A Dynamical Systems Approach to the Pleistocene Climate - Part II of II

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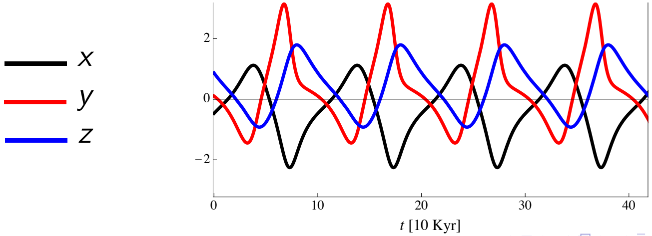
Outline

- Background
- Slow-fast approximation
- Center manifold approximation
- Slow passage(s)

The 1990 Maasch-Saltzman Model

- ODE system for $\mathbf{x}$ = anomalies of ice mass, $\mathbf{y}$ = atmospheric CO_2 and $\mathbf{z}$ = North Atlantic (NA) deep water
- Dimensionless; parameters p, q, r, $s > 0$ are $\mathcal{O}(1)$

$$\boxed{\begin{aligned} \dot{x} &= -x - y \\ \dot{y} &= (r - z^2)y - (p - sz)z \\ \dot{z} &= -qx - qz \end{aligned}}$$



Reductions in this Talk

Full MS $\xrightarrow{s=0}$ **Symmetric MS**

$$\boxed{\begin{aligned}\dot{x} &= -x - y\\ \dot{y} &= (r - z^2)y - (p - sz)z\\ \dot{z} &= -qx - qz\end{aligned}}
\qquad \xrightarrow{s=0} \qquad
\boxed{\begin{aligned}\dot{x} &= -x - y\\ \dot{y} &= (r - z^2)y - pz\\ \dot{z} &= -qx - qz\end{aligned}}$$

$\downarrow\ q \gg 1$ $\qquad$ $\downarrow\ q \gg 1$

Asymmetric 2-D $\xrightarrow{s=0}$ **Symmetric 2-D**

$$\boxed{\begin{aligned}\dot{x} &= -x - y\\ \dot{y} &= (r - x^2)y + (p + sx)x\end{aligned}}
\qquad \xrightarrow{s=0} \qquad
\boxed{\begin{aligned}\dot{x} &= -x - y\\ \dot{y} &= (r - x^2)y + px\end{aligned}}$$

Slow-Fast System

- $q > 1$ is a ratio of time scales.
- Consider the case $q \gg 1$ or $\varepsilon = \frac{1}{q} \ll 1$.
- Then x and y are slow, and z is fast.

$$\begin{aligned} \dot{x} &= -x - y \\ \dot{y} &= ry - pz + (s - y)z^2 \\ \varepsilon \dot{z} &= -x - z \end{aligned}$$

- Invariant, normally attracting mfd $\mathcal{M}_0 = \{z = -x\}$ for $\varepsilon = 0$
- For small ε, invariant, normally attracting manifolds $\mathcal{M}_\varepsilon = \{z = h_\varepsilon(x, y)\}$ persist (*Fenichel Theory*).

Slow Manifold and Invariance Equation

Describe $\mathcal{M}_\varepsilon$ with *invariance equation*

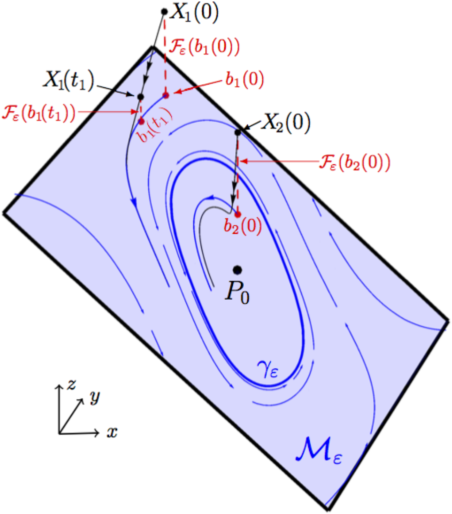
$$\varepsilon \frac{d}{dt} h_\varepsilon(x, y) = -x - h_\varepsilon(x, y),$$

Expand $h_\varepsilon(x, y) = h_0(x, y) + \varepsilon h_1(x, y) + \varepsilon^2 h_2(x, y) + \dots$ and find the h_i.

$$\begin{aligned} h_0(x, y) &= -x \\ h_1(x, y) &= -(x + y) \\ h_2(x, y) &= -(x + y) + (ry + px + (s - y)x^2) \end{aligned}$$

Slow-Fast Decomposition of Typical Solutions

- A solution $X(t)$ starts on the fast stable fiber $\mathcal{F}_\varepsilon(b(0))$.
  - It decomposes into a fast component decaying along $\mathcal{F}_\varepsilon(b(t))$
  - and a slow component that moves with the base point $b(t)$.
- Thus $b(t) \in \mathcal{M}_\varepsilon$ represents $X(t)$ faithfully.



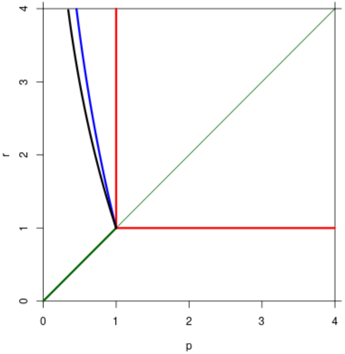
The Slow–Fast System

The system on $\mathcal{M}_\varepsilon$ becomes

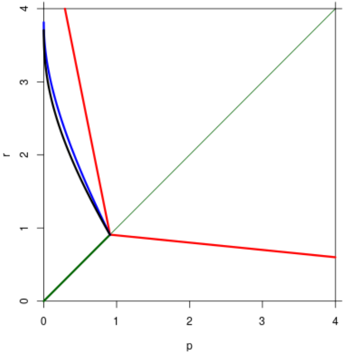
$$\boxed{\begin{aligned}\dot{x} &= -x - y \\ \dot{y} &= ry - ph_\varepsilon(x,y) + (s-y)(h_\varepsilon(x,y))^2\end{aligned}}$$

- Consider the *symmetric case* $s = 0$ and use first order approximation $h_\varepsilon(x,y) = -x - \varepsilon(x+y) + \mathcal{O}(\varepsilon^2)$.
- The result is very similar to the case $\varepsilon = 0$.
- Equilibria at $P_0 = (0,0)$ for all (p,r) and at $P_{1,2} = (\pm\sqrt{r-p}, \mp\sqrt{r-p})$ if $r > p$
- At $(p,r) = (\frac{1}{1+\varepsilon}, \frac{1}{1+\varepsilon})$, P_0 undergoes a $\mathbb{Z}_2$-symmetric BT bifurcation (organizing center).
- All bifurcation curves emanate from this point.

Bifurcation of the Symmetric Slow–Fast System



$q = \infty$

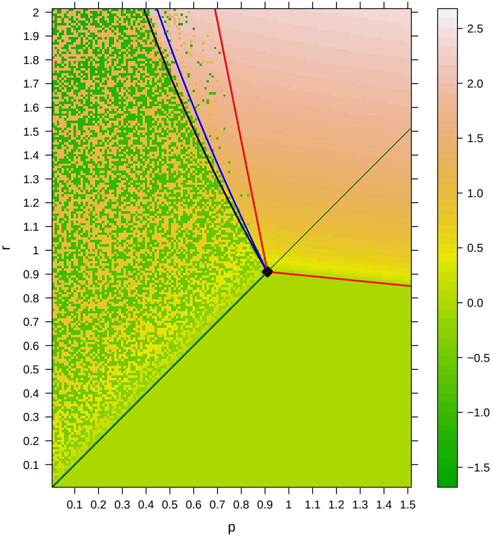


$q = 10$

- Linear stability analysis is similar (Hopf bifurcations)
- Bogdanov-Takens unfolding is similar (homoclinic bifurcation, saddle-node bifurcation)

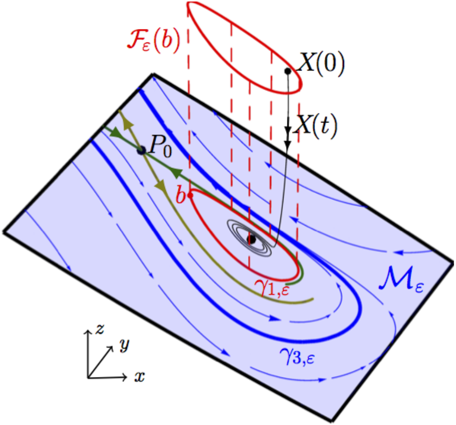
Slow–Fast System: Limit Cycles

- Integrate the full system for $q = 10$ ($\varepsilon = 0.1$) with random initial data and plot $\overline{x}(p, r) = \limsup_t x(t)$.
- Also shown are the bifurcation curves of the reduced system, using an $\mathcal{O}(\varepsilon^3)$ approximation of h_ε.



Basin of Attraction $\mathcal{B}_1$ of P_1

- (p, r) is between the Hopf curve and the homoclinic bifurcation curve.
- P_1 is stable and is surrounded by an unstable limit cycle $\gamma_{1\varepsilon}$ in $\mathcal{M}_\varepsilon$.
- Also shown: a large stable limit cycle $\gamma_{3\varepsilon}$
- The fast stable fibres with base points on $\gamma_{1\varepsilon}$ form the boundary of $\mathcal{B}_1$.



Back to the Full System

$$\boxed{\begin{aligned} \dot{x} &= -x - y \\ \dot{y} &= (r - z^2)y - (p - sz)z \\ \dot{z} &= -qx - qz \end{aligned}}$$

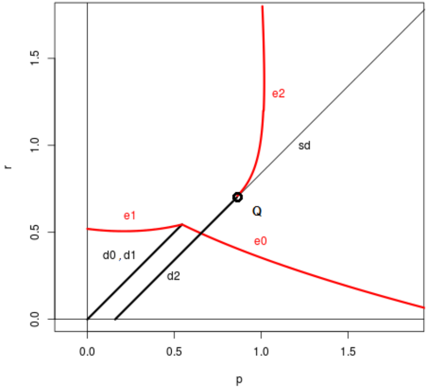
- Trivial equilibrium $P_0 = (0, 0, 0)$
- Two additional equilibria if $\rho = s^2 + 4(r - p) > 0$:

$$P_1 = x_1^* \cdot (1, -1, -1), \quad P_2 = x_2^* \cdot (1, -1, -1)$$

  with $x_{1,2}^* = \frac{1}{2}\left(-s \pm \sqrt{\rho}\right)$
- *P_2 is a "warm" state, P_1 is a "cold" state.*

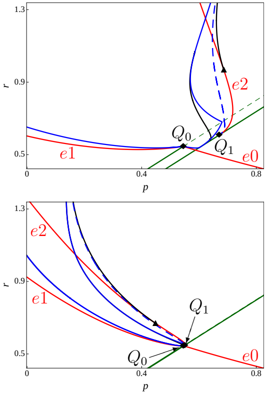
Linear Stability

- P_1, P_2 exist above **d2/sd** ($\rho = 0$)
- P_0 is stable below **d0/d1** ($r = p$) and **e0**
- Hopf bifurcations off $P_{0,1,2}$ on **e0, e1, e2**
- Supercritical on **e0**, subcritical on **e1**, sub- to supercritical on **e2**
- Two organizing centers at Q_0 (where **e0/e1** and **d0/d1** meet) and at $Q = Q_1$



$q = 1.2, s = .8$

Symmetry Breaking Near the Organizing Centers



- Fix $q = 1.2$ and reduce $s \to 0.5 \to 0.2 \to 0.1 \to 0.05$. The two organizing centers coalesce and the curves **e1, e2** collapse into one curve.
- Homoclinic and **fold** bifurcation curves also collapse.

Center Manifolds

- Center manifolds associated with P_i exist near both organizing centers.
- Main steps near $Q_0 = (\frac{q}{1+q}, \frac{q}{1+q})$, for P_0 (or P_1):
- Shift Q_0 to $(0,0)$. Then p becomes $\tilde{p}$, r becomes $\tilde{r}$. The system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ n \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & \frac{q}{1+q} & -\frac{q}{1+q} \\ -q & 0 & -q \end{pmatrix}$$

where $n = n(x, y, z, \tilde{p}, \tilde{r}, s, q)$.

Center Manifolds II

- Jordan normal form for A is

$$J = F^{-1}AF = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

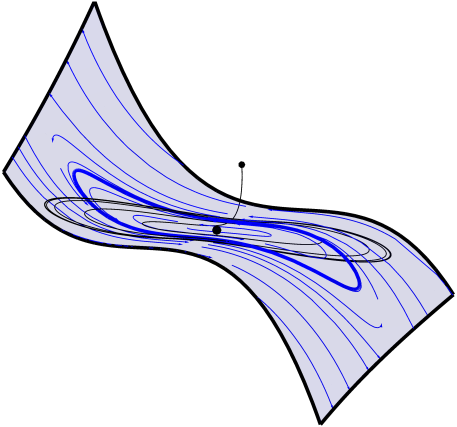
- Transform to Jordan normal form with new variables (u, v, w). The nonlinear term is now complicated but still has rank 1.
- The center manifold is $\mathcal{W}^c = \{w = h(u, v, \dots)\}$.
- There is an invariance equation for h that can be exploited.

Center Manifold III

- Write $h = h_2 + h_3 + \ldots$ as sum of homogeneous polynomials in $u, v, \tilde{p}, \tilde{r}$, with coefficients depending on q, s.
- *Consistency check:* Transform the expansion for $\mathcal{W}^c$ back to (x, y, z) coordinates and compare to the slow manifold expansion. There is agreement as expected (to suitable powers of $\varepsilon = q^{-1}$ and of the state variables).

Dynamics on Center Manifold

- The blue surface is $\mathcal{W}^c$, the blue streamlines indicate the flow on $\mathcal{W}^c$, the thick blue line is the stable limit cycle there.
- The black curve is a solution of the full system.

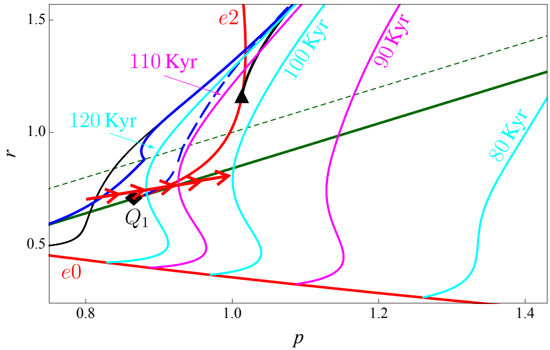


$q = 1.2, s = 0, \tilde{p} = .15, \tilde{r} = .1$

Region of Validity

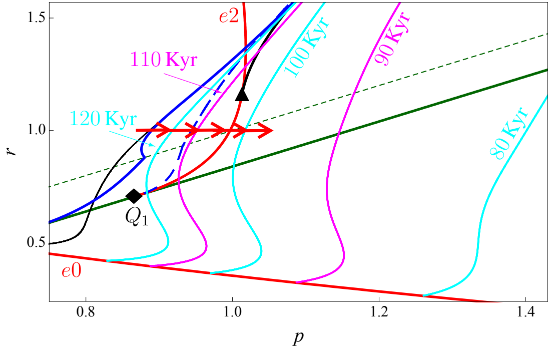
- For $q < q_c = q_c(p, r, s)$, expect both Lyapunov type numbers to become < 1, and $\mathcal{W}^c$ loses smoothness.
- Our approach to approximate q_c numerically:
  - Compute the eigenvalues at all three equilibrium points, for the system on $\mathcal{W}^c$ (three pairs).
  - Compare their real parts to λ_3, the transverse eigenvalue at Q_0 (six ratios).
  - $q \approx q_c$ when one of these ratios becomes 1
- For $0 < p < 2, 0 < r < \frac{3}{2}$, this suggests $q_c < 1$.
- **The reduced systems provide reliable qualitative information about the full dynamics, over the entire parameter range.**

Outlook: Slow Passage



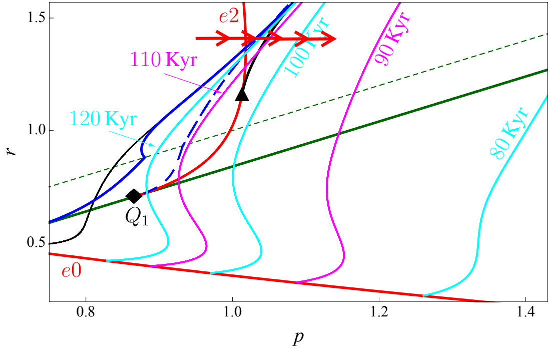
Maasch and Saltzman changed (p, r) slowly from $(0.8, 0.7)$ to $(1.0, 0.8)$ over 2Myr (red path), crossing several bifurcation loci. *Limit cycles with the right periods were then observed.*

Slow Passage



A less complicated path. The fold where P_1 and P_2 disappear is not crossed. The Hopf line is crossed in a subcritical place.

Slow Passage



Another alternative. The fold where P_1 and P_2 disappear is not crossed. The Hopf line is crossed in a supercritical place.

Thank You!