Stochastic Arnold diffusion of deterministic systems

V. Kaloshin

May 25, 2017

Plan of the talk

- Motivation:
  - Hamiltonian systems and failure of ergodicity
  - Systems with mixed behavior: quasiperiodic and stochastic.
  - Examples of mixed behavior: Bunimovich mushroom, the Solar system systems.
  - (Nearly) integrable Hamiltonian, KAM theory, Arnold diffusion.

- The first example: the Asteroid belt and Kirkwood gaps.
  - Quasiperiodic (KAM) behavior away from Kirkwood gaps
  - Stochastic behavior in Kirkwood gaps

- The second (Arnold's) example:
  the pendulum $\times$ rotor $+$ small coupling.
  - Our result about stochastic diffusion inside instability layers.

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Hamiltonian systems and Ergodicity

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function, $(q, p) \in \mathbb{R}^n \times \mathbb{R}^n$. Let Φ_H be the Hamiltonian flow of H

$$\begin{cases} \dot{q} = \ \partial_p H \\ \dot{p} = -\partial_q H. \end{cases}$$

For example, $n = 1$ and

$$H(p, q) = \text{Kinetic energy} + \text{Potential energy} = \frac{p^2}{2} + U(q)$$

for some potential $U(x)$.

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Denote by Φ_H^t the time t flow. Let $S_E = \{(q, p) \in \mathbb{R}^{2n} : H(q, p) = E\}$ be an energy surface. Assume S_E is compact.

- Φ_H^t preserves energy $H(q, p) = H(\Phi_H^t(q, p)) = E$;
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Ergodic Hypothesis (Boltzmann, Maxwell) Is generically Φ_H^t ergodic on S_E?

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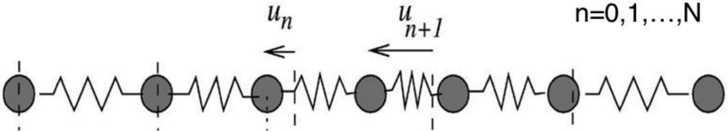
Numerical doubts (Fermi-Pasta-Ulam '53) Chains of nonlinear springs



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the α-term — nonlinearity. Most "small" solutions are quasi-periodic!

KAM theory gave mathematical proof that ergodicity fails! (to be discussed)

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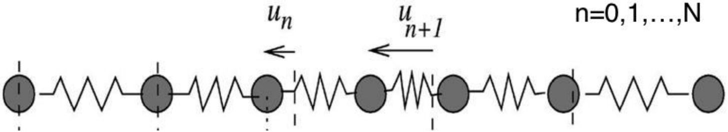

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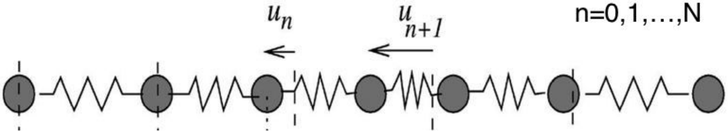

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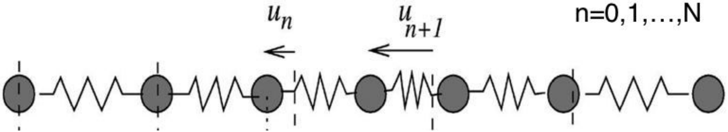

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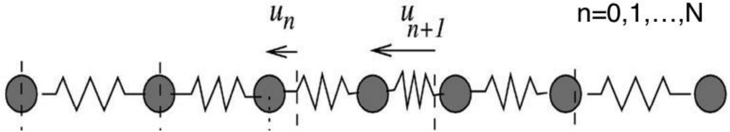

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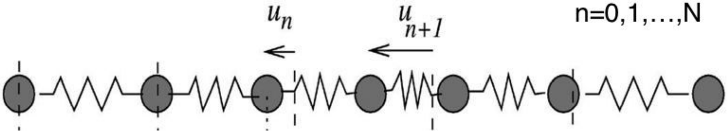

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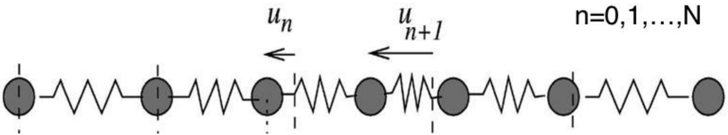

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- A tiny fraction of differential equations have explicit solutions.
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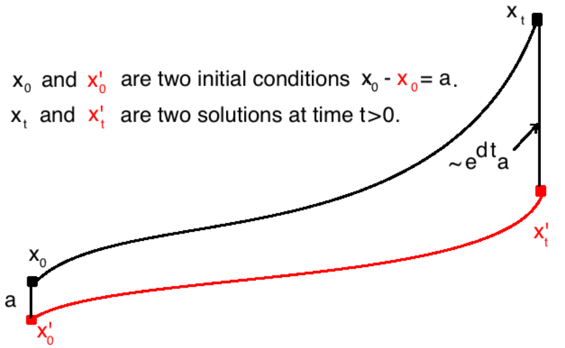

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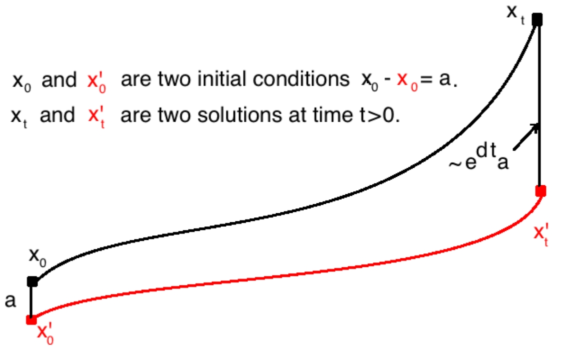

Figure: No practical hope to describe an individual solution!

Enselmble of solutions of ODEs.

$$\dot{x} = f(x), \qquad x \in M \text{ — a manifold, e.g. } \mathbb{R}^n, \mathbb{T}^n \ldots$$

- Consider an emseble of initial conditions. For example, a grid of initial conditions in a region of the phase space. Then study statistics of evolution of this ensemble.
- More generally, consider a probability measure μ of initial conditions. Then study distributions of the pushforward of this measure μ.
- Modified goal: Analyze long time behavior statistically.

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Ergodic systems and systems with mixed behavior

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Let μ be the volume and the flow is volume preserving.

- The system is ergodic if for a μ-almost every initial condition long time behavior is the same, i.e. time and space averages coincide.
- The system has mixed behavior if there are at least two sets of positive μ-measure of initial conditions with different long time behavior.

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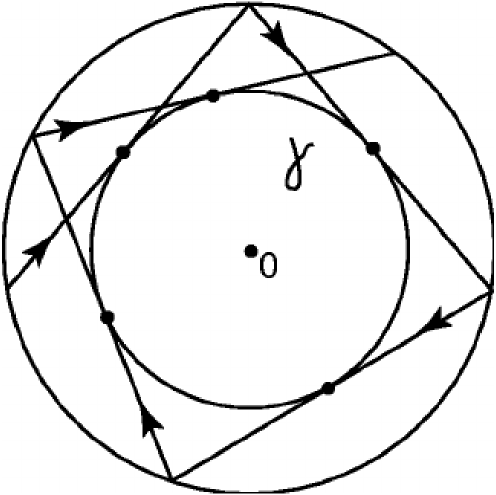
- The system is ergodic if for a μ-almost every initial condition long time behavior is the same, i.e. time and space averages coincide.
- The system has mixed behavior if there are at least two sets of positive μ-measure of initial conditions with different long time behavior.

Mixed Behavior: Bunimovich Mushroom



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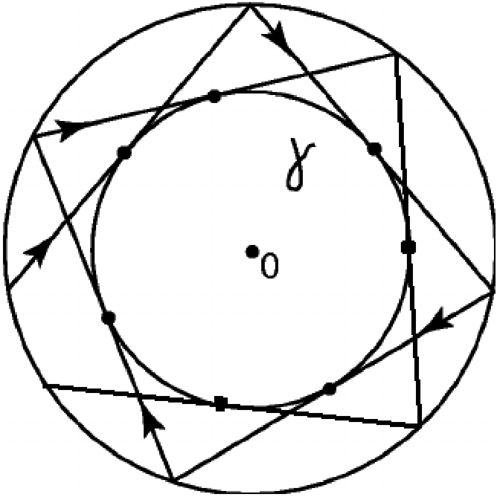

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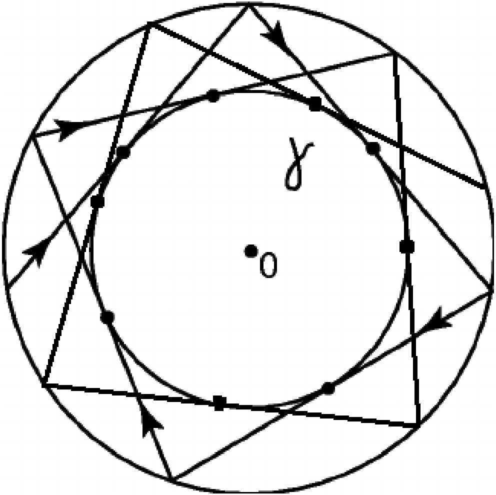

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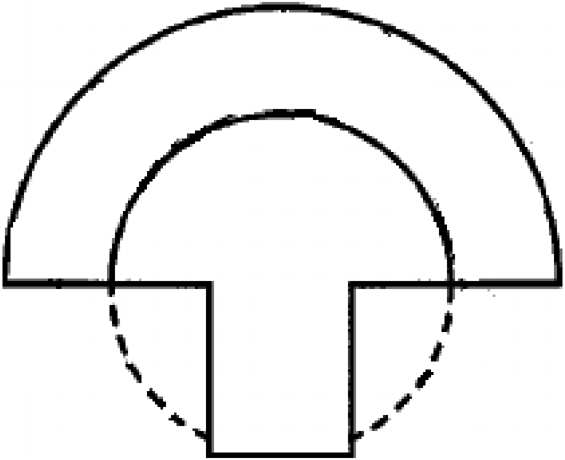

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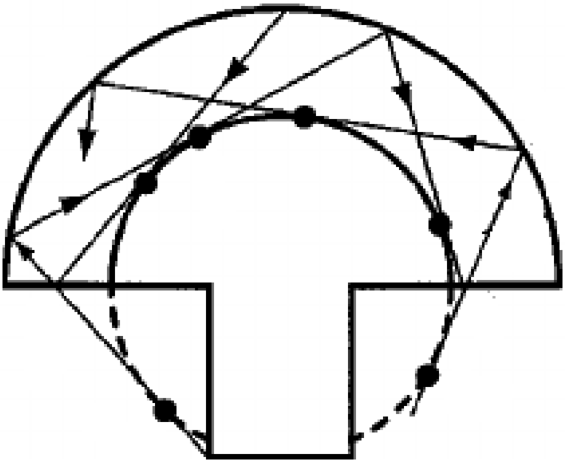

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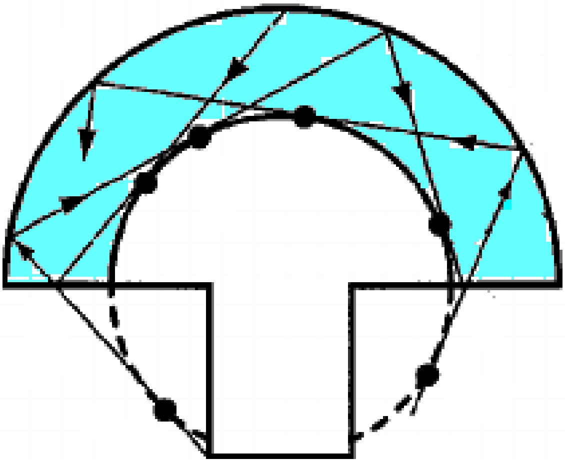

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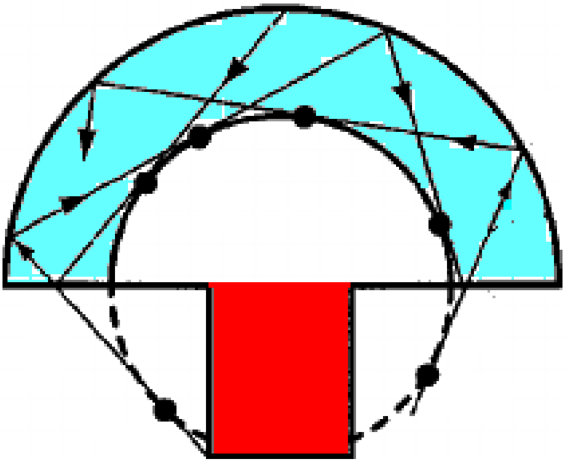

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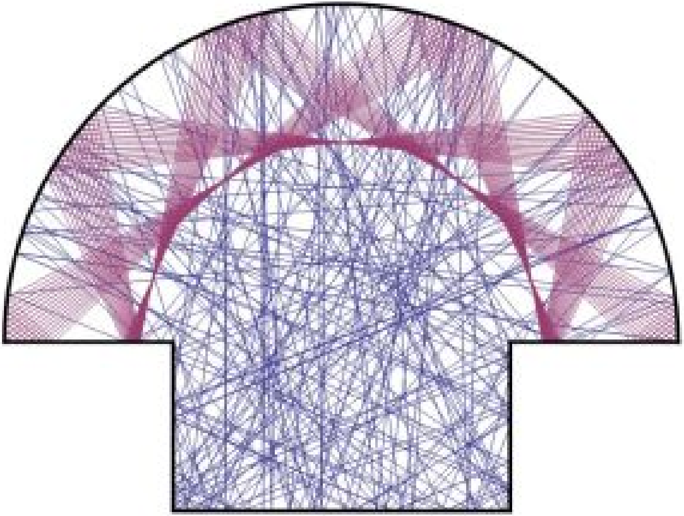

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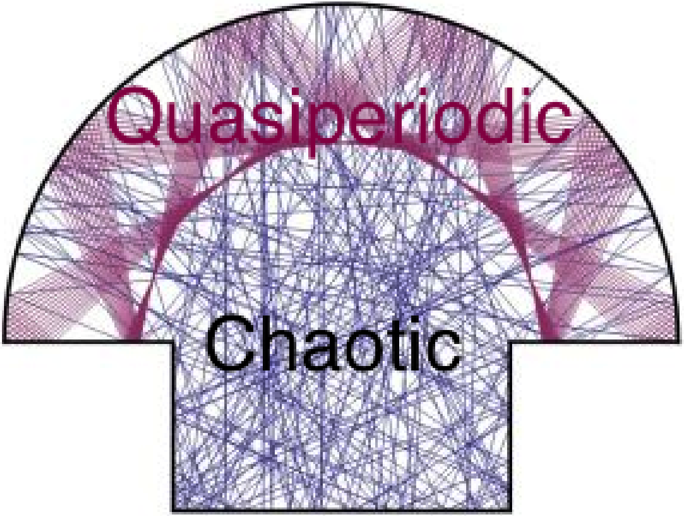


Laskar simulations on instability of the Solar system

BBC NEWS

'Tiny chance' of planet collision

By Pallab Ghosh
Science correspondent, BBC News

Astronomers calculate there is a tiny chance that Mars or Venus could collide with Earth - though it would not happen for at least a billion years.

Astronomers had thought that the orbits of the planets were predictable. But 20 years ago, researchers showed that there were slight fluctuations in their paths.

The researchers carried out more than 2,500 simulations. They found that in some, Mars and Venus collided with the Earth.

"It will be complete devastation," said Professor Laskar.

"The planet is coming in at 10km per second - 10 times the speed of a bullet - and of course Mars is much more massive than a bullet."

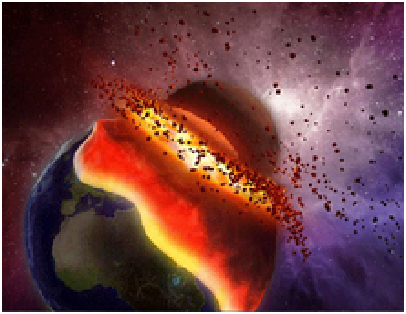
Professor Laskar's calculations also show that there is a possibility of Mercury crashing into Venus. But in that scenario, the Earth would not be significantly affected.

Laskar simulations on instability of the Solar system

Figure: Venus and Earth collide

Laskar simulations on instability of the Solar system

Figure: Venus and Earth collide



Mars and Earth collide

Integrable systems & action-angles coordinates

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian, $\varphi \in \mathbb{T}^n$ be angle, $I \in \mathbb{R}^n$ be action.

A Hamiltonian system is **Arnold-Liouville integrable** if for an open set $U \subset \mathbb{R}^n$ there exists a symplectic map $\Phi : \mathbb{T}^n \times U \to \mathbb{R}^{2n}$ s. t. $H \circ \Phi(\varphi, I)$ depends only on I and

$$\begin{cases} \dot{\varphi} = \partial_I (H \circ \Phi)(I) = \omega(I), \\ \dot{I} = 0. \end{cases}$$

(φ, I)**–action-angle coordinates**

In particular, $\Phi(\mathbb{T}^n \times U)$ is foliated by invariant n-dimensional tori and on each torus $\mathbb{T}^n \times I_0$ the flow is linear.

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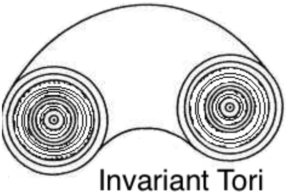

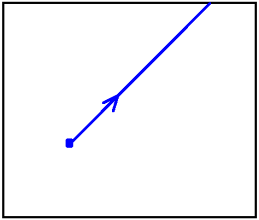

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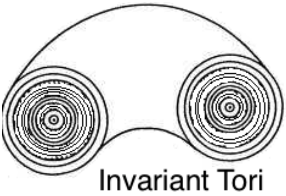

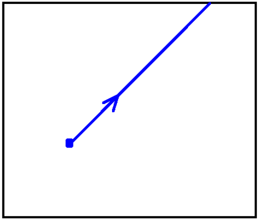

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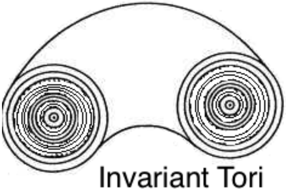

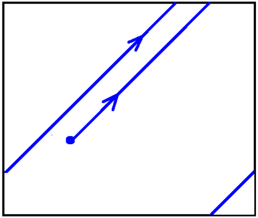

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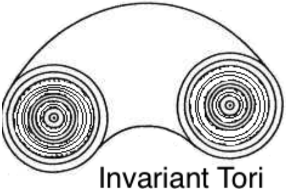

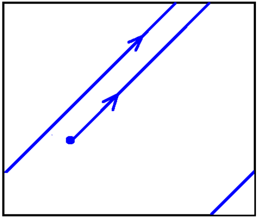

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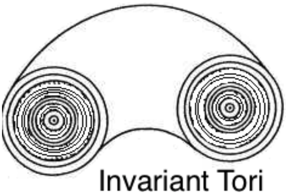

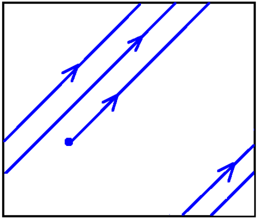

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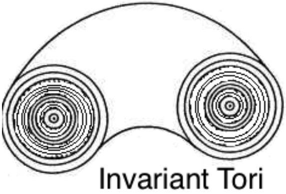

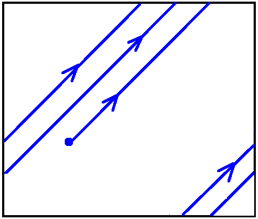

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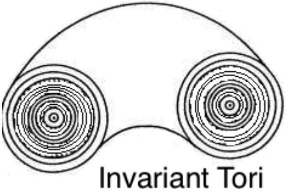

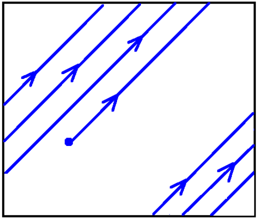

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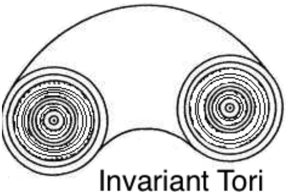

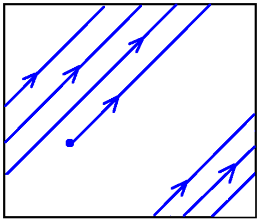

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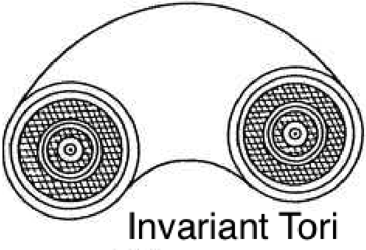

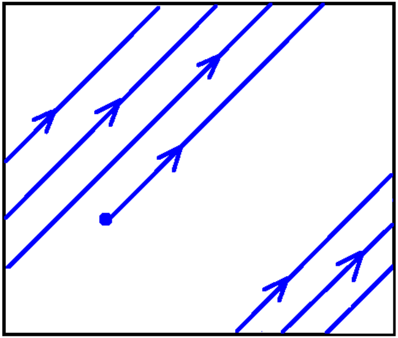

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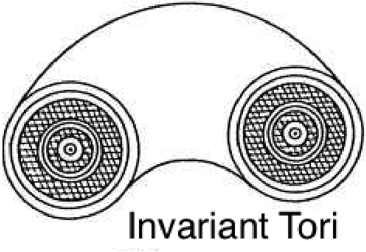

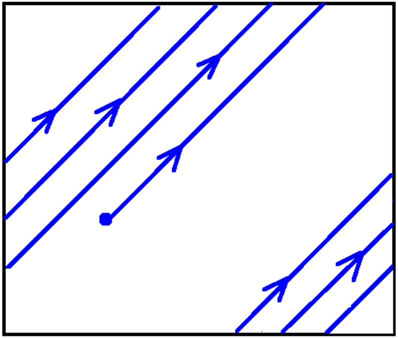

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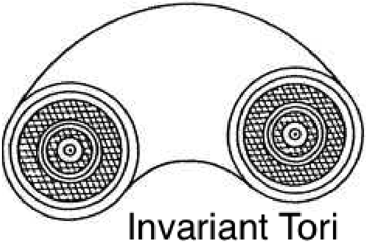

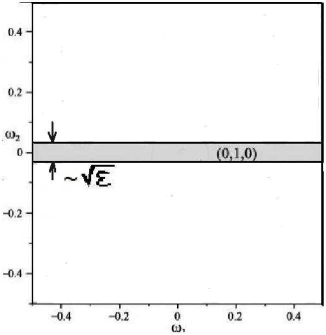

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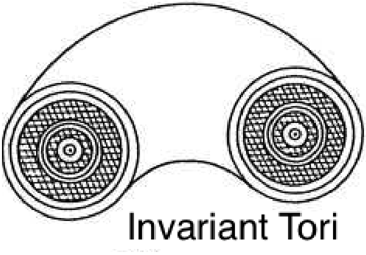

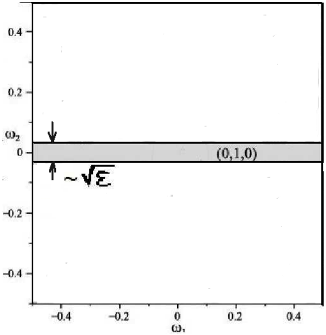

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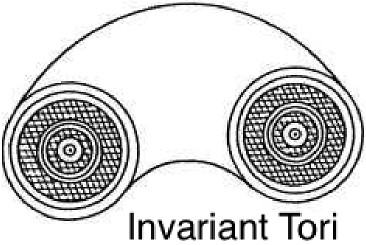

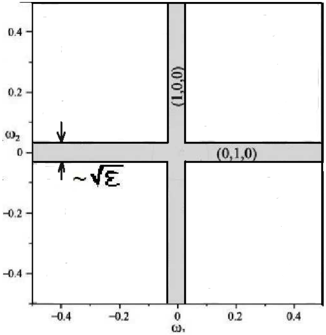

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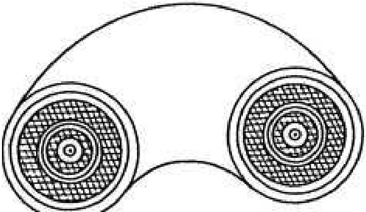

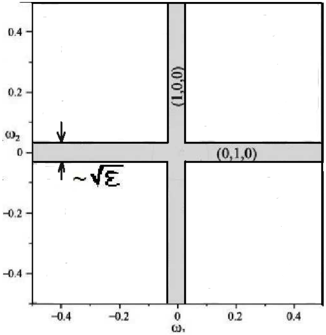

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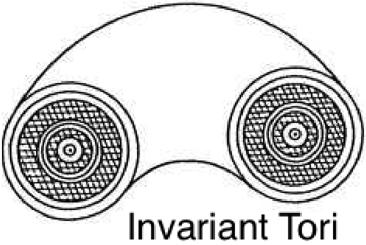

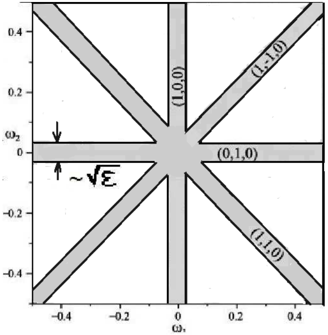

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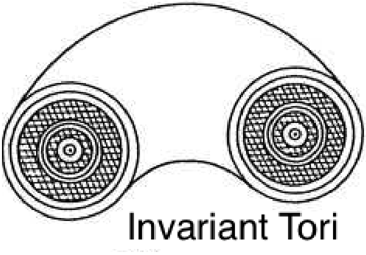

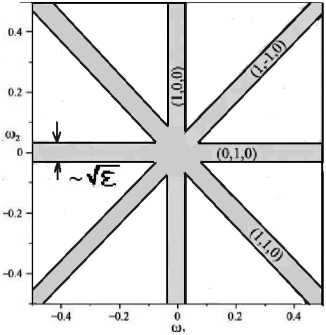

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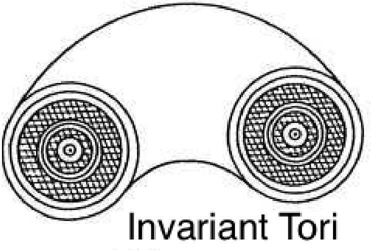

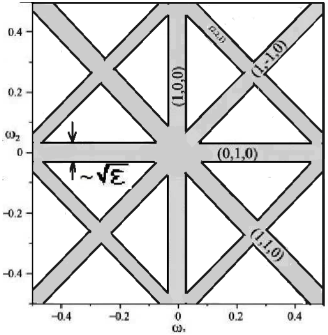

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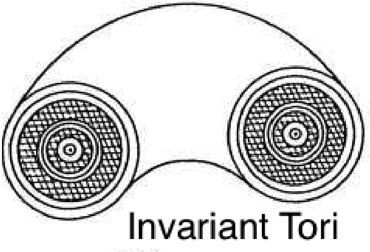

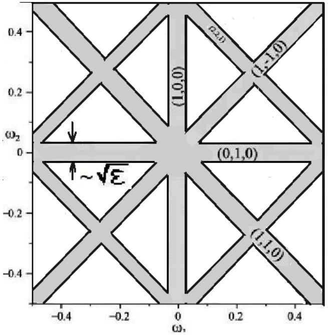

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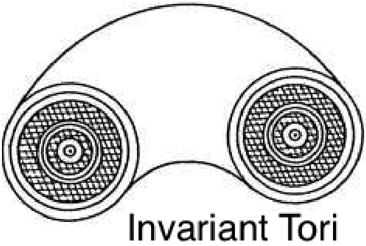

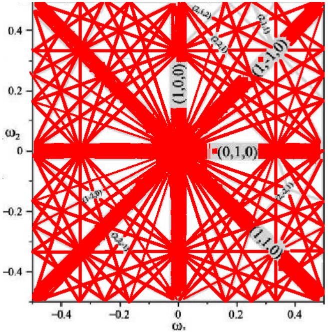

Width $\sim \sqrt{\varepsilon}$

Stochastic layers

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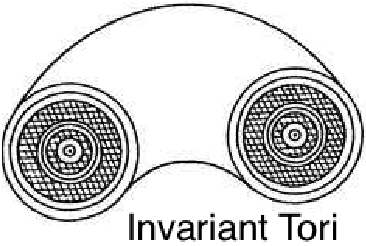

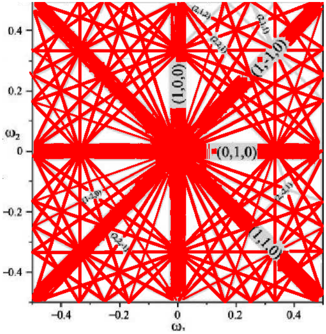

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Arnold's Conjecture

Soviet Mathematics–Doklady 5 581–5 (1964)

INSTABILITY OF DYNAMICAL SYSTEMS WITH SEVERAL DEGREES OF FREEDOM

V. I. ARNOL'D

Arnold conjecture *For a generic perturbation* $H_\varepsilon(\varphi, I) = H_0(I) + \varepsilon H_1(\varphi, I)$ *does there exist "diffusing orbits" whose action component* $I(t)$ *can "travel"* $O(1)$*, i.e.* $|I(t) - I(0)| > O(1)$ *for some* $t > 0$*? In particular, such orbits (if exist) are not quasiperiodic.*

$$\begin{cases} \dot\varphi = \partial_I H_0(I) + \varepsilon \partial_I H_1(\varphi, I), \\ \dot I = -\varepsilon \partial_\varphi H_1(\varphi, I). \end{cases}$$

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PHYSICS REPORTS (Review Section of Physics Letters) 52, No. 5 (1979) 263–379. NORTH-HOLLAND PUBLISHING COMPANY

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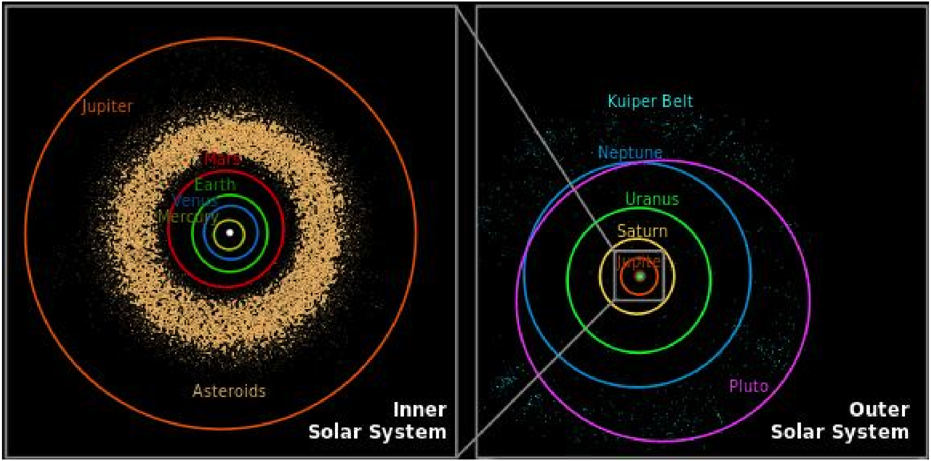

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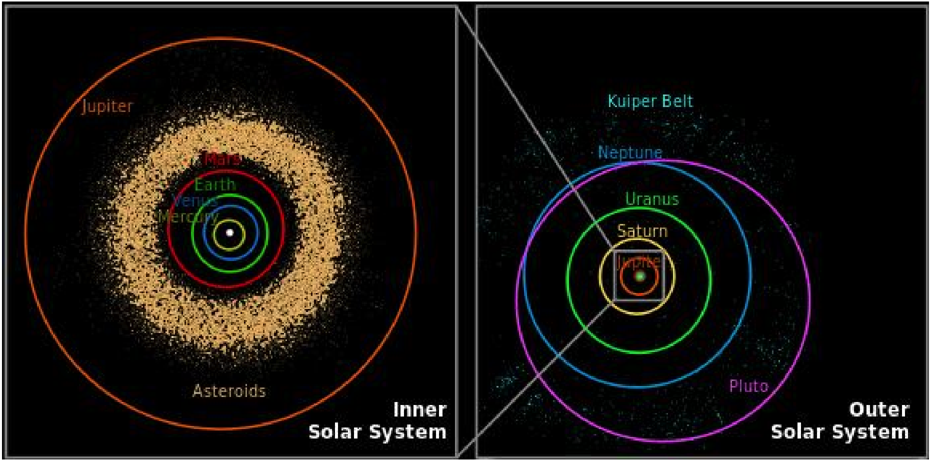

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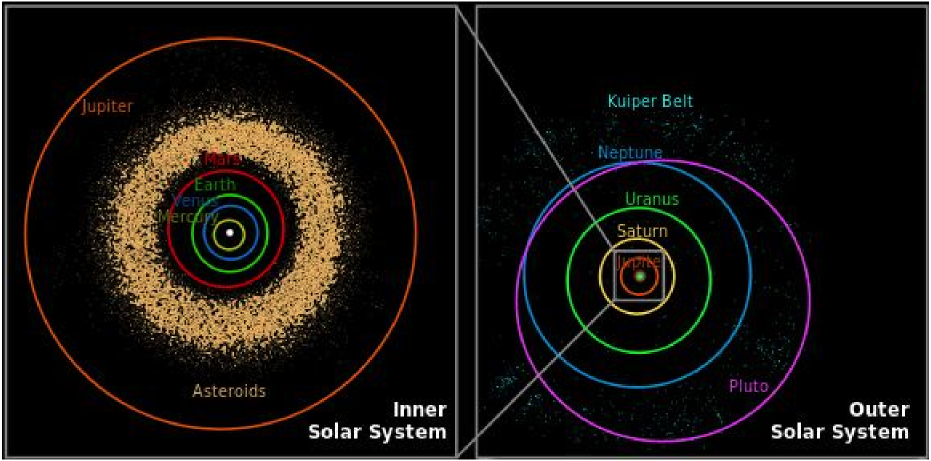

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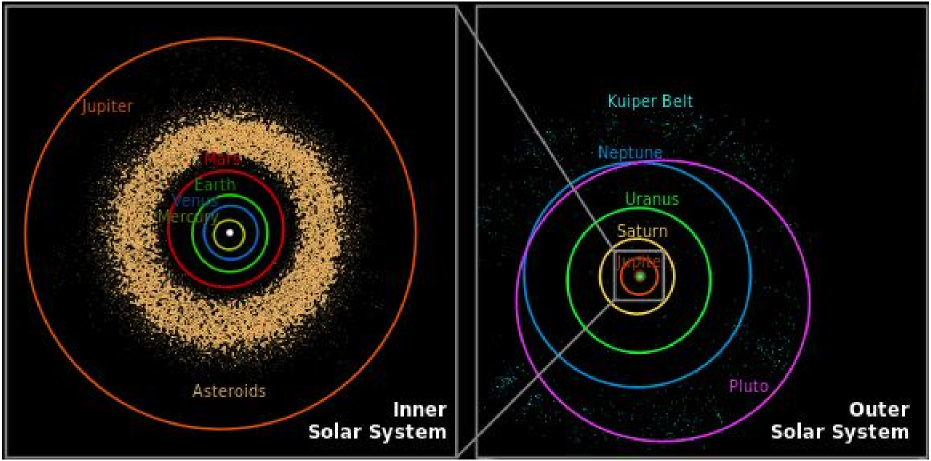

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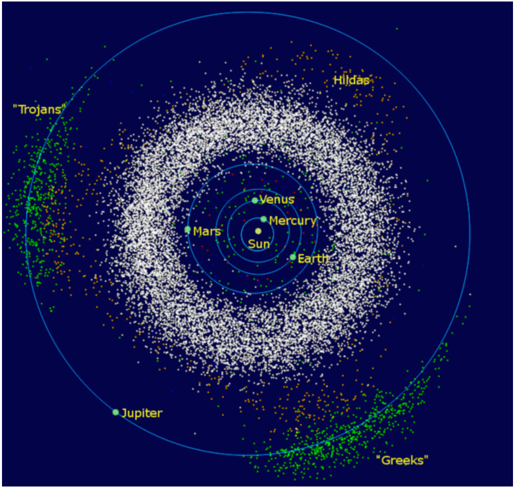
It is occupied by over a million of asteroids with diameter at least 1km.



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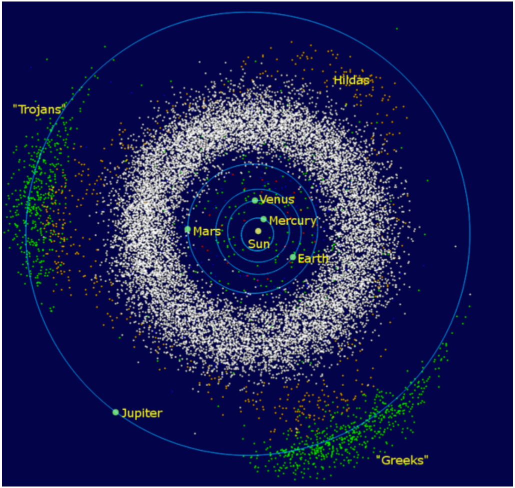

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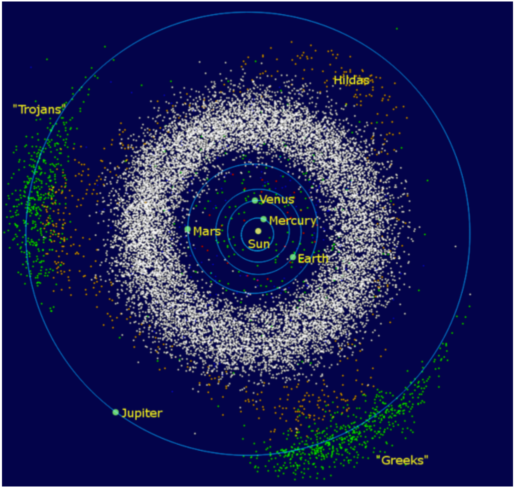

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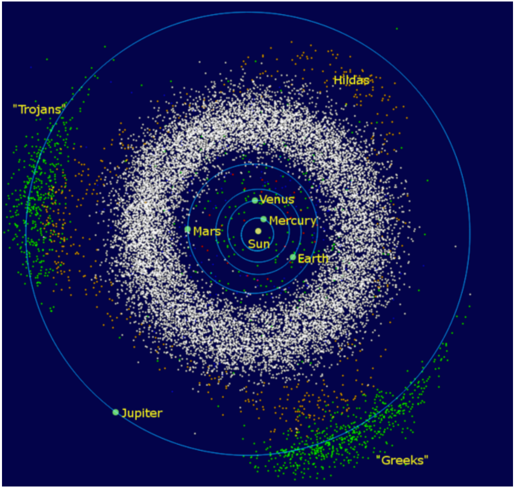
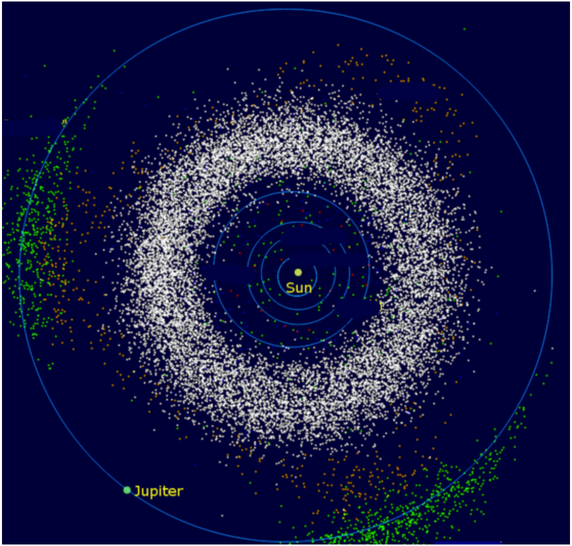


The Deterministic Model: the Sun-Jupiter-Asteroid



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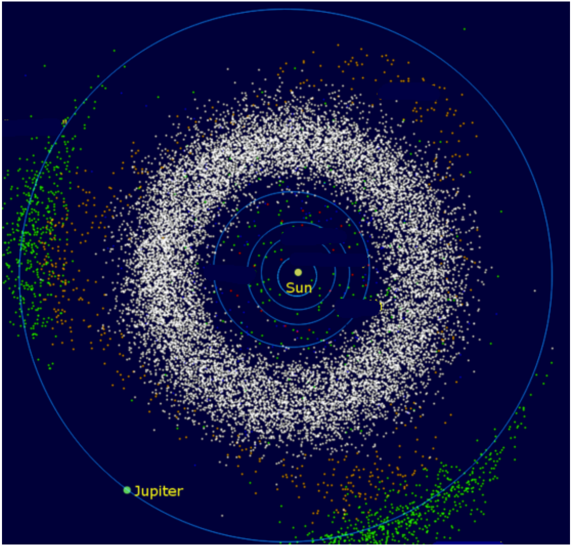

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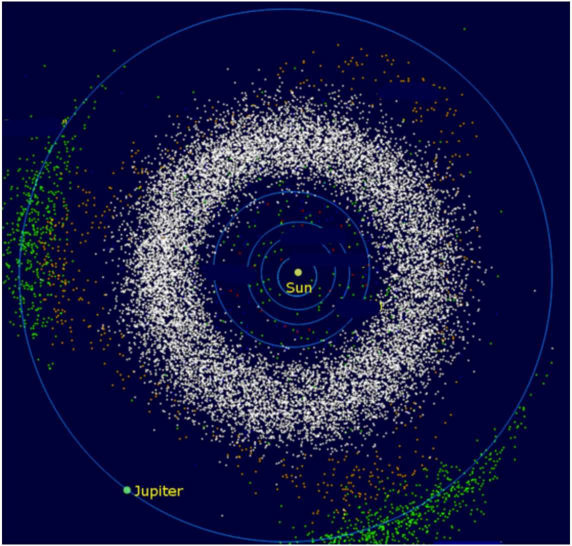

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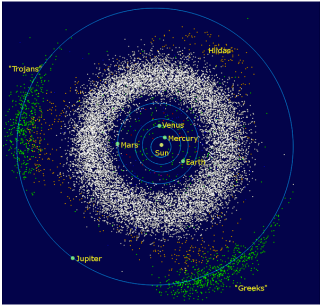
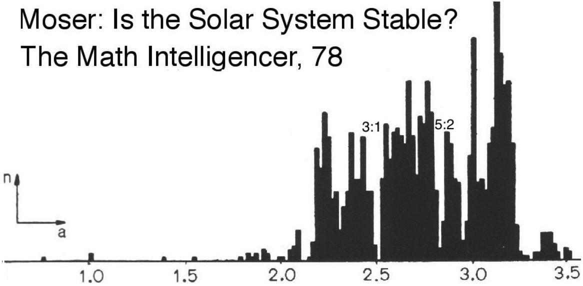
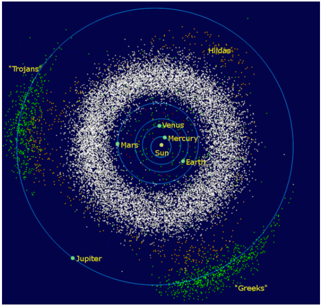
Kirkwood gaps in the Asteroid Belt



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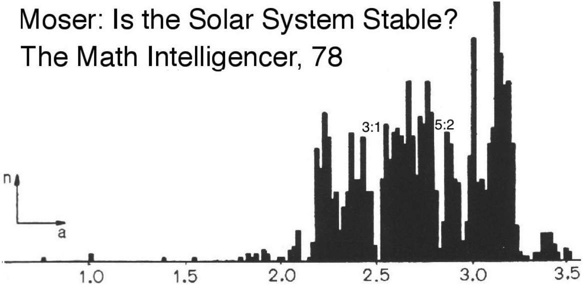
Venus
Mercury
Mars
Sun
Earth
Jupiter
"Greeks"


Moser: Is the Solar System Stable?
The Math Intelligencer, 78
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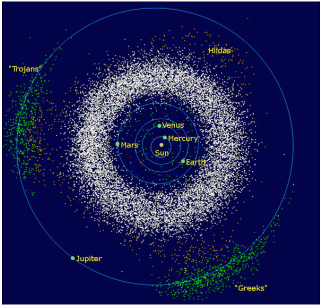

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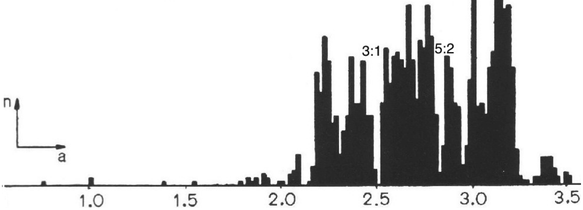

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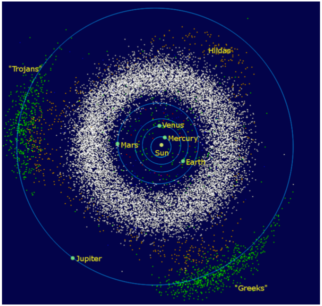

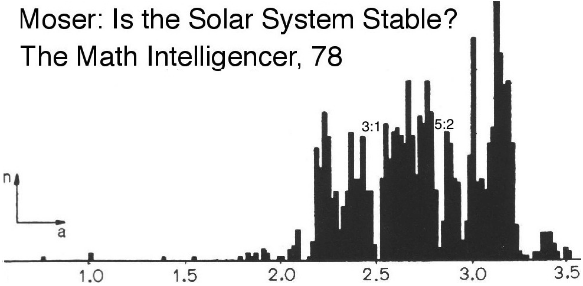

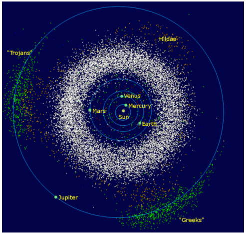

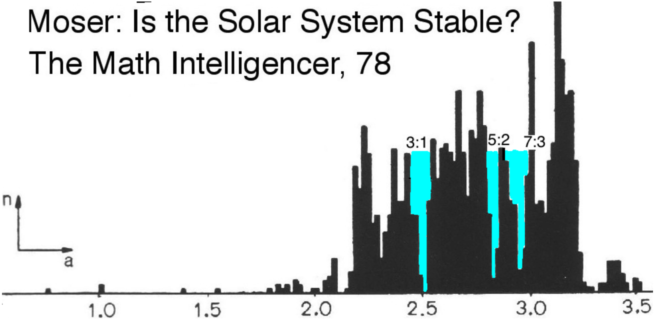

Kirkwood gap occurs at *mean-motion resonance*, i.e. when period of Jupiter and of Asteroid are in small rational relation, e.g. 3:1, 5:2, 7:3.

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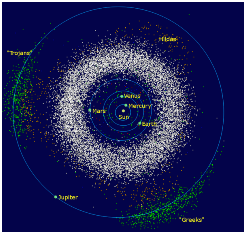

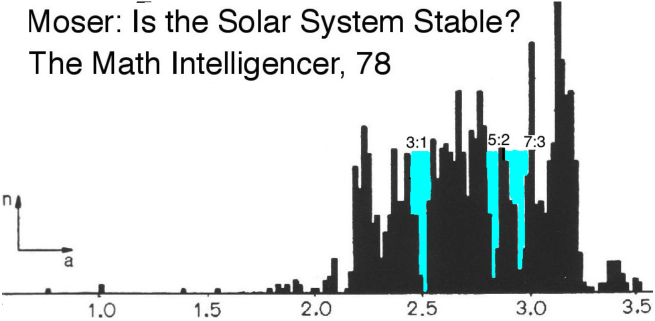

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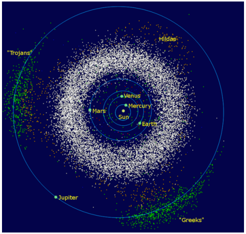

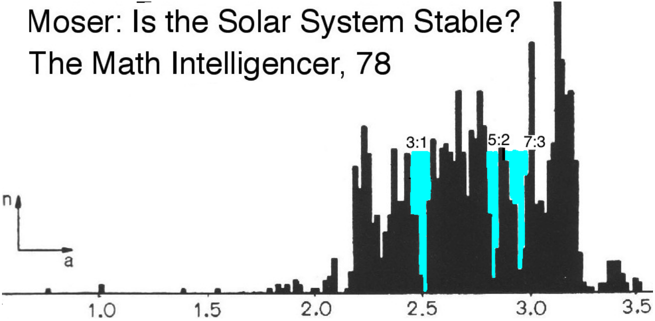

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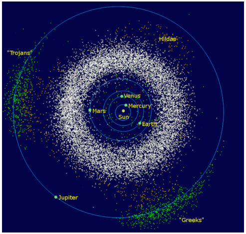

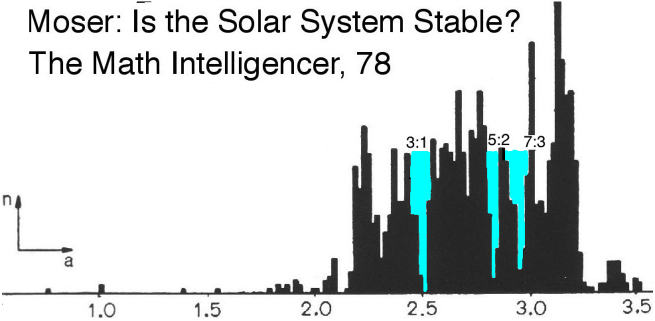

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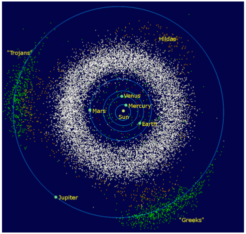

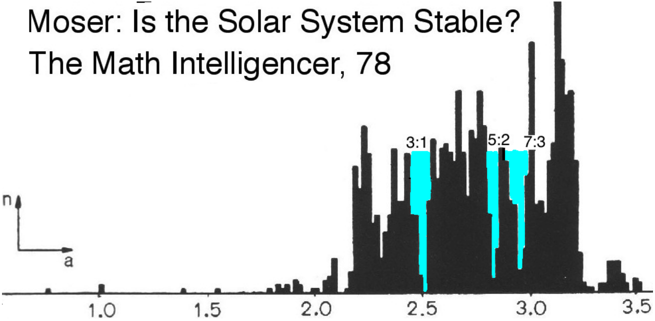

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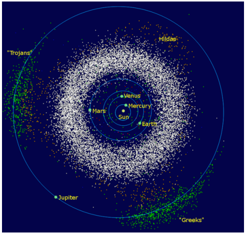

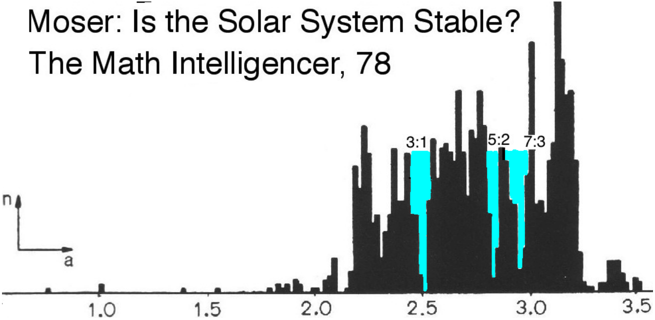

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Numerics and Chirikov's diffusion conjecture

K-Roldan Fix mass ratio μ, small Jupiter eccentr e_J, Asteroid's $a_A^{3/2}(0) = \frac{1}{3}$, $e_A(0) = e^*$. Pick 10^4 initial angles. Run $T_t = \frac{t}{\mu^2 e_J^2}$. Plot histogram for $e_A(T)$.

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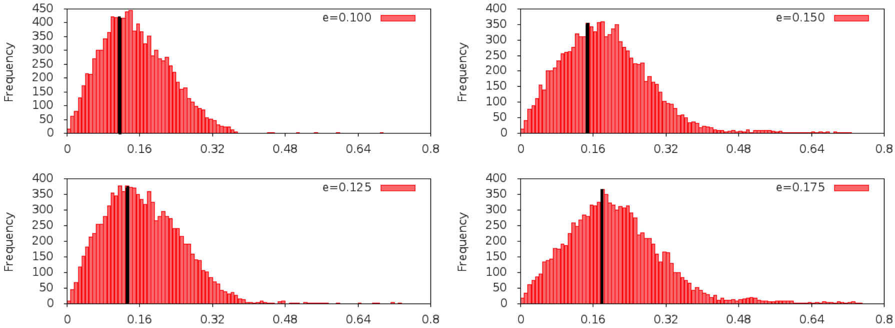

Diffusion conjecture Inside Kirkwood gaps as $\mu \to 0$ distributions of eccentricity $e_A(T_t)$ in a certain time scale weakly converge to distributions of a diffusion process $e_t = e_0 + \int_0^t \sigma(e_s)dw_s$, where dw_s is the white noise and $\sigma(e)$ is a smooth function.

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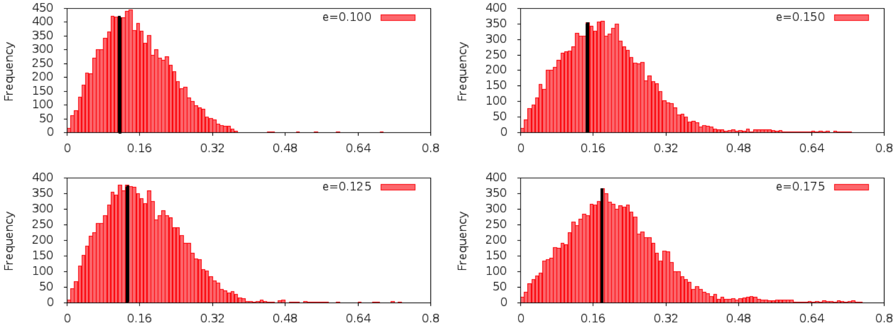

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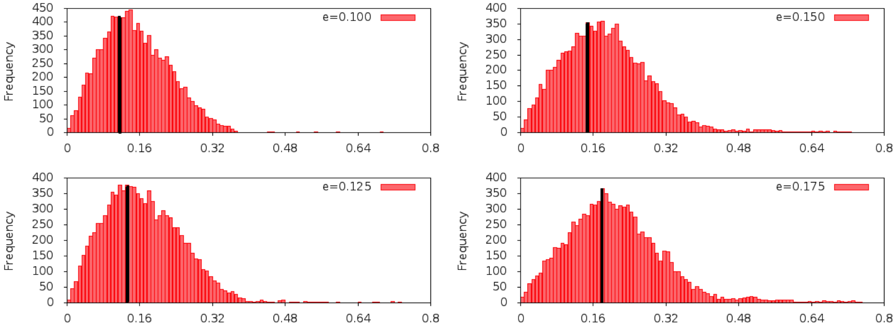

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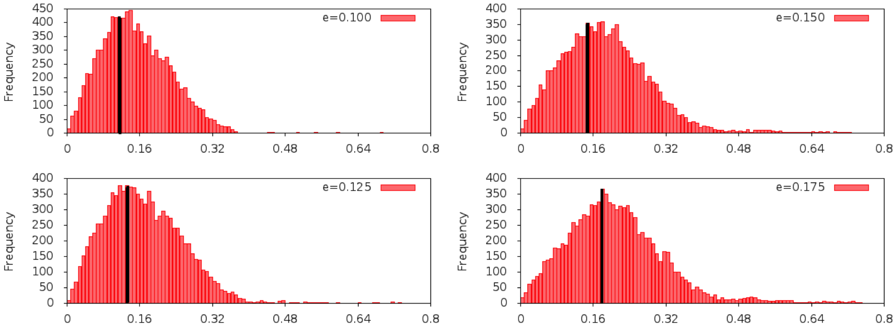

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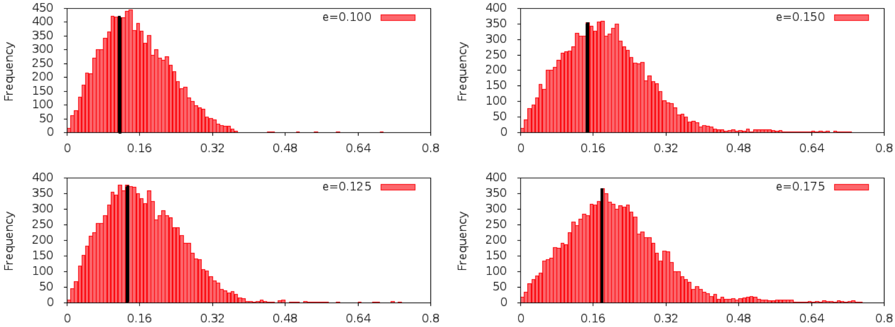

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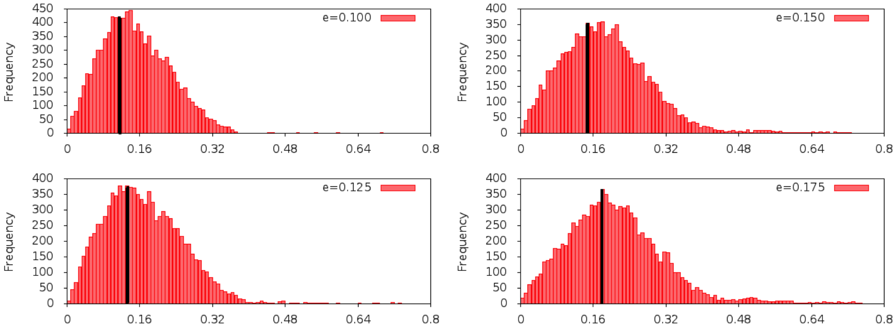

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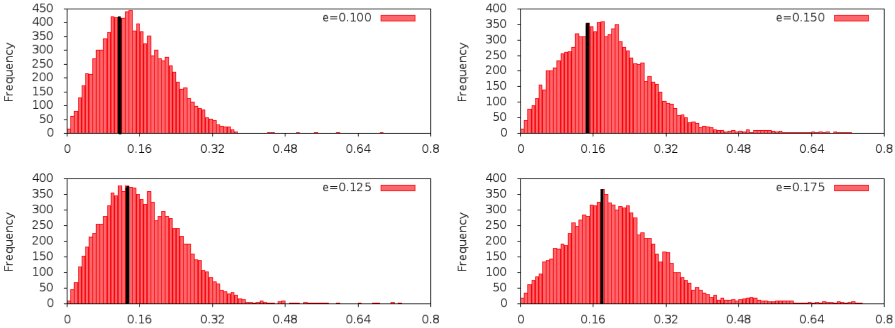

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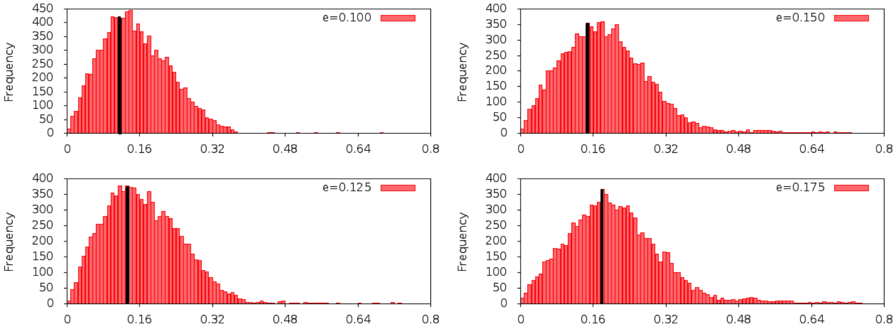

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Arnold's example

Soviet Mathematics–Doklady 5 581–5 (1964)

INSTABILITY OF DYNAMICAL SYSTEMS WITH SEVERAL DEGREES OF FREEDOM

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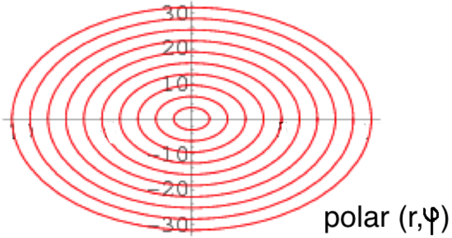
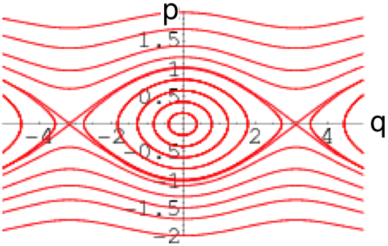
$$H_0(p,q,r,\varphi,t) = \underbrace{\frac{r^2}{2}}_{rotor} + \underbrace{\frac{p^2}{2} + (\cos q - 1)}_{pendulum},$$

where $q, \varphi, t \in \mathbb{T}$ are angles, $p, r \in \mathbb{R}$.

Rotor



Mathematical Pendulum



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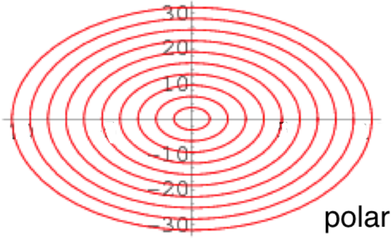

polar (r,φ)

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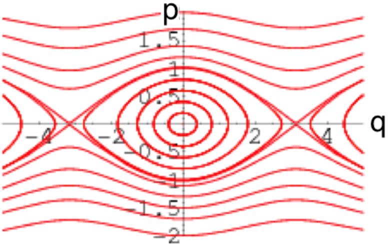

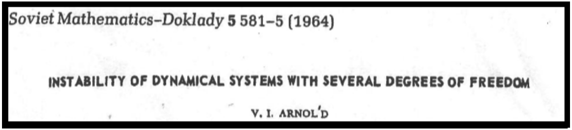

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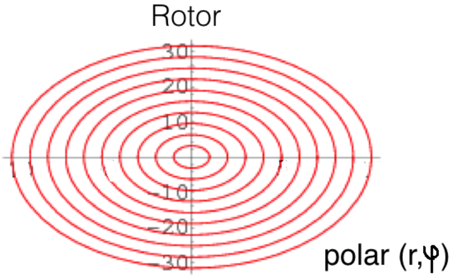

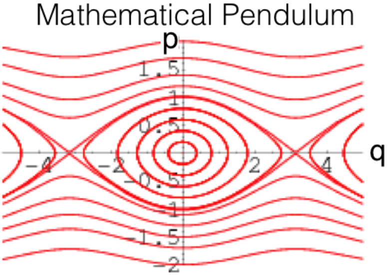

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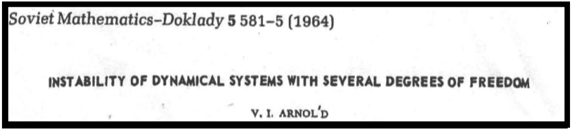

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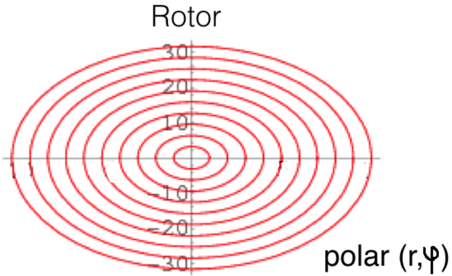

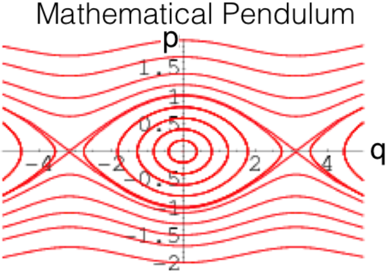

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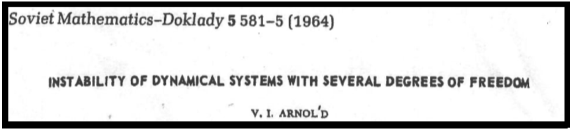

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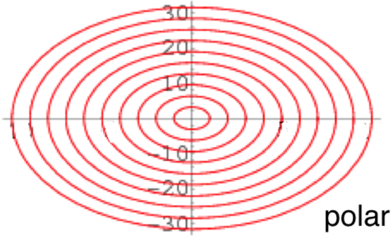

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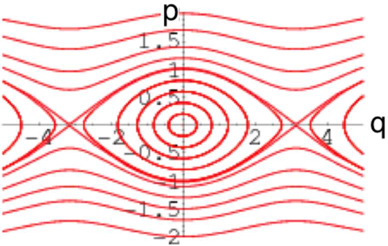

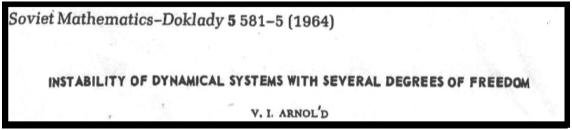

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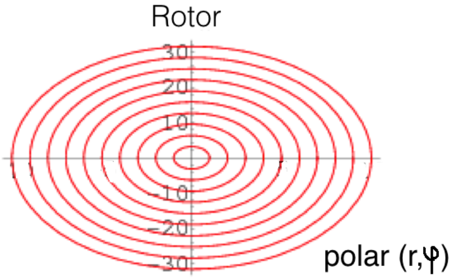

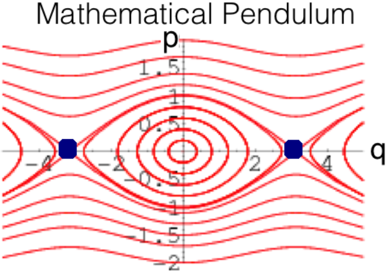

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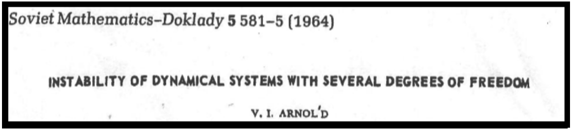

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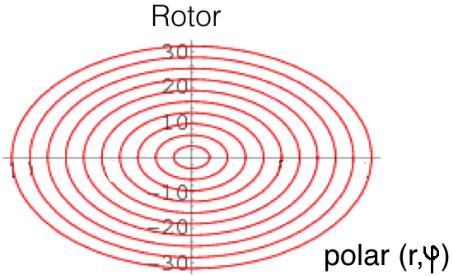

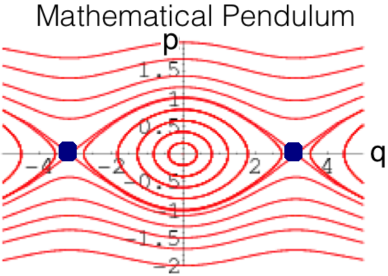

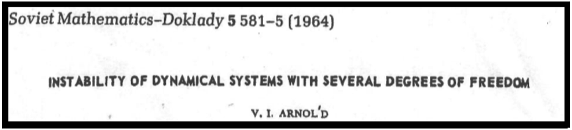
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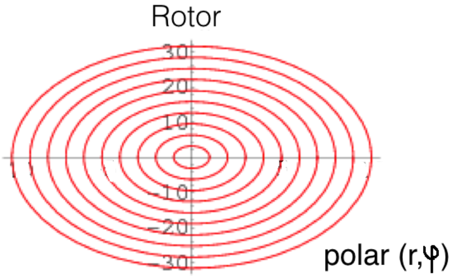

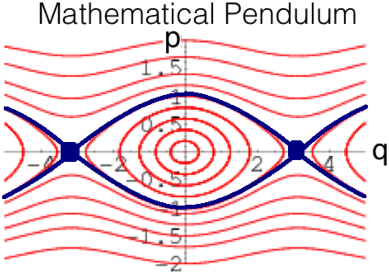

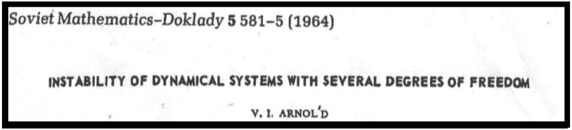

Soviet Mathematics–Doklady 5 581–5 (1964)

INSTABILITY OF DYNAMICAL SYSTEMS WITH SEVERAL DEGREES OF FREEDOM

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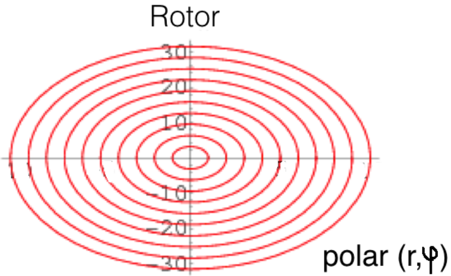

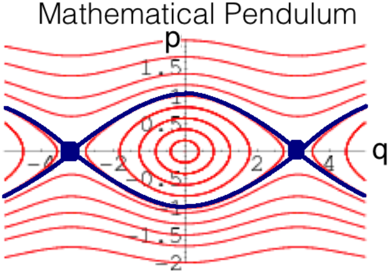

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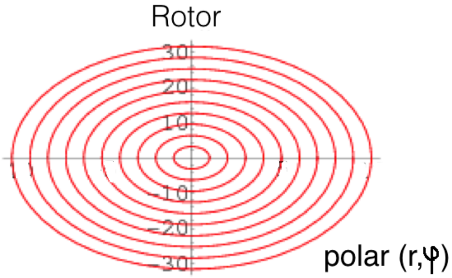

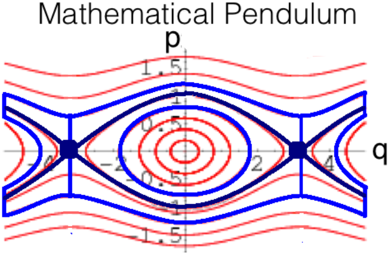

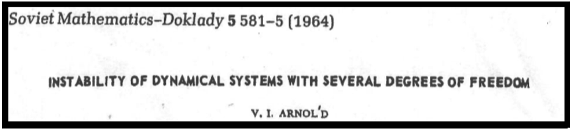
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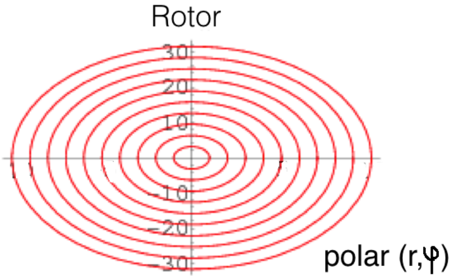

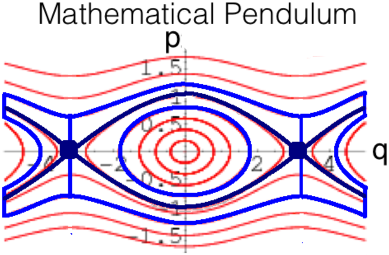

Arnold's example

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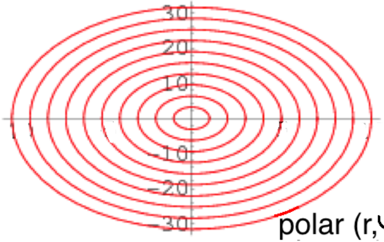
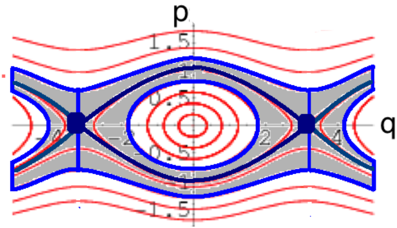
Rotor



Mathematical Pendulum



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Soviet Mathematics–Doklady 5 581–5 (1964)

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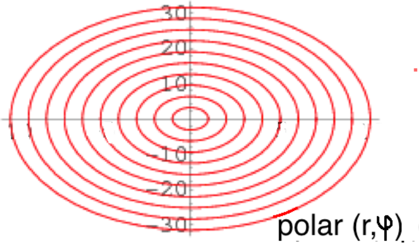

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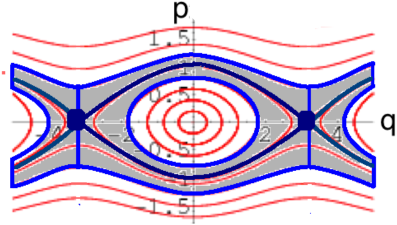

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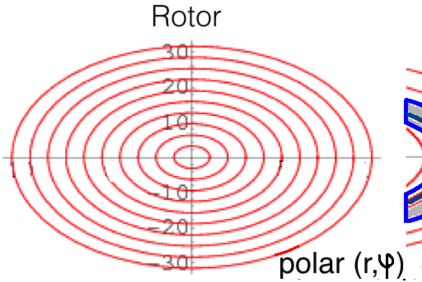

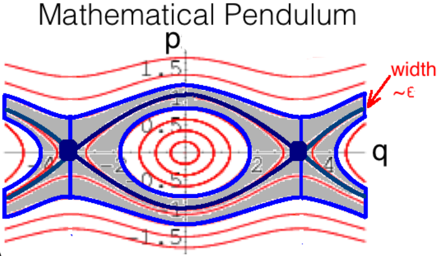

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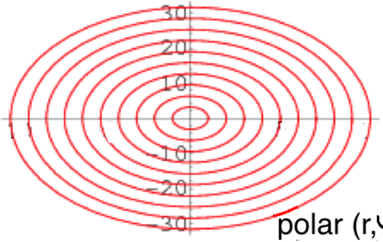
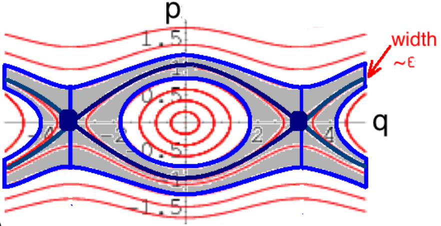
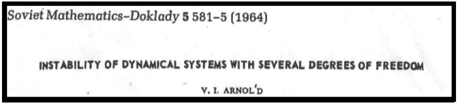


polar (r,φ)

Mathematical Pendulum



Arnold’s example



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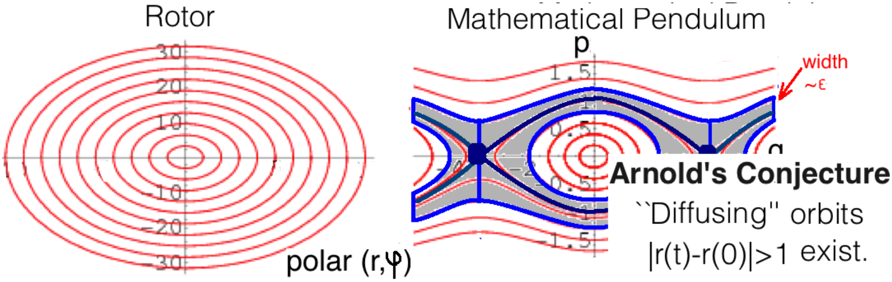

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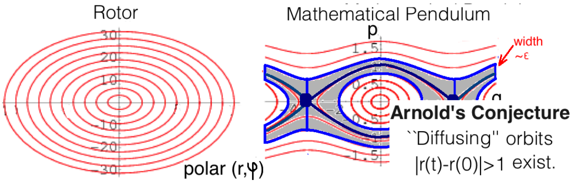

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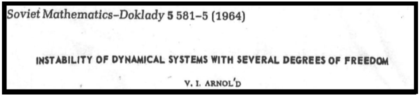

Soviet Mathematics–Doklady 5 581–5 (1964)

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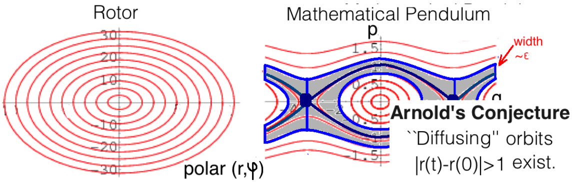

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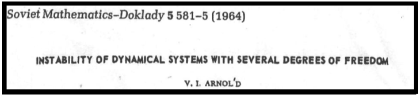
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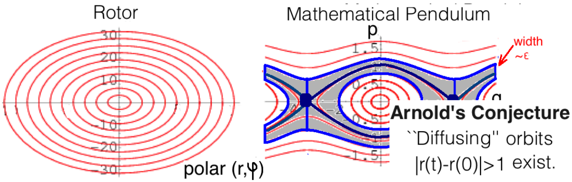

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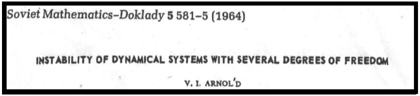
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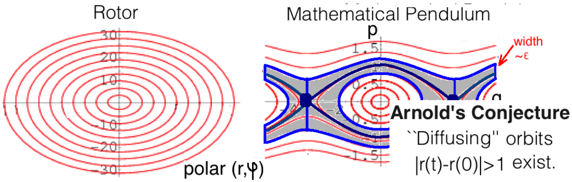

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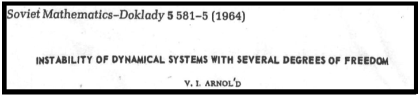
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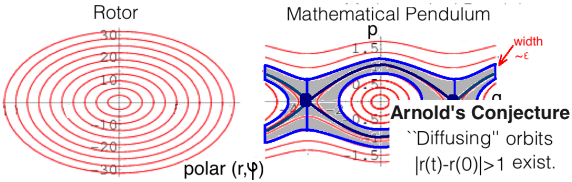

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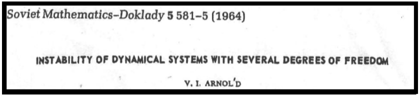
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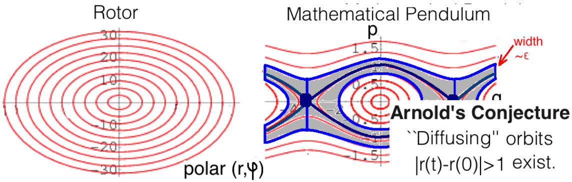

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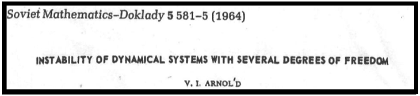

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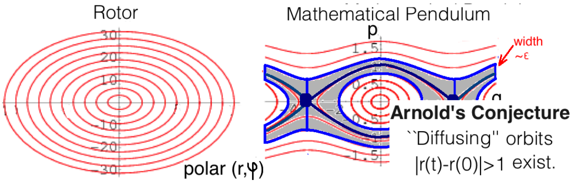

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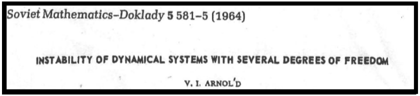
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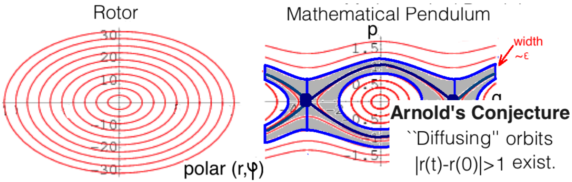

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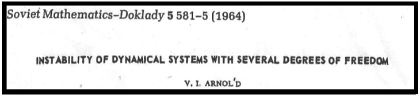
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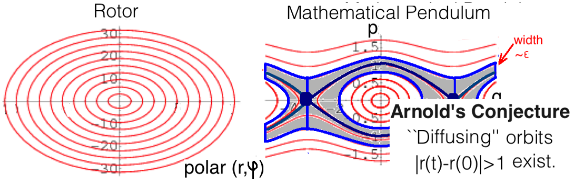

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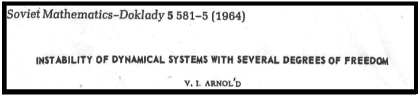
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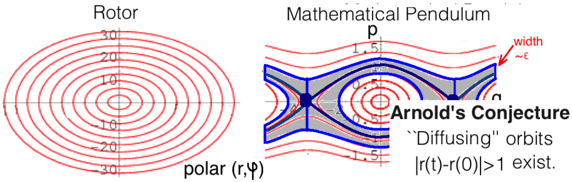

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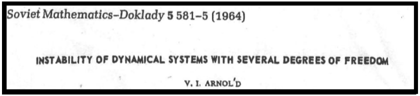

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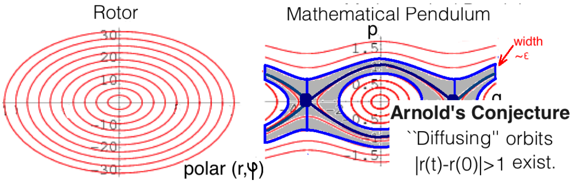

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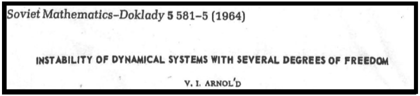

Soviet Mathematics–Doklady 5 581–5 (1964)

INSTABILITY OF DYNAMICAL SYSTEMS WITH SEVERAL DEGREES OF FREEDOM

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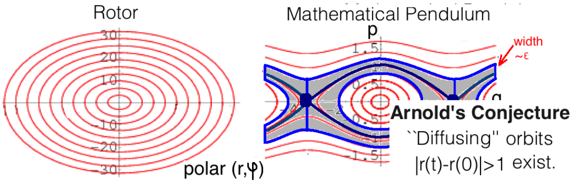

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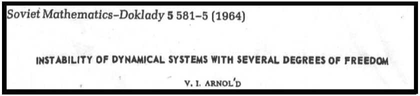
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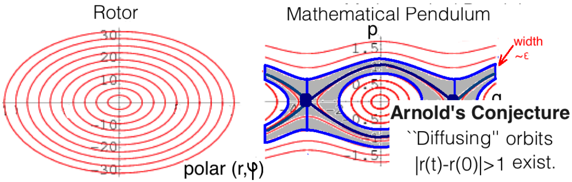

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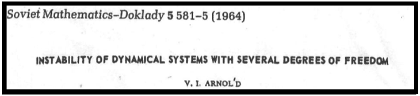

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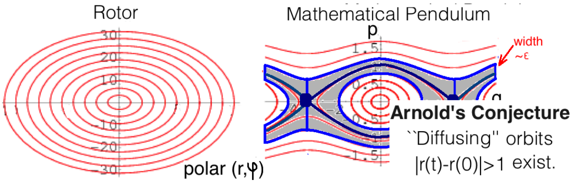

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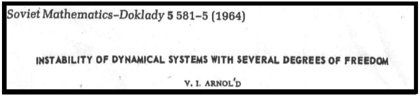
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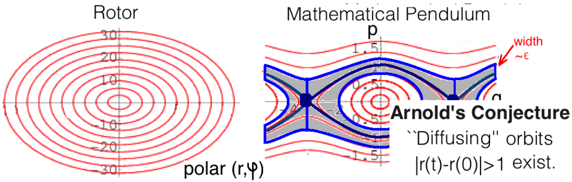

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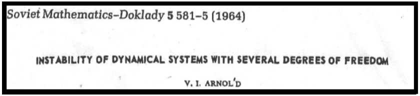
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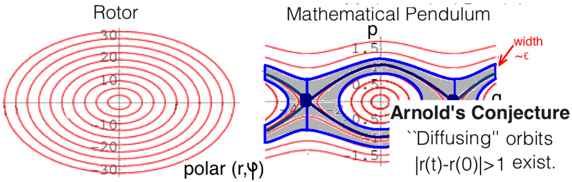

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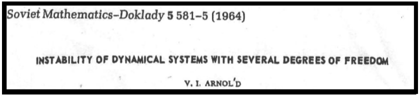

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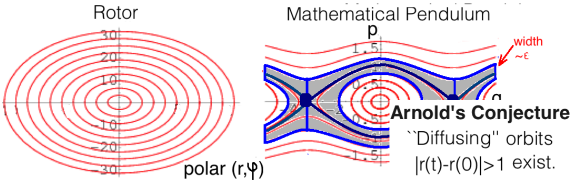

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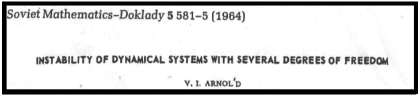

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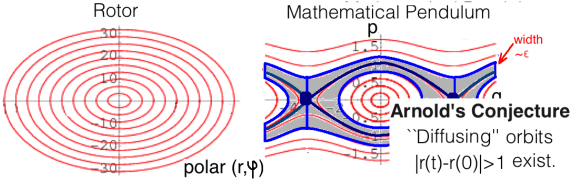

Chirikov's conjecture: Inside stochastic layer $r(t\,\varepsilon^{-2}\ln 1/\varepsilon)$ behaves as a stochastic diffusion process $x(t)$.

Numerics for Arnold's example

(K-Roldan) Fix $r = r^*$. Pick 10^6 initial conditions: (p_i, q_i) near 0, $\varphi_i \in \mathbb{T}$. Run $T_t = t\varepsilon^{-2} \log 1/\varepsilon,\ \varepsilon = 0.01$.

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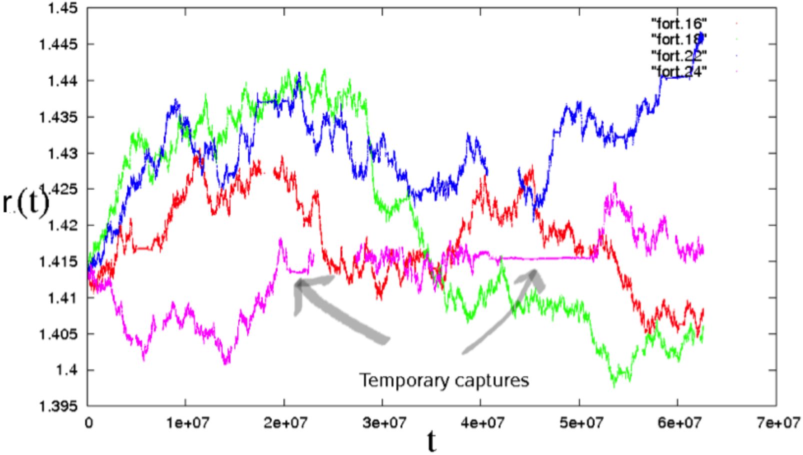

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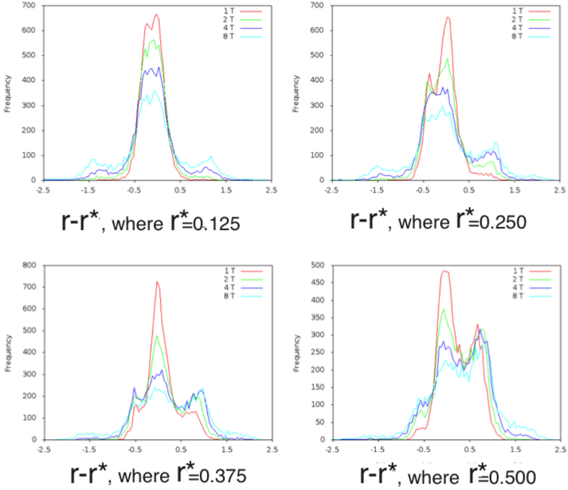

Main Result

$$H_\varepsilon(p,q,r,\varphi,t) = \frac{r^2}{2} + \frac{p^2}{2} + (\cos q - 1) + \varepsilon P_N(q,\varphi,t),$$

where P_N is a trigonometric polynomial, $N \geq 2$. Let $r_0 \in \mathbb{R}$, $\delta(r_0)$ be the δ-measure at r_0, Π_r be the projection on r, $T_t = t\varepsilon^{-2}\log 1/\varepsilon$.

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Thanks!