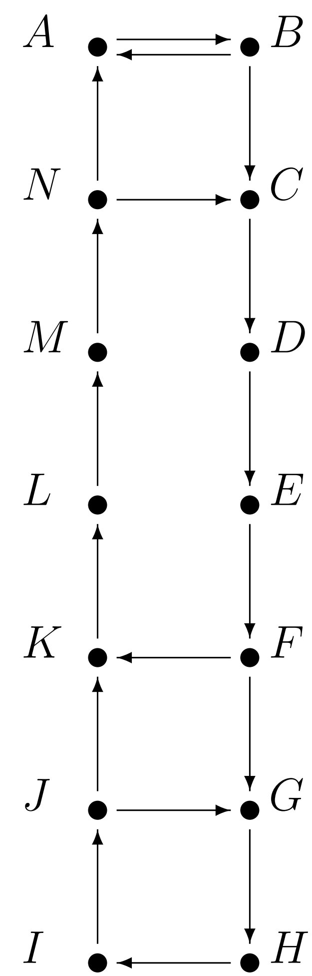
labeling regions





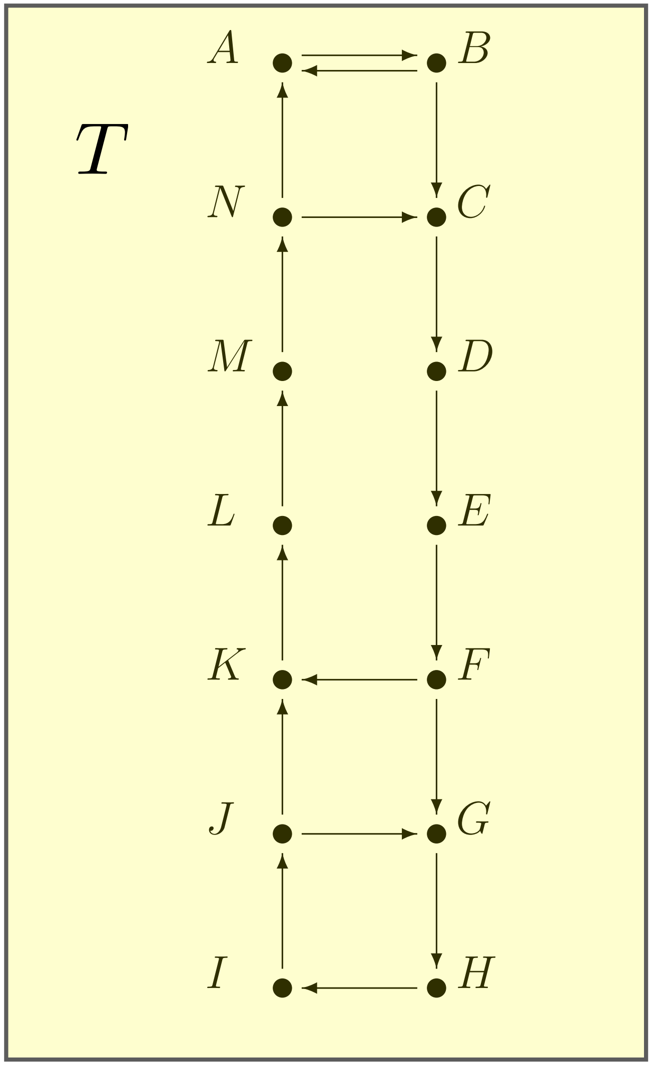
building an index pair

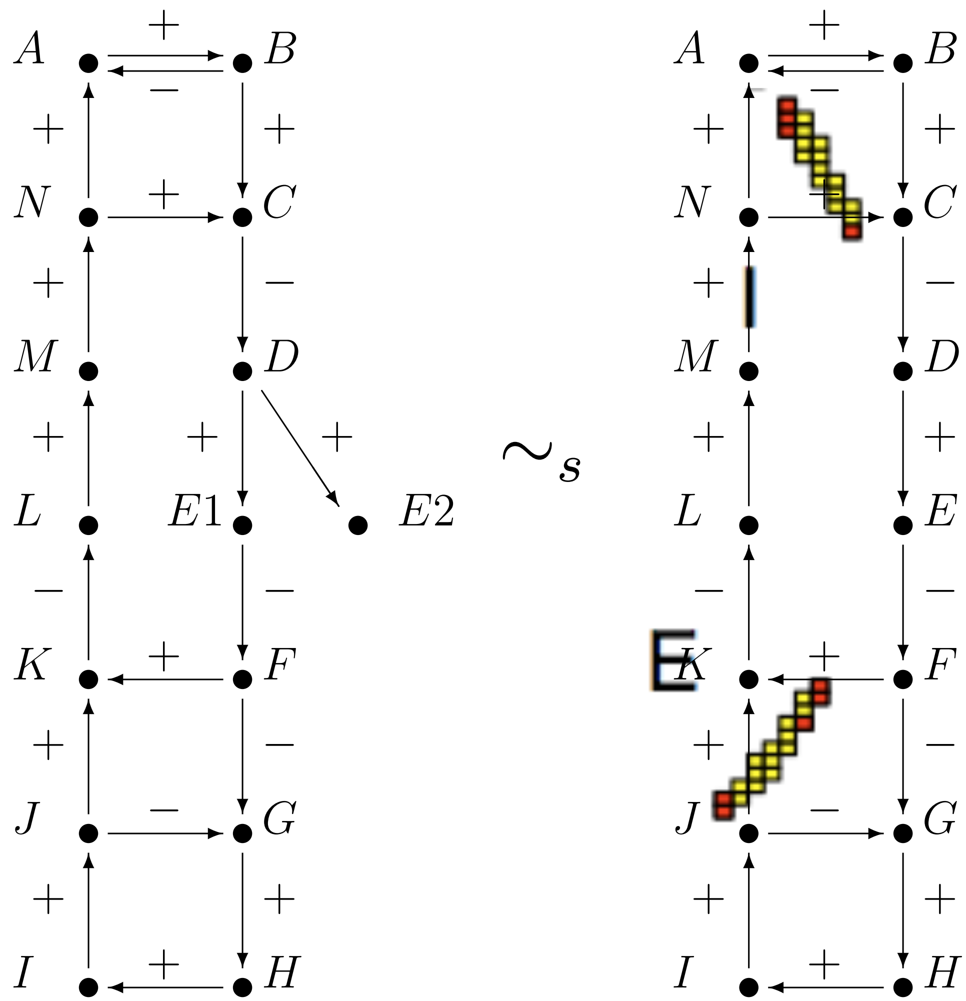




computing the index

$$H_*(P_1, P_0) \cong (0,\ \mathbb{Z}^{15},\ 0,\ 0,\ \ldots)$$

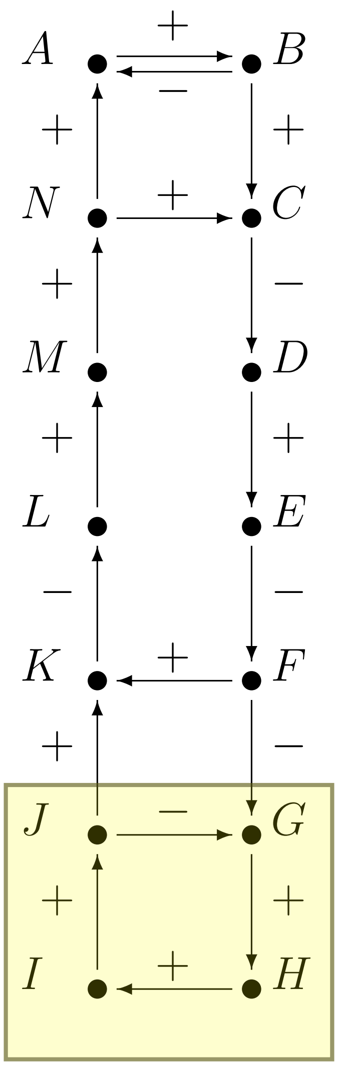




transition graph on components

index map on generators

computing the index



$$
\begin{aligned}
\mathrm{Con}_*(S', h|_J \circ h|_I \circ h|_H \circ h|_G) \\
&= ([-1][1][1][1])_s \\
&= [-1]_s \\
&\neq [0]_s \qquad \text{(since not nilpotent)}
\end{aligned}
$$

Therefore, $\overline{GHIJ} \in \mathrm{Im}(\rho)$.

verifying symbolic dynamics

$$\begin{array}{ccc} S & \xrightarrow{h} & S \\ \downarrow \rho & & \downarrow \rho \\ \Sigma_T & \xrightarrow{\sigma} & \Sigma_T \end{array}$$

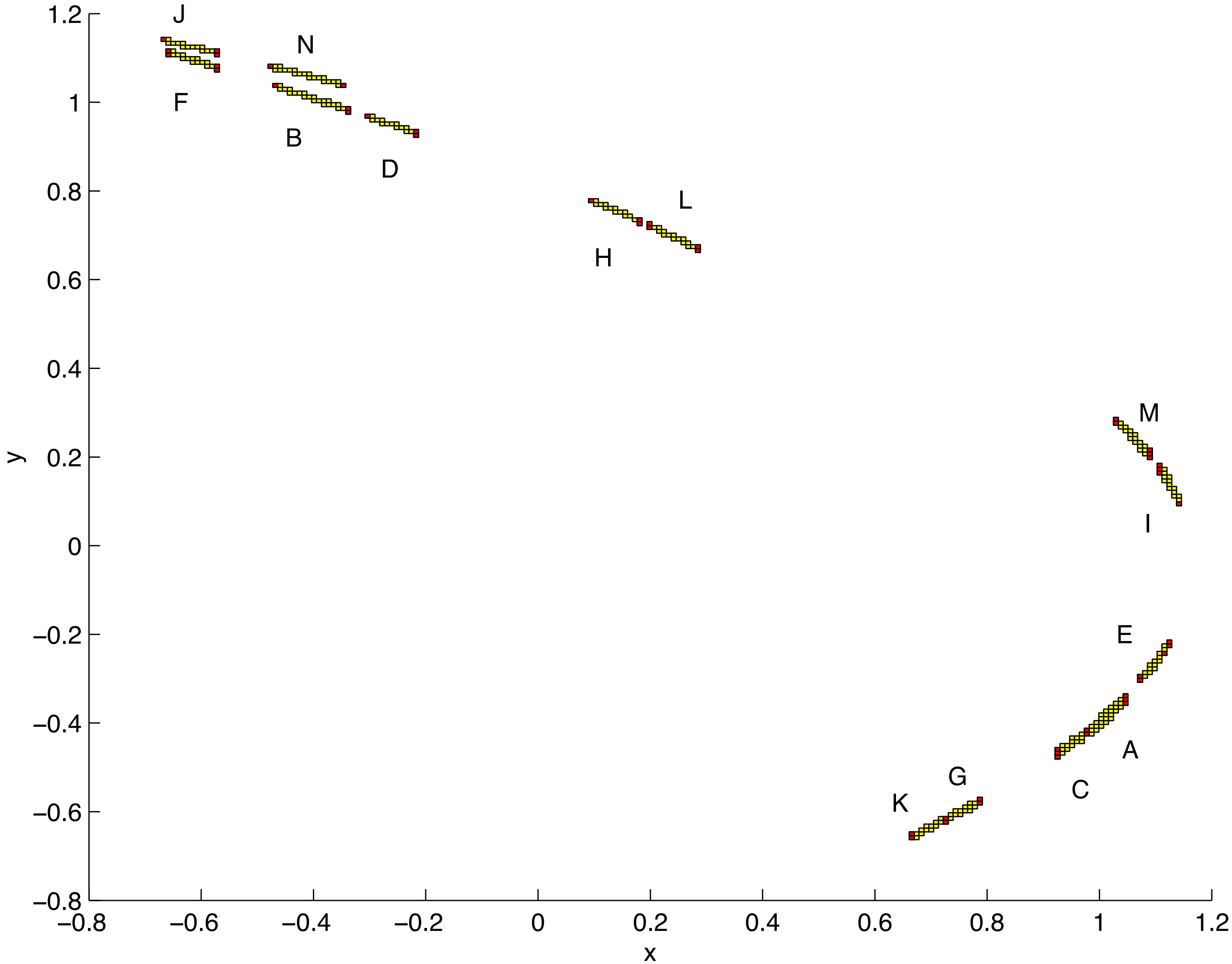
$$\Sigma_T = \{(s_i)_i | (s_i, s_{i-1}) \text{an edge in } T\}$$

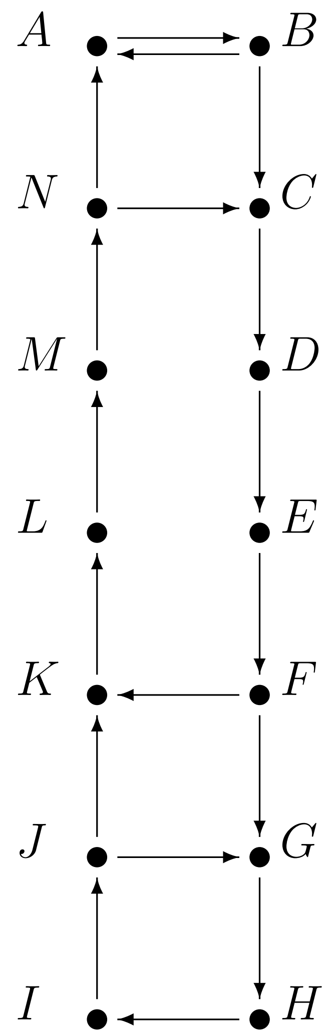
$$\rho(x) = \{(s_i)_i | h^i(x) \in s_i\}$$

Since the Conley index corresponding to each cycle in T is nontrivial, ρ maps onto the set of periodic points in Σ_T.

ρ is continuous and S is compact. Therefore, ρ maps onto Σ_T, the closure of the set of periodic points in Σ_T.

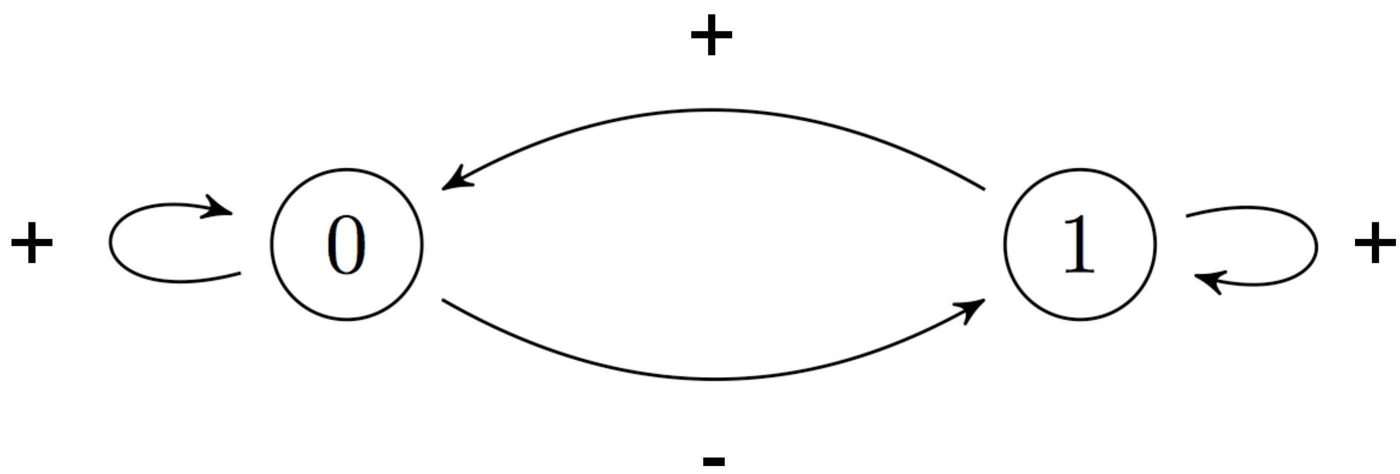
Theorem. [D., Junge, Mischaikow] There is a semi-conjugacy from h on $S := \mathrm{Inv}(P_1 \setminus P_0, h)$ to the symbol subshift given by the transition graph.





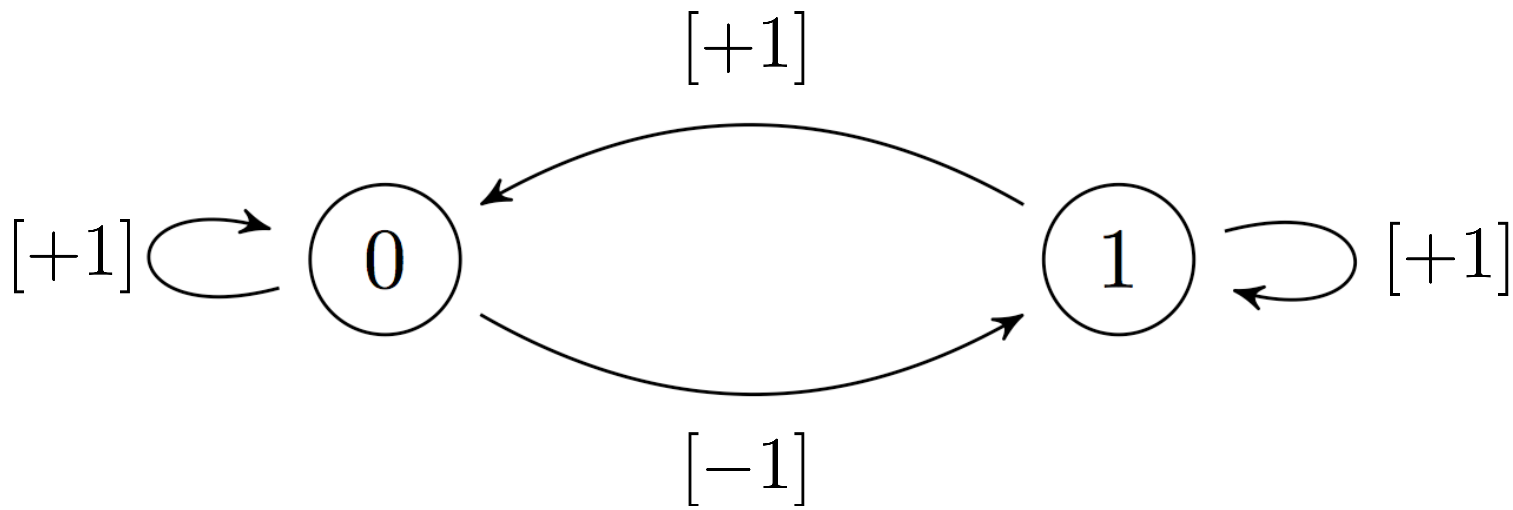
Consider a symbol transition graph weighted by matrices (individual Conley index maps).

Before:



Consider a symbol transition graph weighted by matrices (individual Conley index maps).

Before:

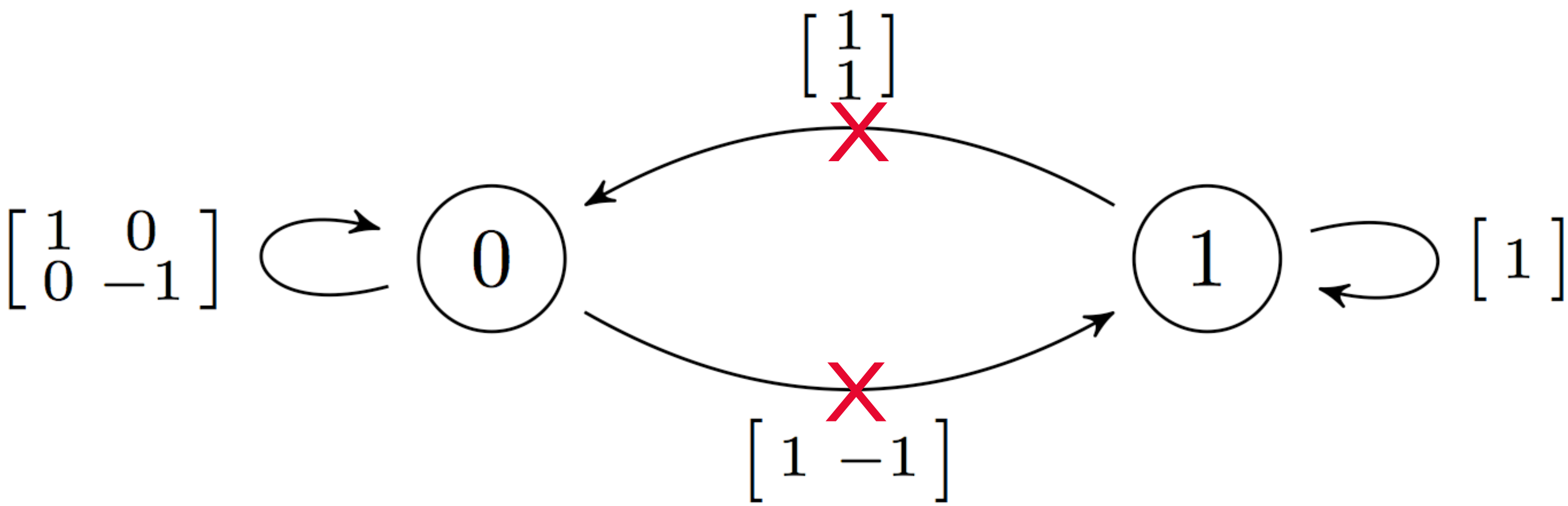


Consider a symbol transition graph weighted by matrices (individual Conley index maps).

Now:

$$
\begin{tikzcd}
0 \arrow[loop left, "{\begin{bmatrix}1&0\\0&-1\end{bmatrix}}"] \arrow[rr, bend right, "{\begin{bmatrix}1&-1\end{bmatrix}}"'] & & 1 \arrow[ll, bend right, "{\begin{bmatrix}1\\1\end{bmatrix}}"'] \arrow[loop right, "{\begin{bmatrix}1\end{bmatrix}}"]
\end{tikzcd}
$$

Note: nilpotency is preserved by the index. If an index map is not nilpotent, the index is nontrivial and the corresponding invariant set is nonempty.



$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^k = \begin{bmatrix} 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$$

problem: $\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

cut at least one edge from $\{(0,1), (1,0)\}$

verified subshift

two fixed points, zero entropy

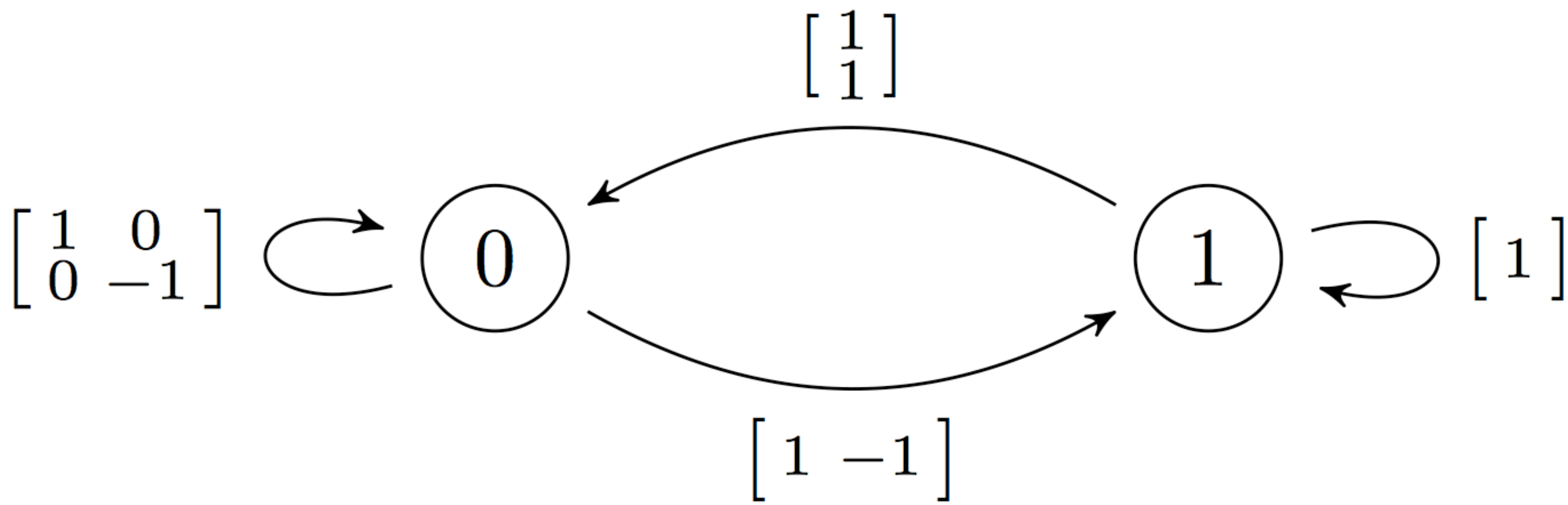
$$\begin{tikzcd}[column sep=huge] 0 \arrow[loop left, "\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}"] \arrow[r, bend right, "\begin{bmatrix}1 & -1\end{bmatrix}"'] & 1 \arrow[l, bend right, "\begin{bmatrix}1\\ 1\end{bmatrix}"'] \arrow[loop right, "\begin{bmatrix}1\end{bmatrix}"] \end{tikzcd}$$

problem: $\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = 0$

prohibit the word 101

the cycle 1001 has matrix product

$$\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix} = 2$$



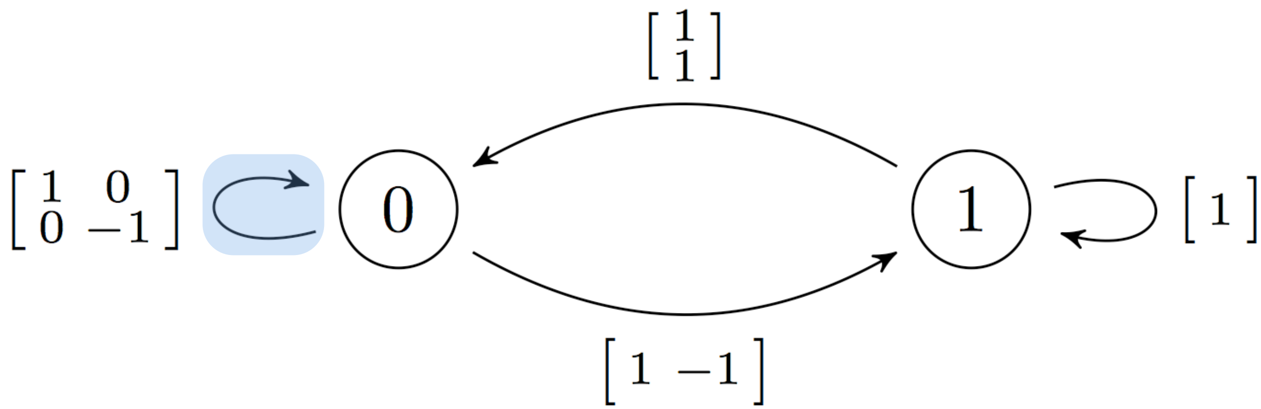
the cycle 10^k1 has matrix product

$$\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1&-1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 2, \text{ if } k \text{ is even}$$

$$\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1&-1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 0, \text{ if } k \text{ is odd}$$

prohibit 10^k1 *for* k *odd*

(this is not a subshift of finite type)



Finite state machine approach (D., Frongillo)

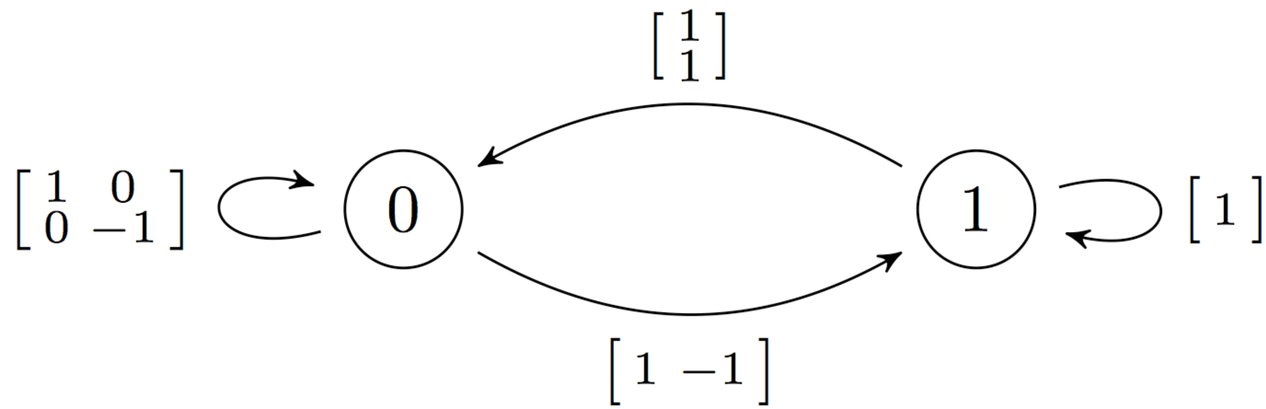
Cocyclic subshifts (Kwapisz 1999)

$$\left\{0, \begin{bmatrix}1&0\\0&1\end{bmatrix}\right\}$$

current symbol

matrix product image space

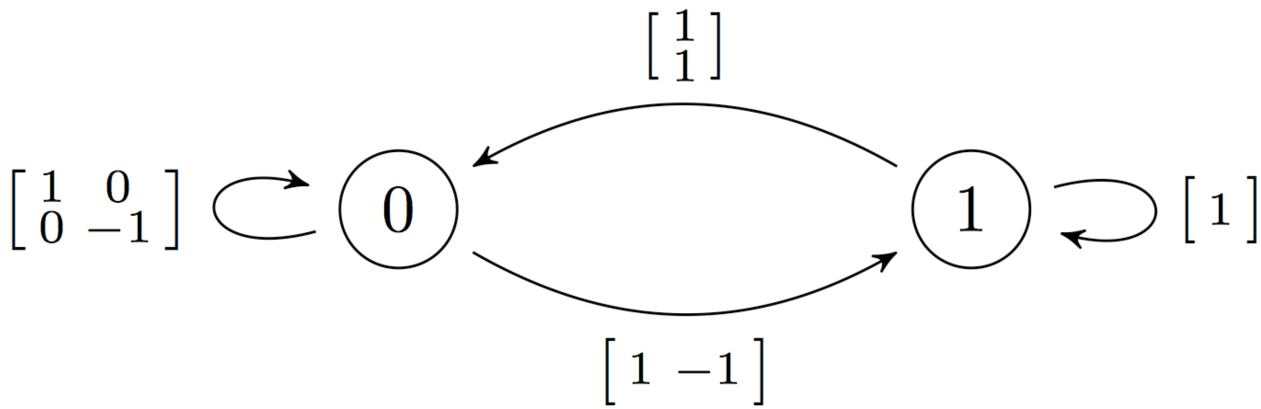
length 1 paths

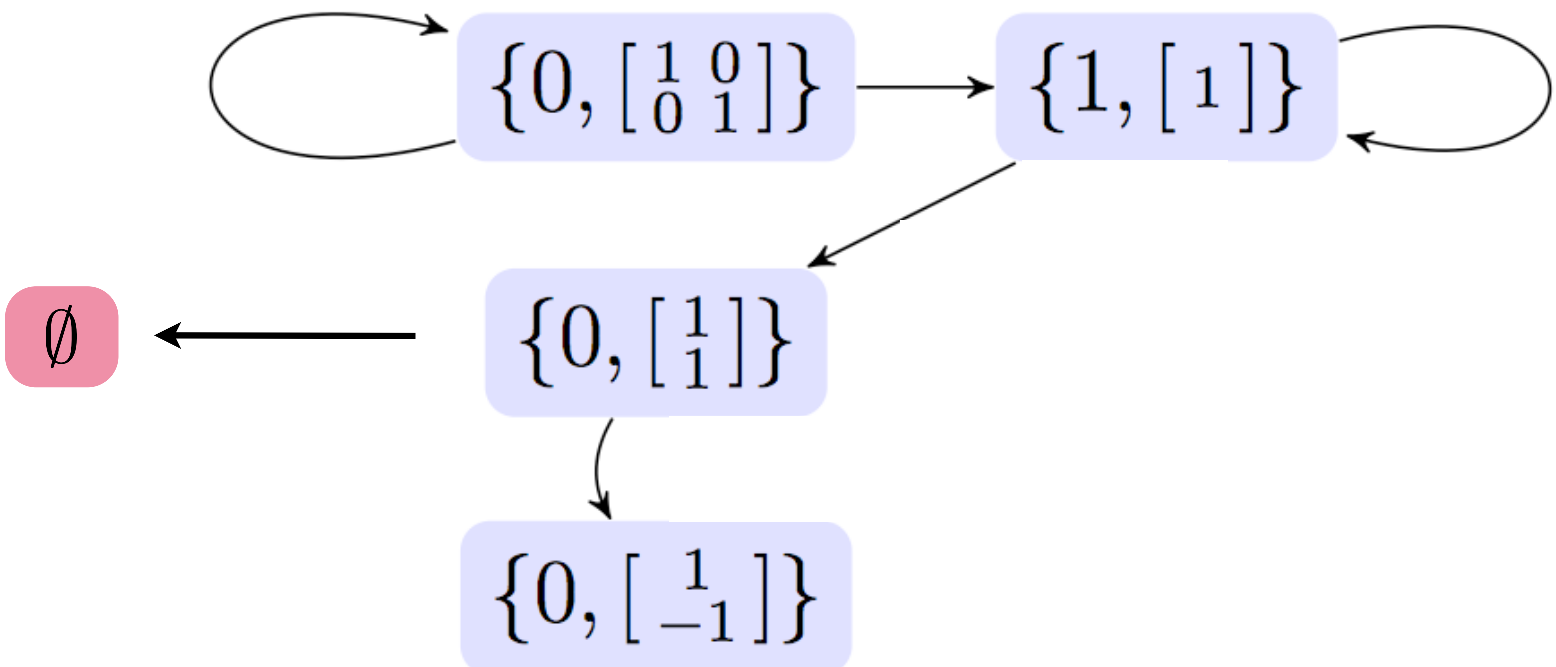


$\{0, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}$ $\quad$ $\{1, \begin{bmatrix} 1 \end{bmatrix}\}$

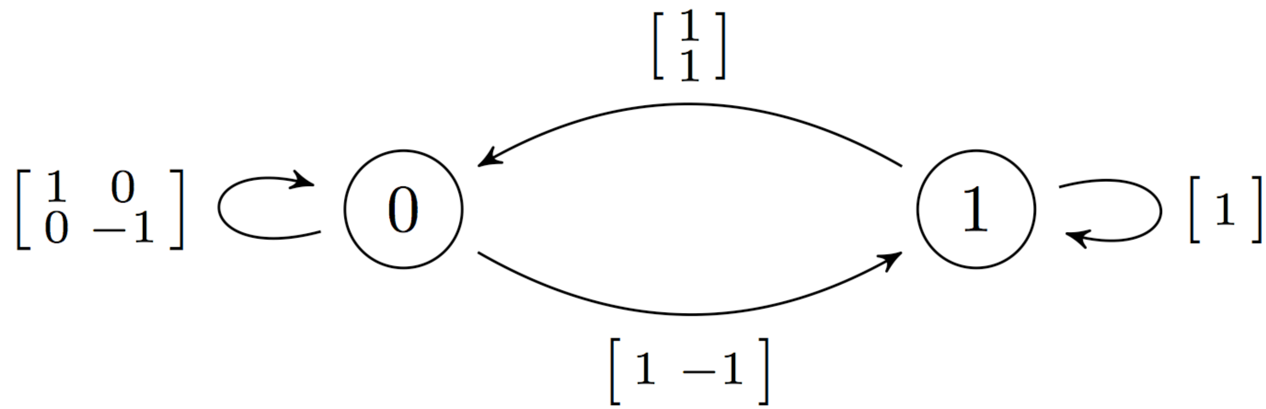
$\{0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$

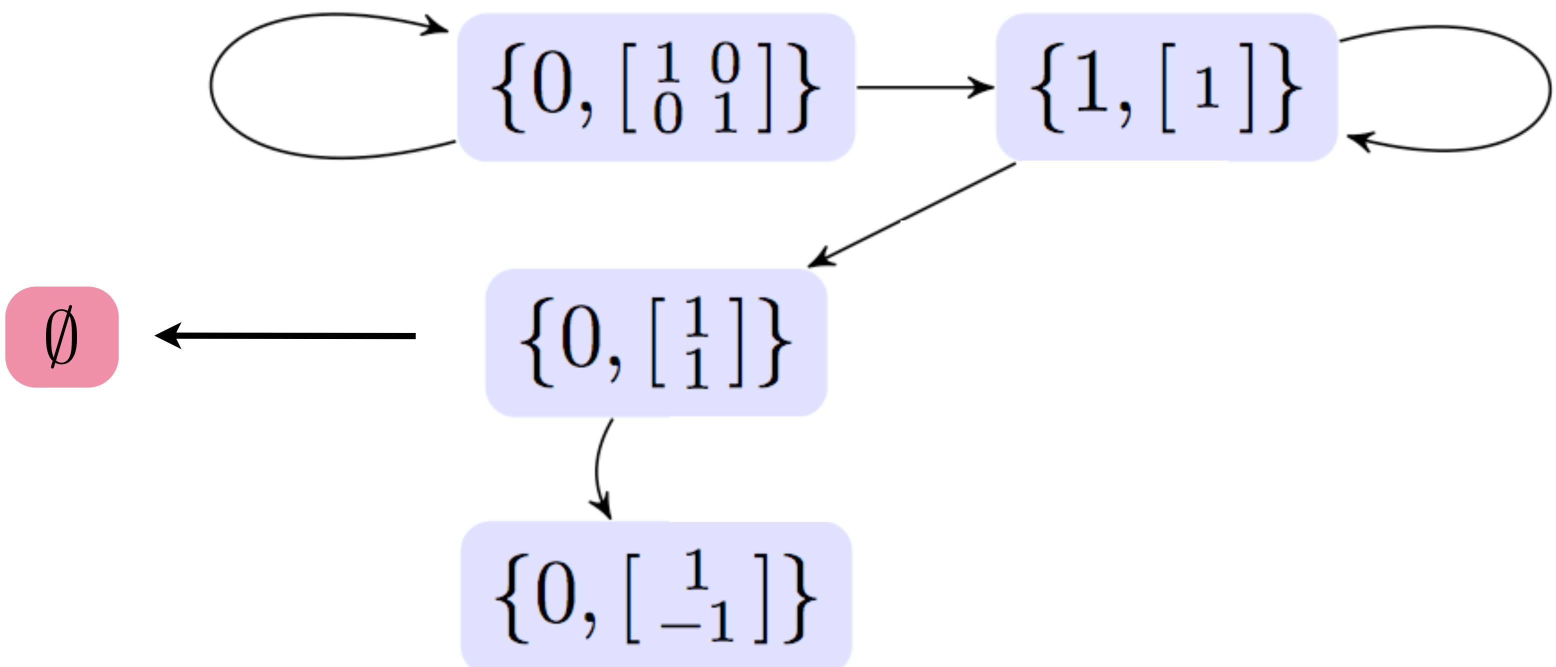
length 1 paths



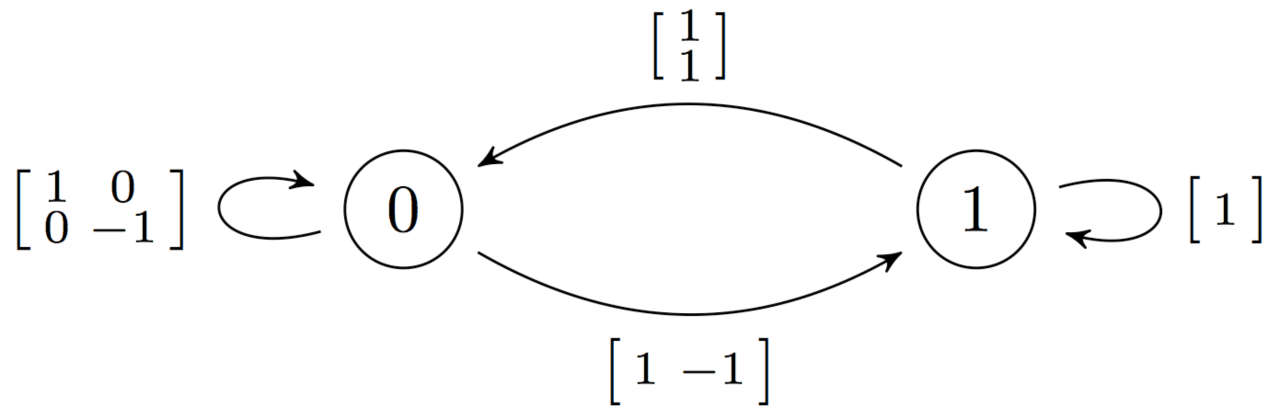


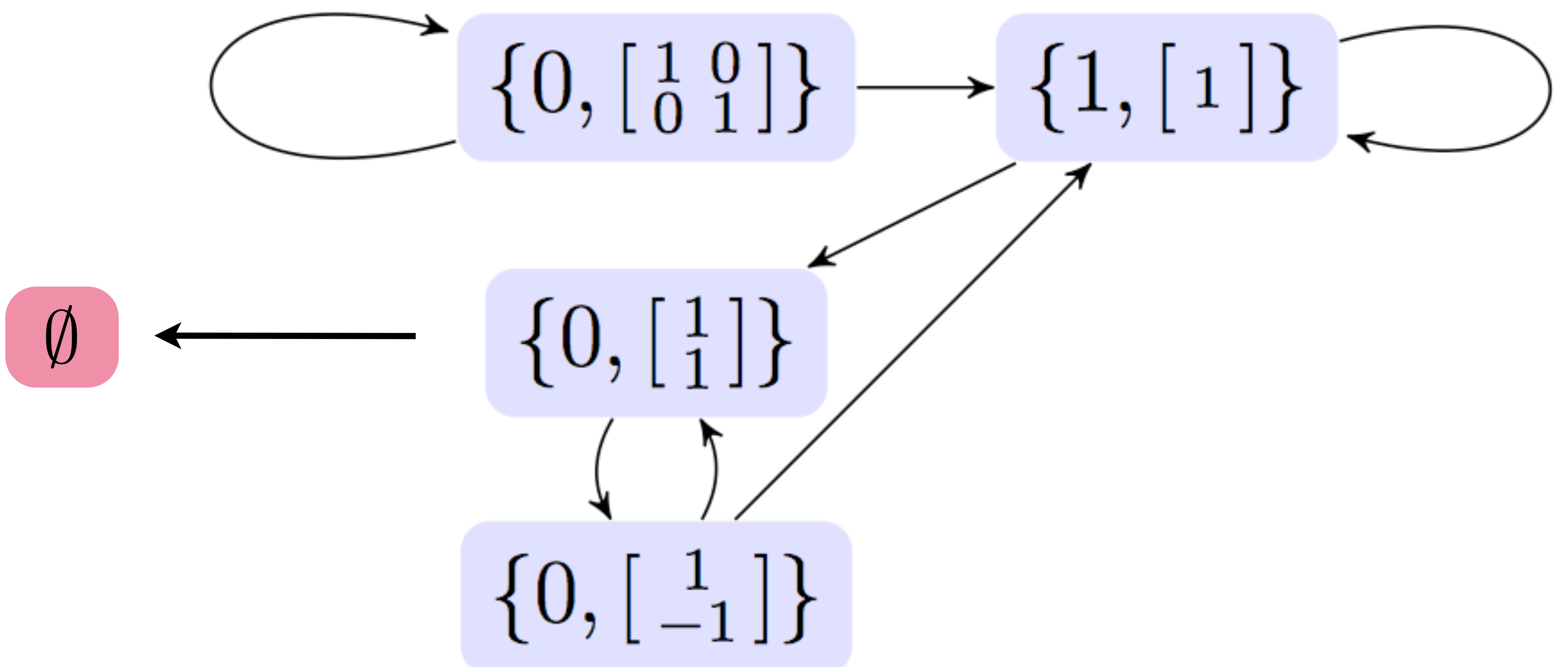
length 2 paths





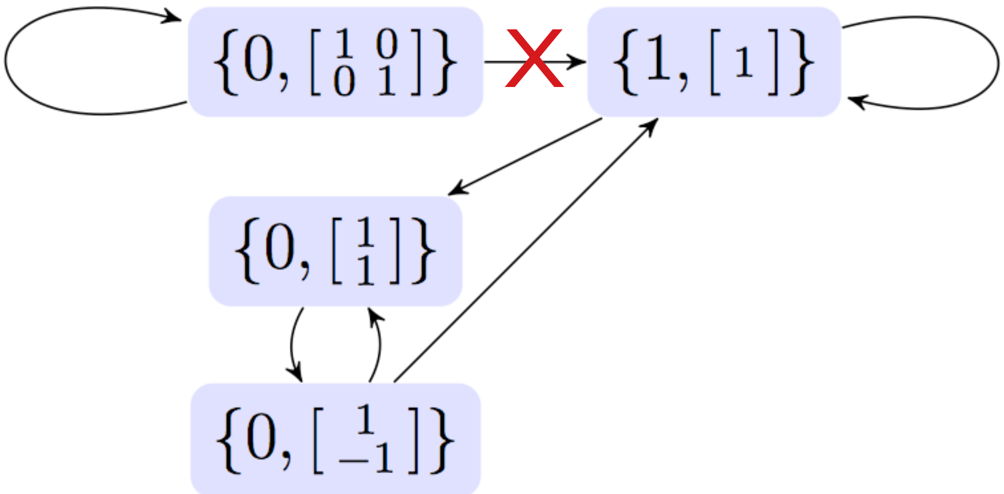
unprocessed length 3 paths

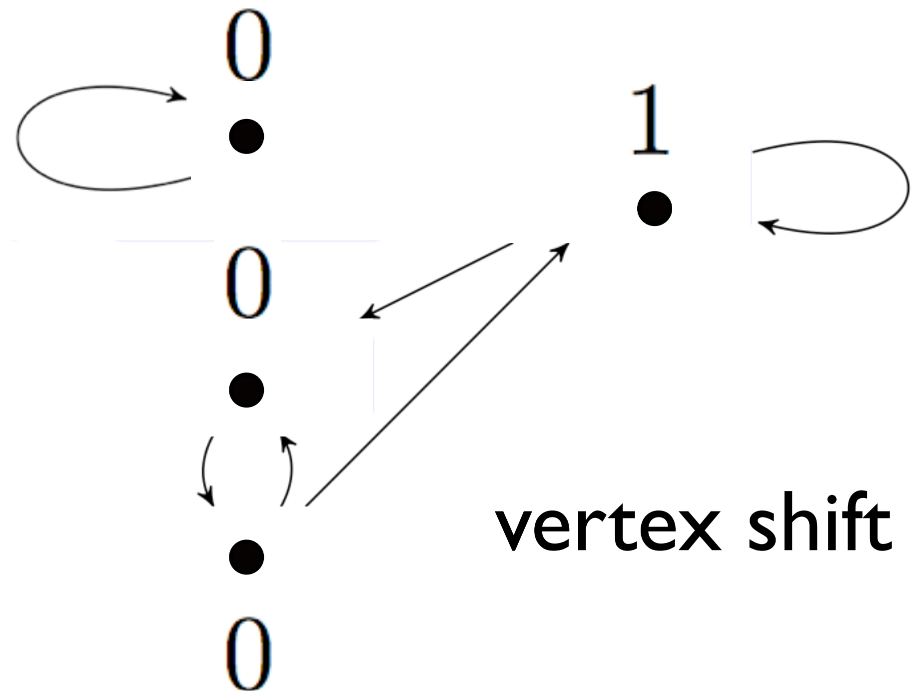




unprocessed length 3 paths

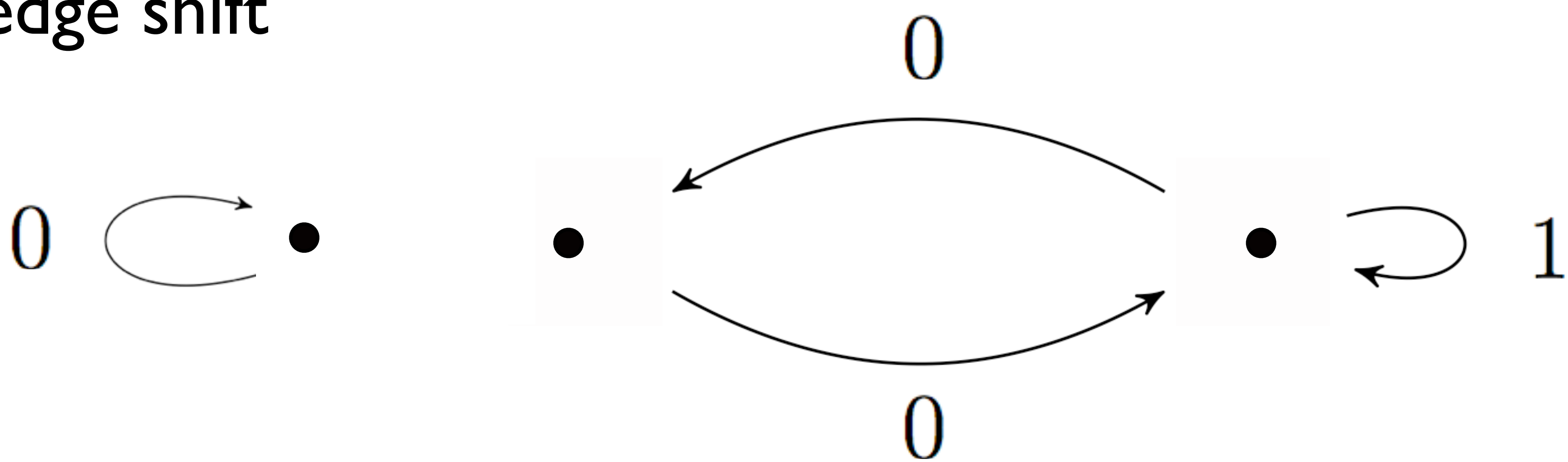
Even Shift and Fixed Point



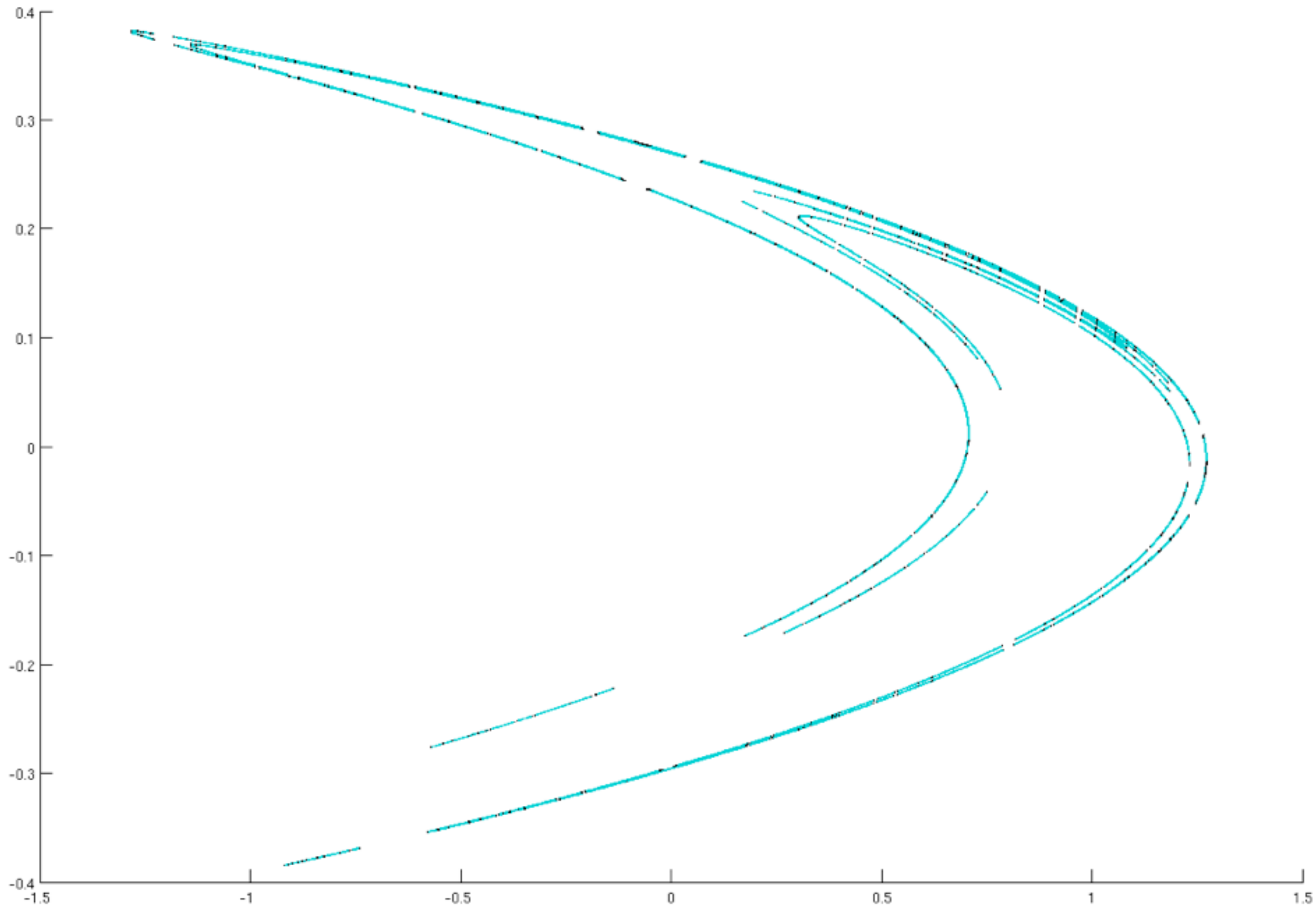


vertex shift

edge shift



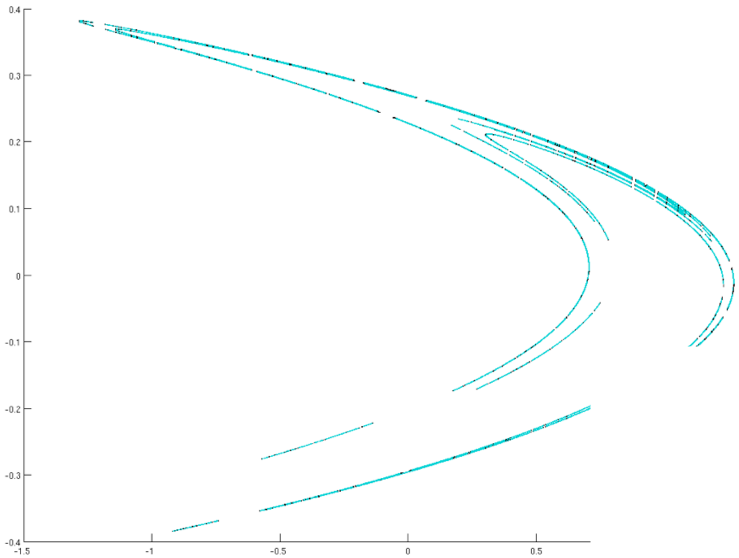
Theorem. [D., Frongillo] The topological entropy for the Henon map is bounded from below by 0.4555 (0.4410 using DFT).

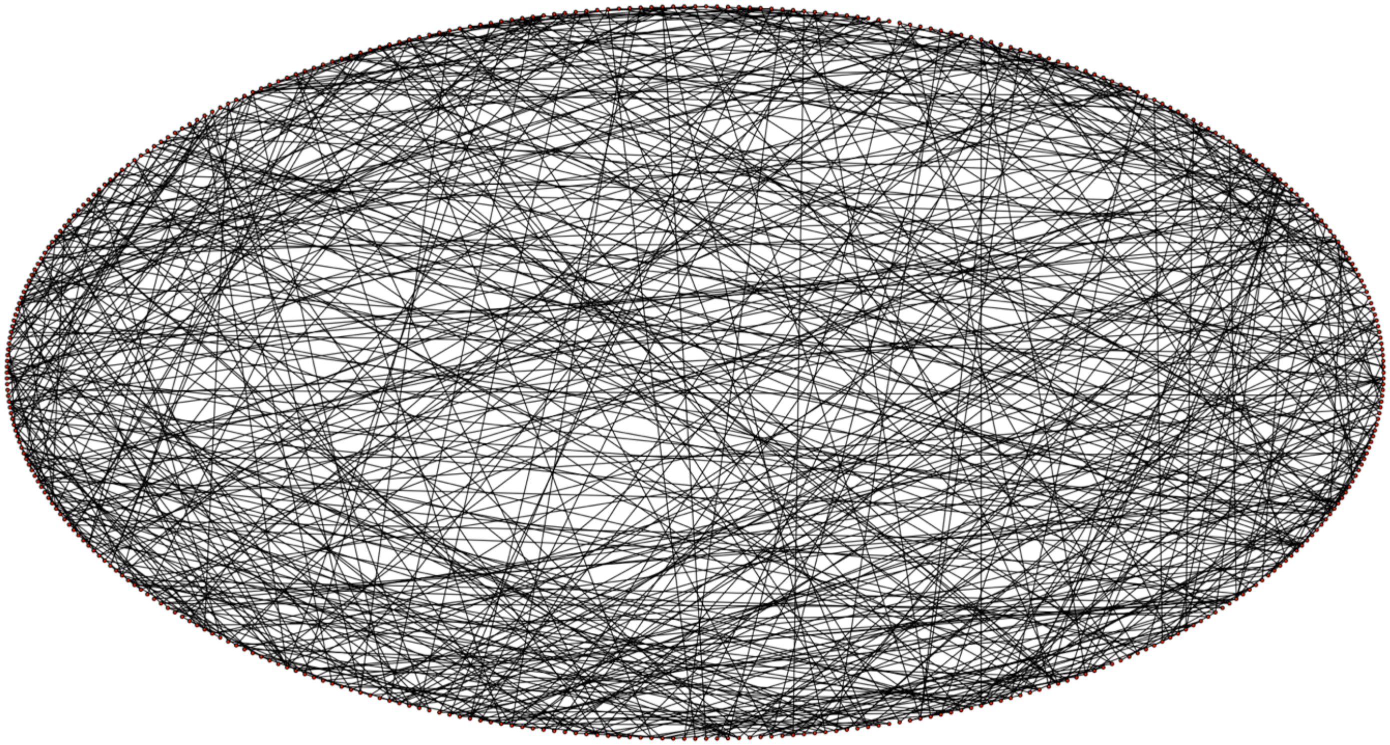


342 symbols/regions of index pair (~24
396 states in sofic shift

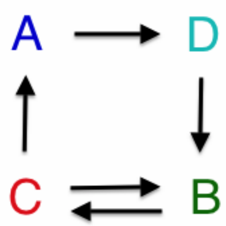
Galias: topological entropy of $h \approx 0.4650$
Newhouse, Berz, Grote, and Makino:
topological entropy of h ≥ 0.4617

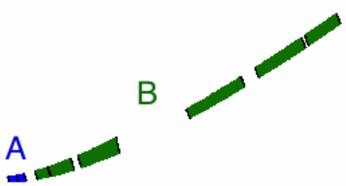
Theorem. [D., Frongillo] The topological entropy for the Henon map is bounded from below by 0.4555 (0.4410 using DFT).





KSR sample result (D., Kalies)

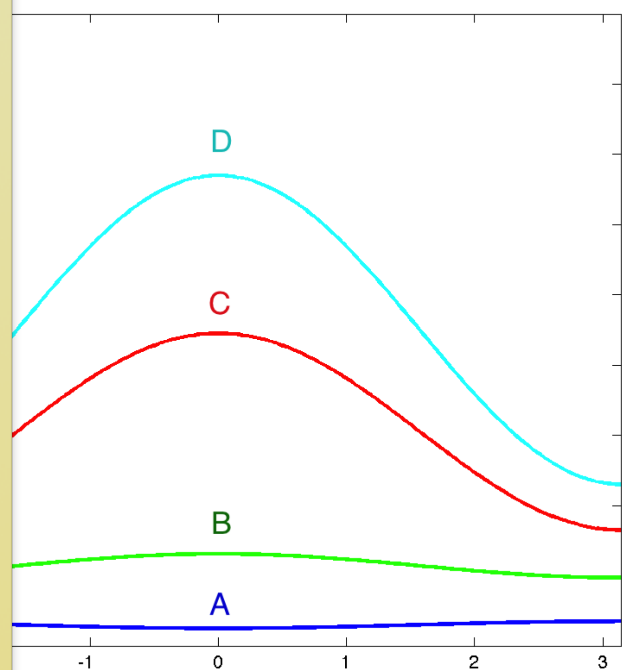




M=30
m=6
s_*=10
10^{-5}<E^{Ch}<10^{-4}
33 subdivisions
alt. between testing pointwise bounds and computing mv map and extracting recurrent set

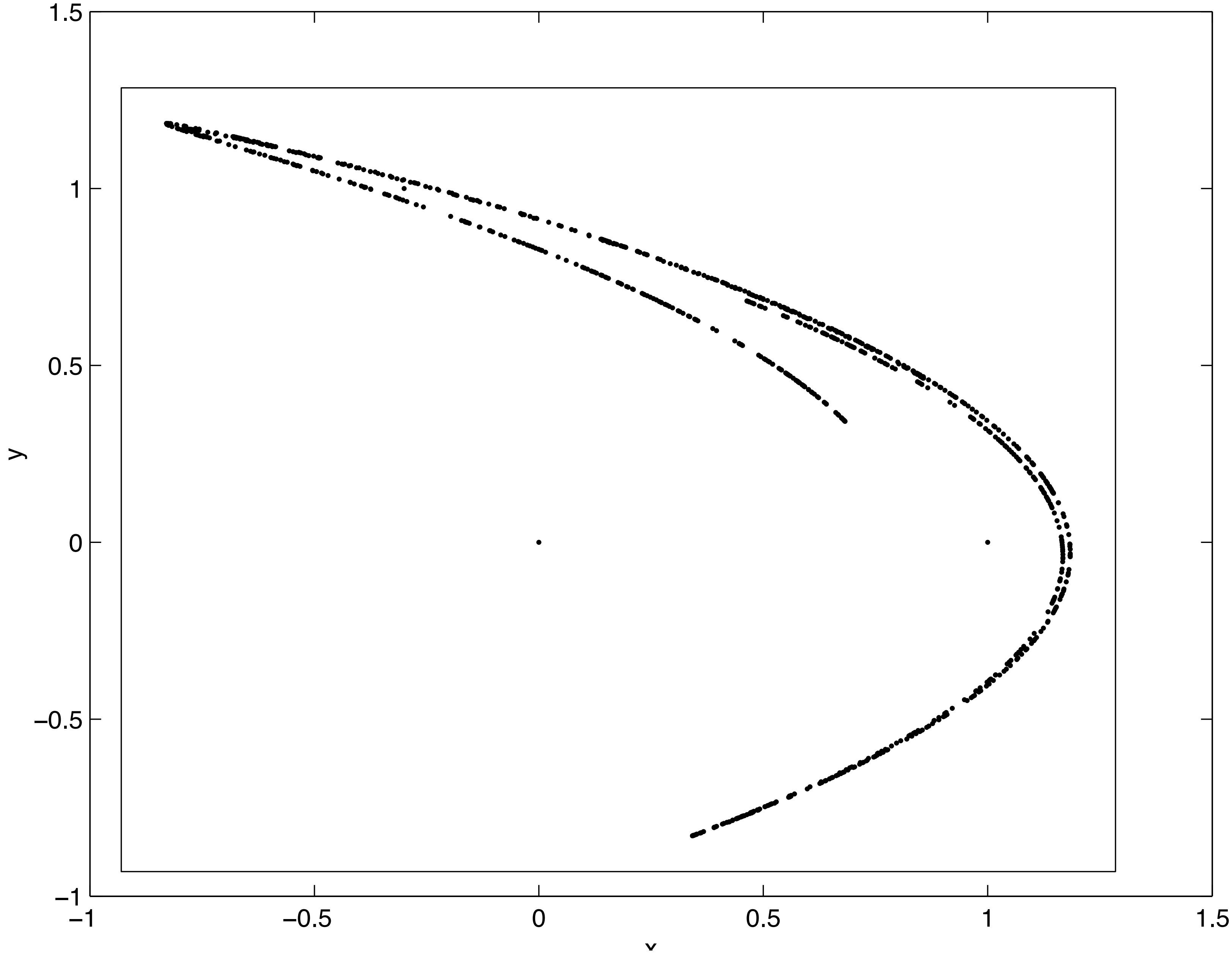
In the end,
M=14, L=14, E^{Ch}~10^{-5}, E^G~10^-7}
recurrent set =134, 311 boxes (coarser in picture)

Use 9 verifiable cycles/orbits in attractor to form isolating neighborhood. Gives 16 connected components which can be amalgamated into 4 sets, labeled A, B, C, D. Use modification of algorithms in DFT to show Conley indices for all periodic orbits in transition graph are nontrivial, giving semiconjugacy to corresponding subshift of finite type. ln(sp(A)) gives the entropy of the subshift and a lower bound (via the semiconjugacy) on the topological entropy of the full system.

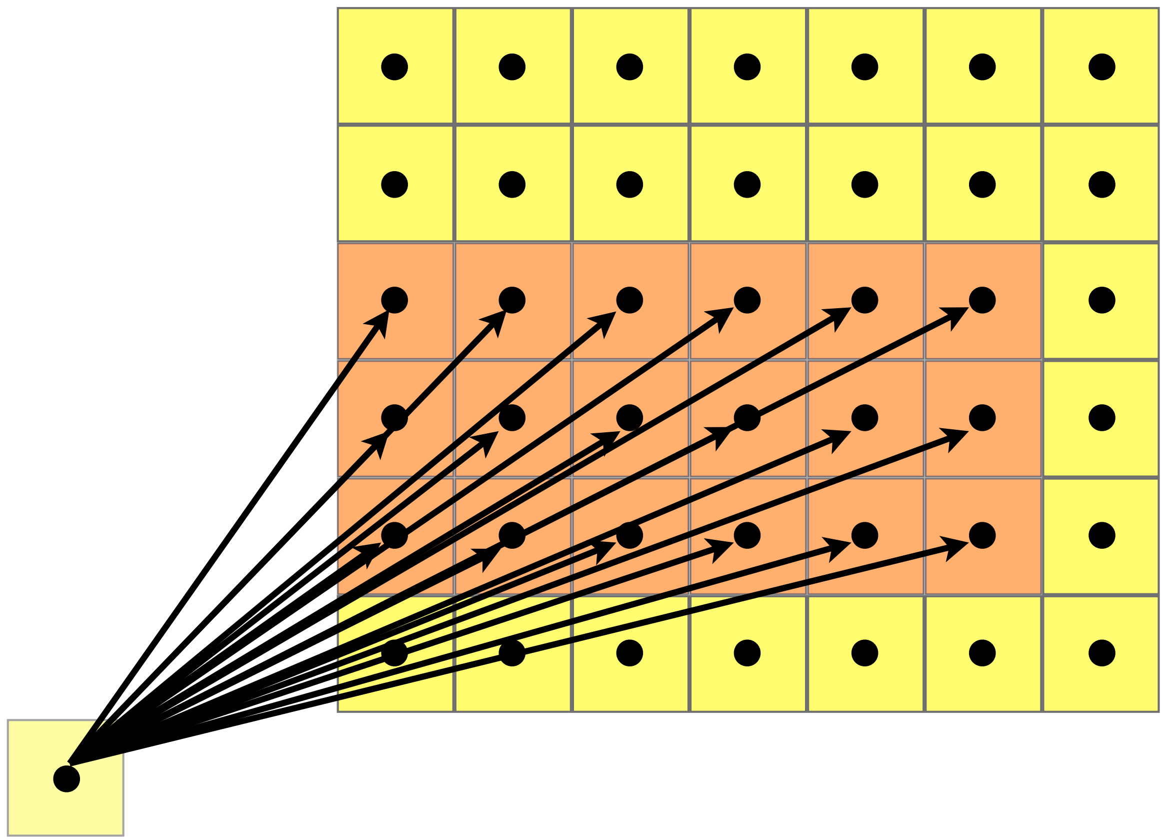


Amalgar … sition graph with lower entropy bound of 0.2406.

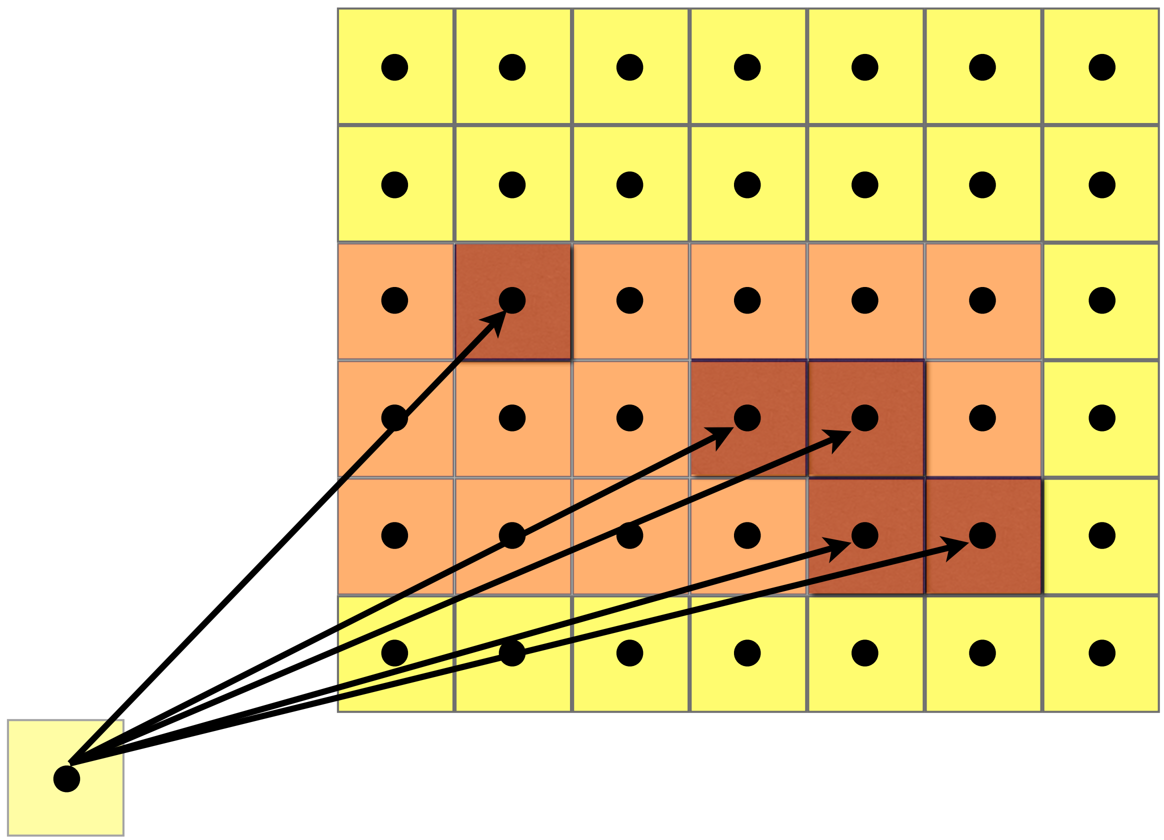
Combinatorial approximation from data



Combinatorial approximation from data



Combinatorial approximation from data



Combinatorial approximation from data

approaches:

correct the problem
holes/gaps
(Harker et al. 201

adapt the grid to t
interpolate to fix th
(Wess 2008, Alex

M. Wess. Computing Topological Dynamics from Time Series. Ph.D. Dissertation, Florida Atlantic University, 2008.

Shaun Harker, Hiroshi Kokubu, Konstantin Mischaikow, and Pawel Pilarczyk. Inducing a map on homology from a correspondence. Proceedings of the American Mathematical Society, 2015.

Zachary Alexander, Elizabeth Bradley, James D. Meiss, and Nicole F. Sanderson. Simplicial multivalued maps and the witness complex for dynamical analysis of time series. SIAM J. Appl. Dyn. Syst., 14(3):1278–1307, 2015.

Thank you

Work supported by NSF DMS 0955604.