Exploring the Risk of Desynchronization

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Seminar
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Outline

I. Networks
  i. Classification
  ii. Network Types
II. Synchronization
  i. Linear Stability
  ii. Basin Stability
III. Lyapunov functions
IV. The Risk of Desynchronization
  i. Reduced Order Model (ROM)
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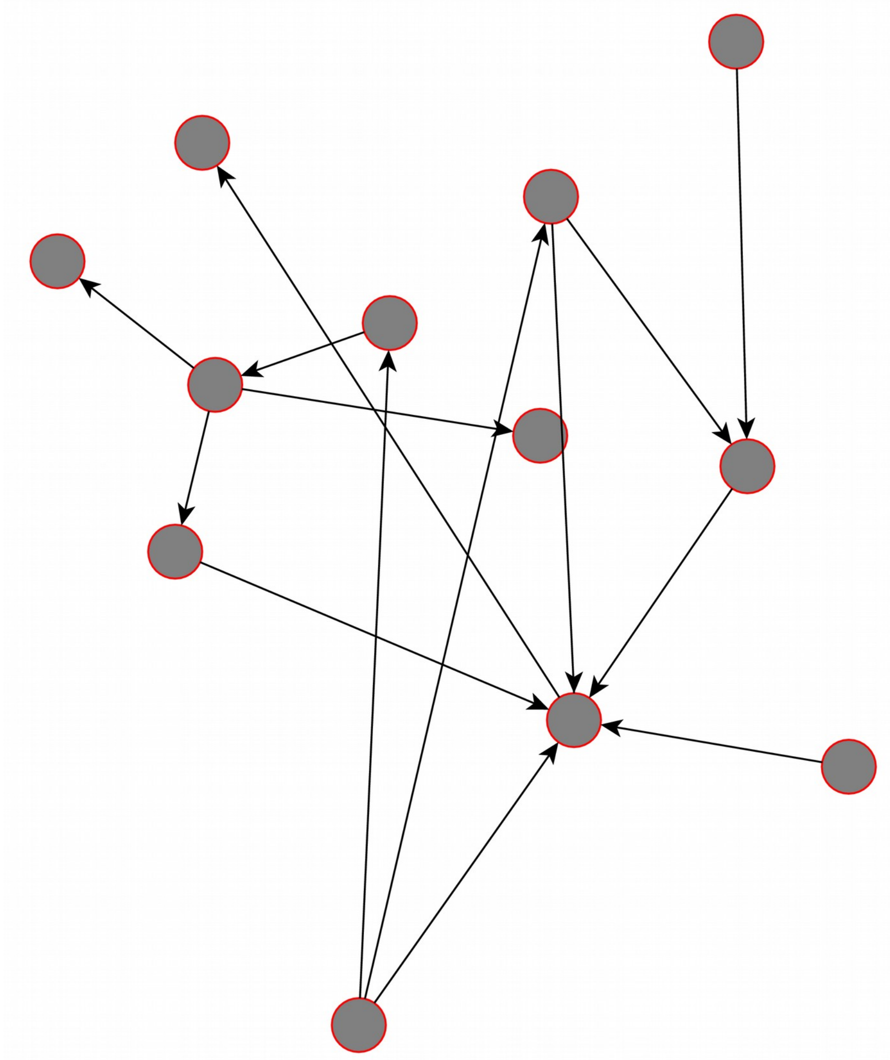
What is a Network?

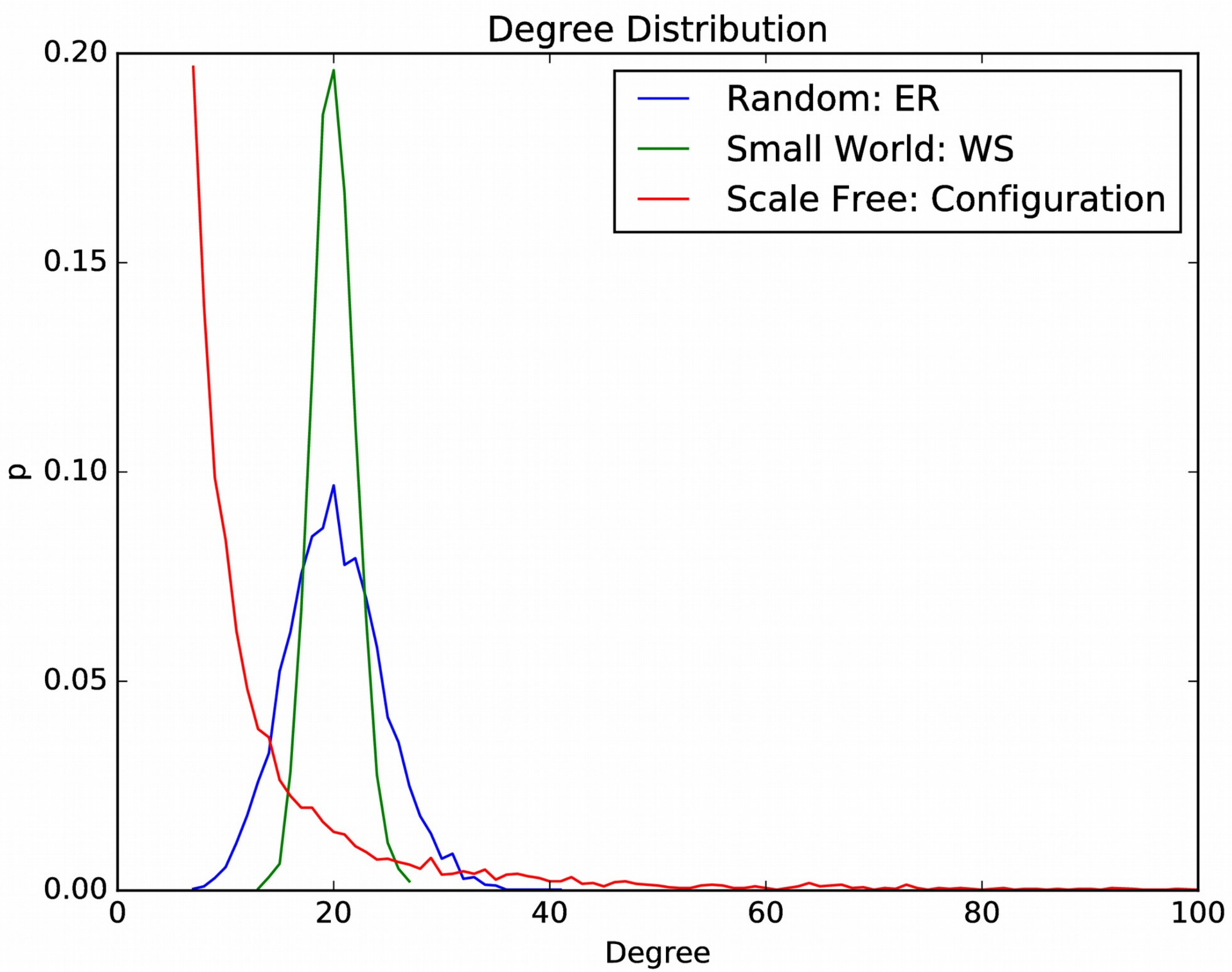
- Nodes
- Edges (Links)
- Directed
- Undirected
- Degree
- Degree Distribution



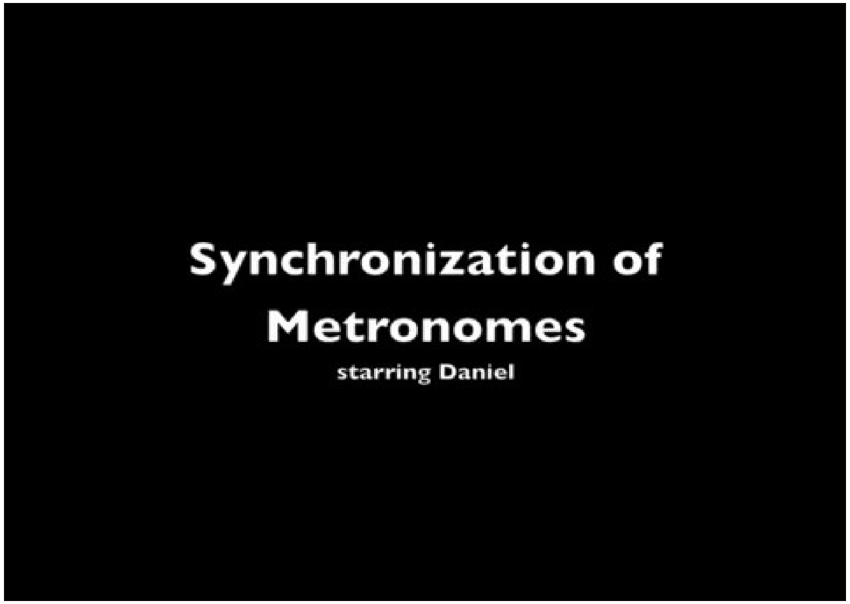
Real world examples of networks include the power grid, LASER networks, facebook and the brain

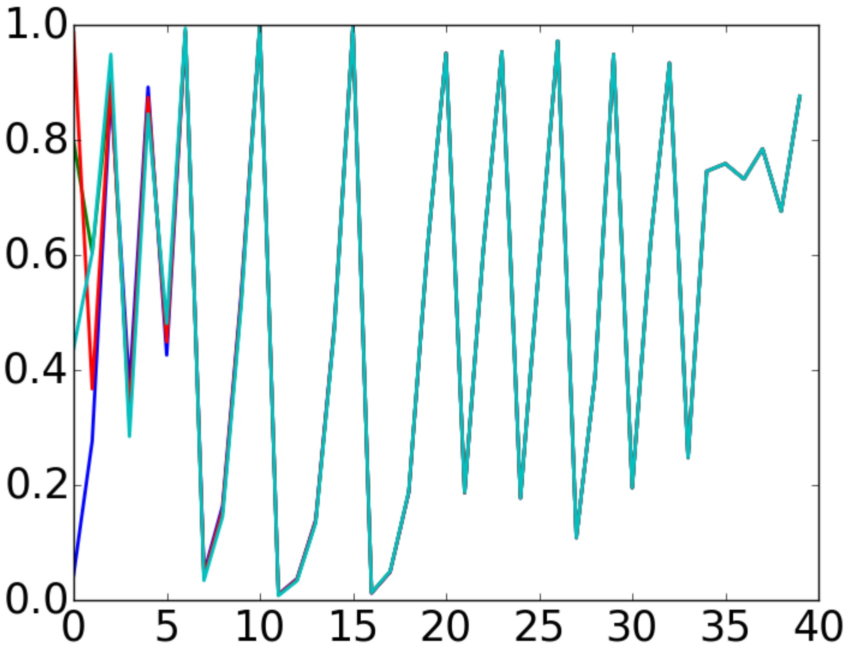
Network Types

- Random
  (ER with N = 5000, p = 0.004)
- Scale Free
  (WS, N = 5000, k = 20, p = 0.25)
- Scale Free
  (Configuration, N = 5000, λ = 2.5, d_{min} = 7



Synchronization





Master Stability Equations

General Equation

$$\dot{x}_i = f_i(x_i) + \sigma \sum_{j=1}^{N} A_{i,j} h_i(x_i, x_j)$$

$$\sigma \in \mathbb{R}, x_i \in \mathbb{R}^m, A \in \mathbb{R}^{N \times N}$$

$$f : \mathbb{R}^m \mapsto \mathbb{R}^m, h : \mathbb{R}^{2m} \mapsto \mathbb{R}^m$$

Under Assumptions (Identical oscillators, with identical coupling functions)

$$\dot{x} = F(x) - \sigma L \otimes E \cdot x$$

$$\sigma \in \mathbb{R}, x \in \mathbb{R}^{mN},$$

$$L \in \mathbb{R}^{N \times N}, E \in \mathbb{R}^{m \times m}$$

After linearization

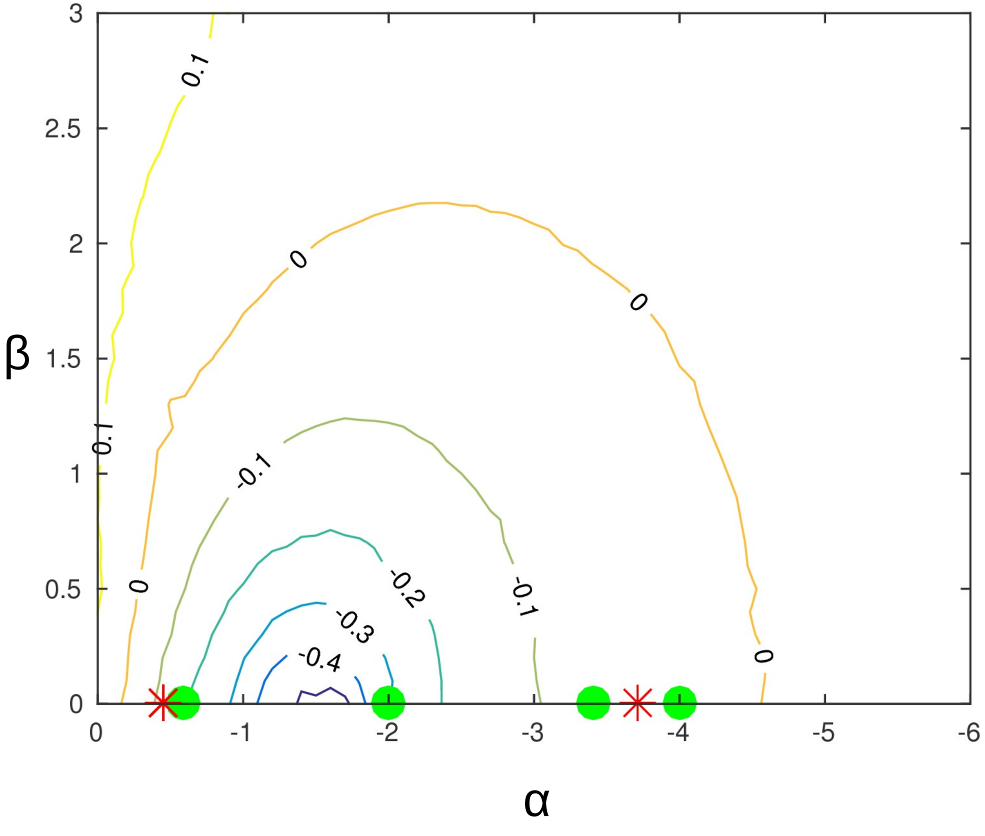
$$\dot{\zeta}_k = [DF + \sigma \lambda_k E] \cdot \zeta_k$$

Master Stability function

We can now define the Master Stability Function (MSF)

$$\dot{\zeta}_k = [DF + (\alpha + i\beta)E] \cdot \zeta_k$$

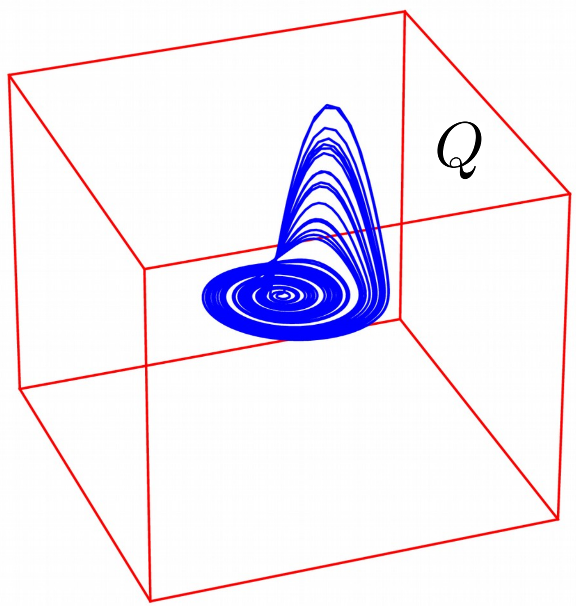
Now performing search over α and β, while calculating the largest Lyapunov exponent (in this case in a chaotic Rossler system)



Basin Stability

$$\mathrm{M} = \mathrm{x}_1(0), \ldots x_N(0) | x_1(t) = x_2(t) = \ldots x_N(t) \in \mathcal{A}$$
$$\Omega(\mathcal{M}) = x_1(0), \ldots x_N(0) | x_1(t) \ldots x_N(t) \to \mathcal{M}, \ as\ t \to \infty$$
$$\mathrm{S}_{\Omega(\mathcal{M}) \cap Q} = \frac{Vol(\Omega \cap Q)}{Vol(Q)} \in [0,1]$$

The Concept of Basin Stability:
Draw initial conditions from a box and see how many synchronize.



Lyapunov Functions

$$V : D \mapsto \mathbb{R}$$

$$V \in C^1$$

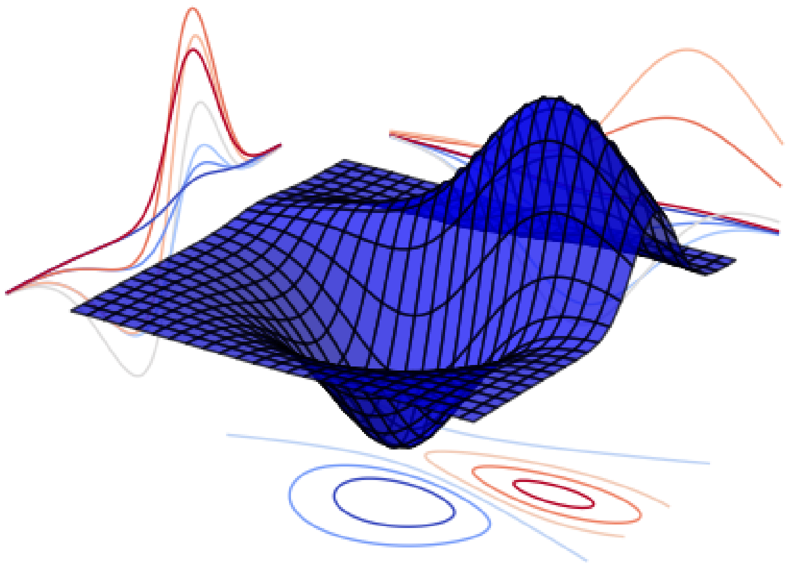
$$V(0) = 0$$

$$V(x) > 0 \in D - 0$$

$$\dot{V} < 0 \in D - 0$$

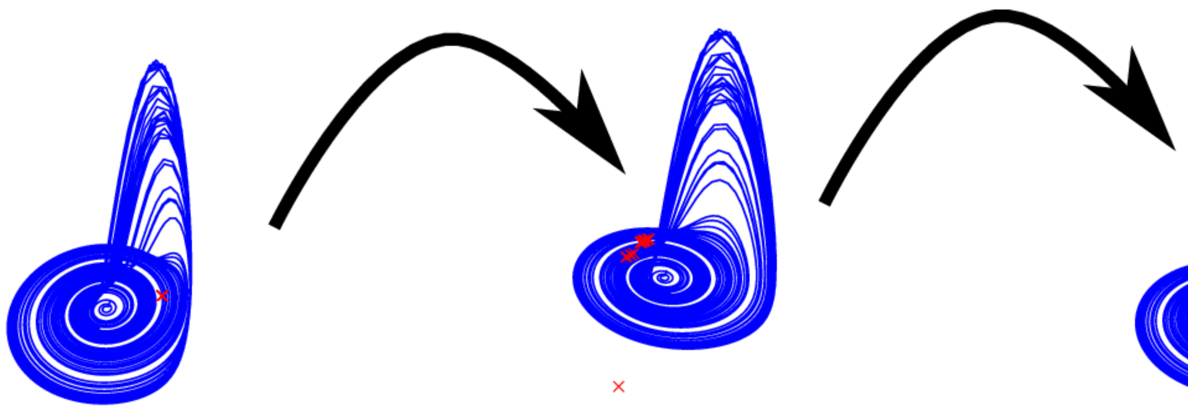
$$\dot{V} = 0 \; at \; x = 0$$

If the above conditions are met, then x = 0 is asymptotically stable



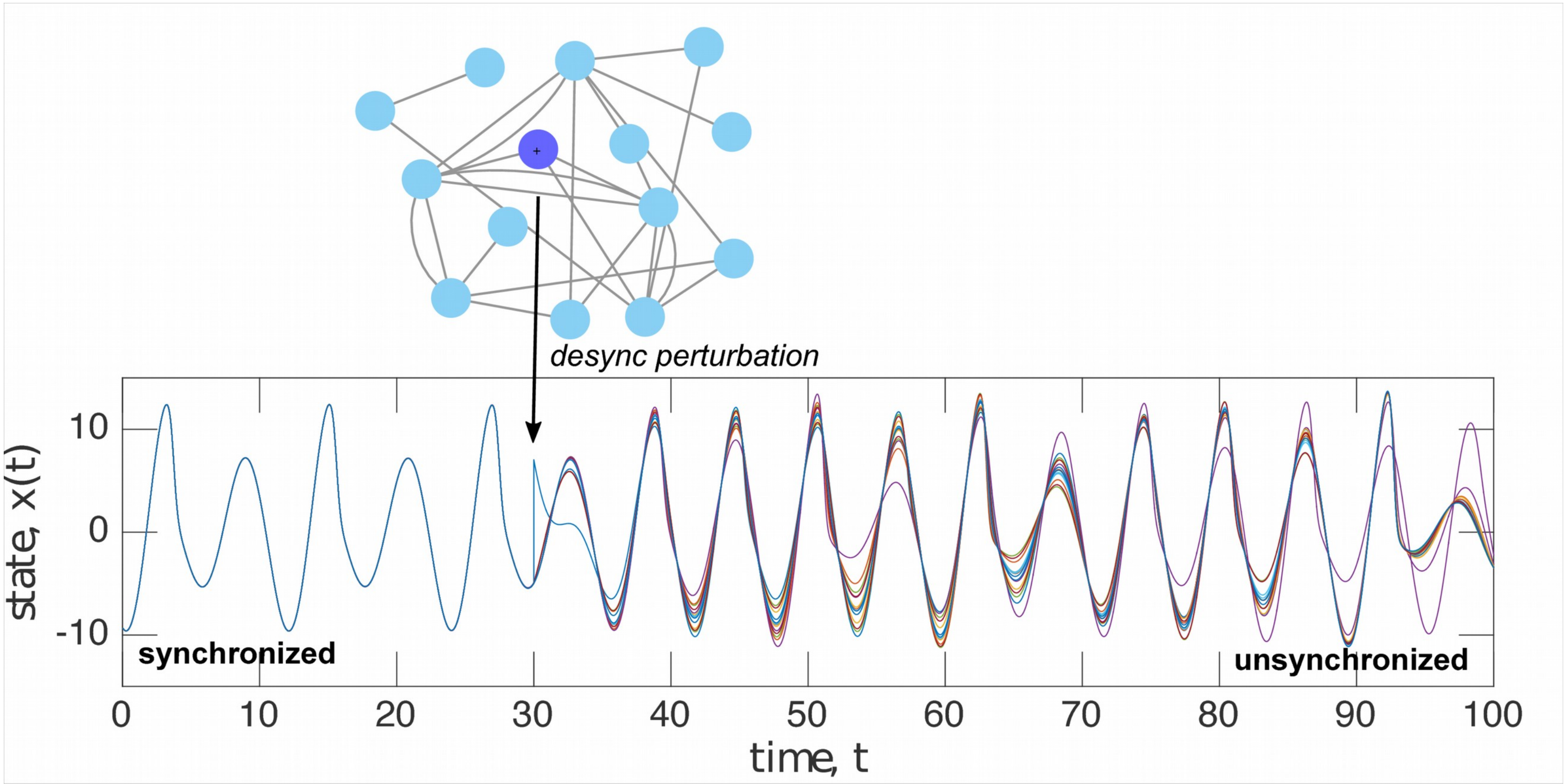
Single Node Perturbations

A single node perturbation: The system starts on a synchronous trajectory, but a single node is perturbed away from the synchronous trajectory.



$$\Omega^{(i)}(s) = x_i(0) | x_j(0) = s \forall j \neq i, x_1(0) \ldots x_N(0) \in \Omega(\mathcal{M})$$

Desynchronization



Reduced Order Model I.

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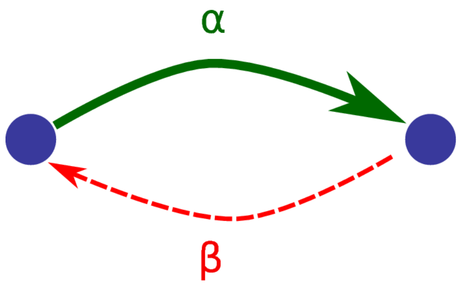
Assuming a Lyapunov function exists, we can map the system equations to a lower dimensional system

$$\dot{x}_i = f(x_i) + \sigma \sum_{j=1}^{N} A_{i,j} h(x_i, x_j) \approx$$

$$[\dot{y}_1, \dot{y}_2] = [f(y_1) + \alpha_i h(y_1, y_2), f(y_2) + \beta_i h(y_2, y_1)]$$

$$\alpha_i = \sigma \sum_{j=1}^{N} A_{i,j},\ \beta_j = \sigma A_{j,i}$$

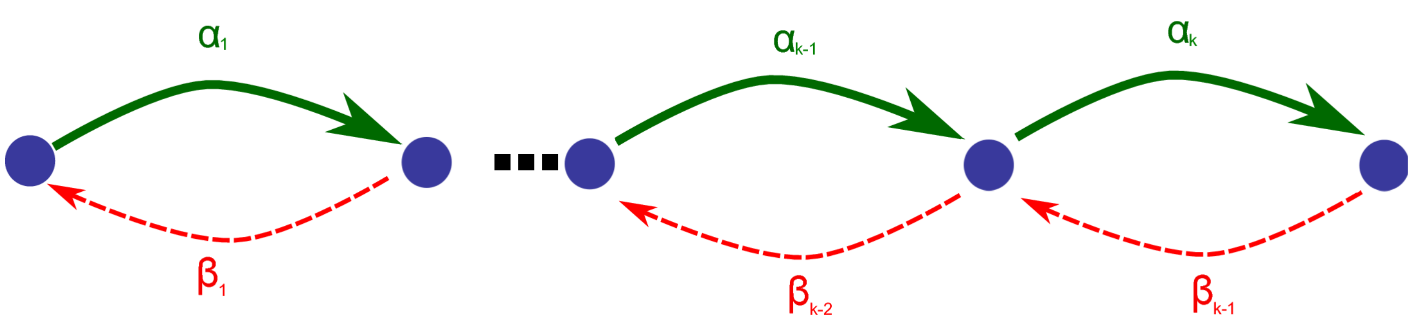


Reduced Order Model II.

For Chaotic systems, specifically the Rossler system, we have discovered that in the high coupling region, the ROM must be expanded

$$\begin{aligned}
\dot{y}_1 &= f(y_1) + \alpha_1 h(y_1, y_2) \\
\dot{y}_2 &= f(y_2) + \beta_1 h(y_2, y_1) + \alpha_2 h(y_2, y_3) \\
\dot{y}_3 &= f(y_3) + \beta_2 h(y_3, y_2) + \alpha_3 h(y_3, y_4) \\
&\vdots \\
\dot{y}_k &= f(y_k) + \beta_{k-1} h(y_k, y_{k-1})
\end{aligned}$$

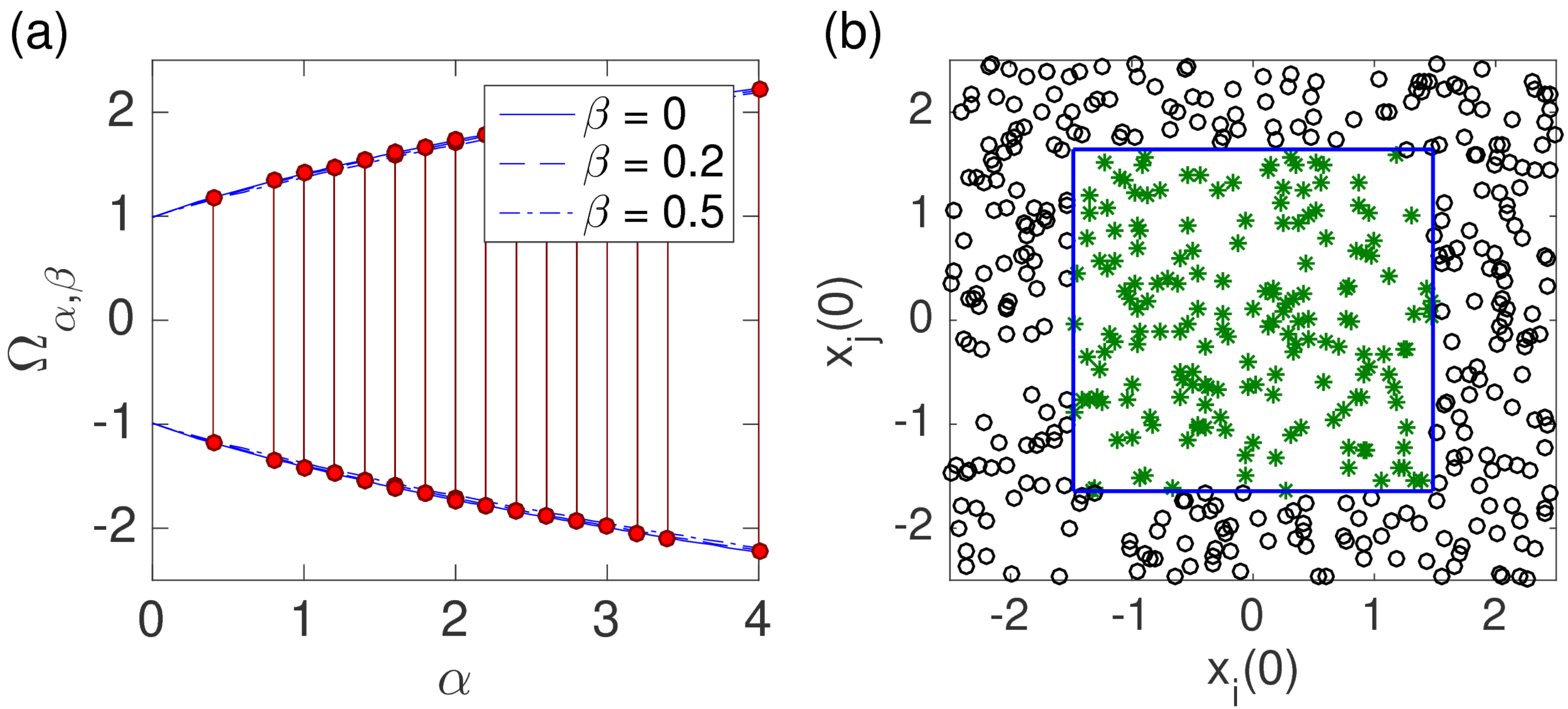
k is determined by the breadth first search from the perturbed node



Confirming the ROM Fixed Point system

The Cubic oscillator system has a stable fixed point at x = 0 (for a single oscillator).

$$f(x_i) = x_i(x_i^2 - 1)$$



Defining Risk

$$\mathcal{R} = \frac{|\Omega(\mathcal{M}) \cap P|}{|P|} \in [0, 1]$$

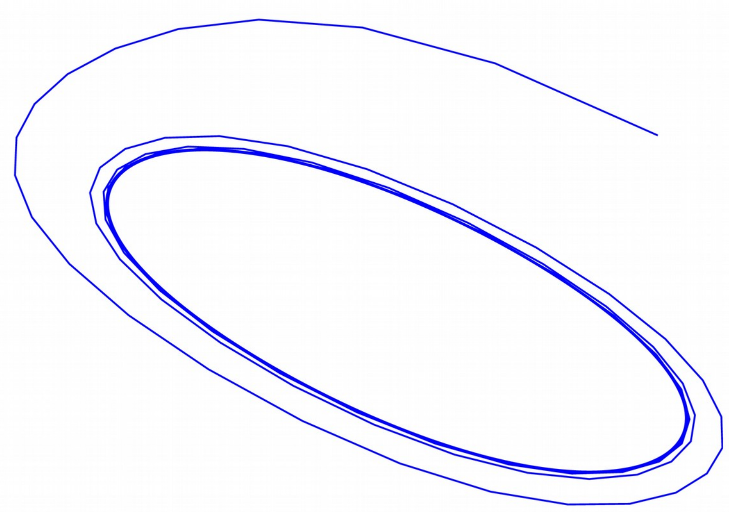
$$\mathcal{R} - Risk,$$

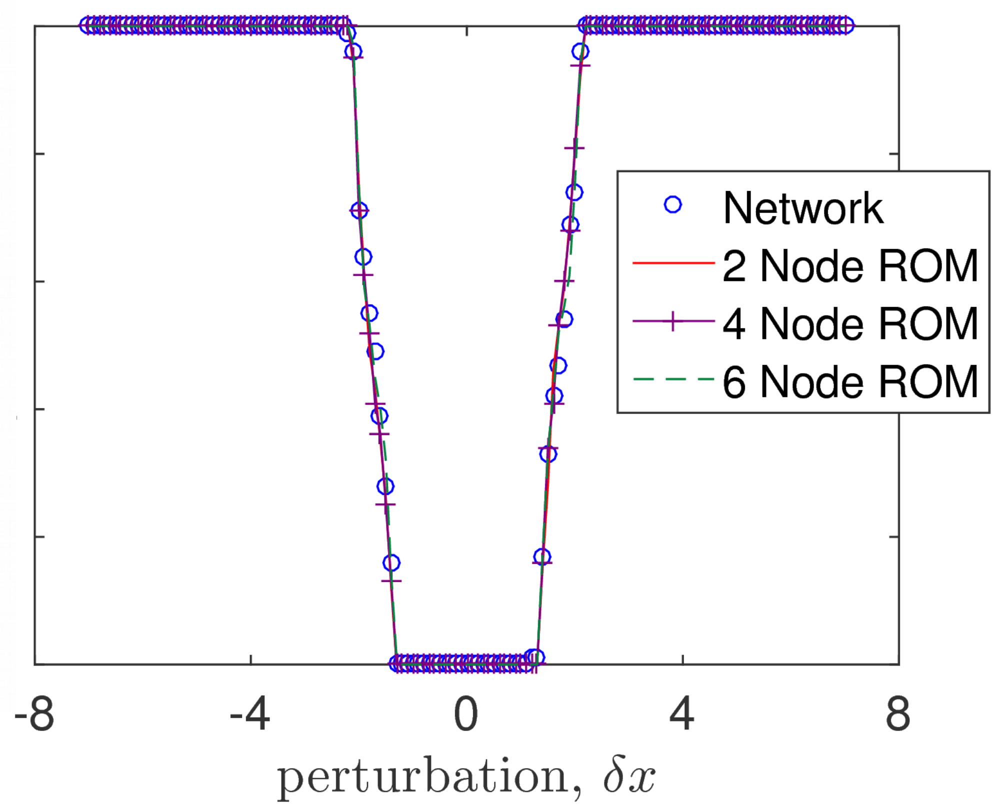
$$\mathcal{P} - Set\ of\ perturbations\ away\ from\ synchrony$$

Confirming the ROM Limit Cycle system

$$\dot{x} = -x + y - (ay - by^3 + +cy^5)$$

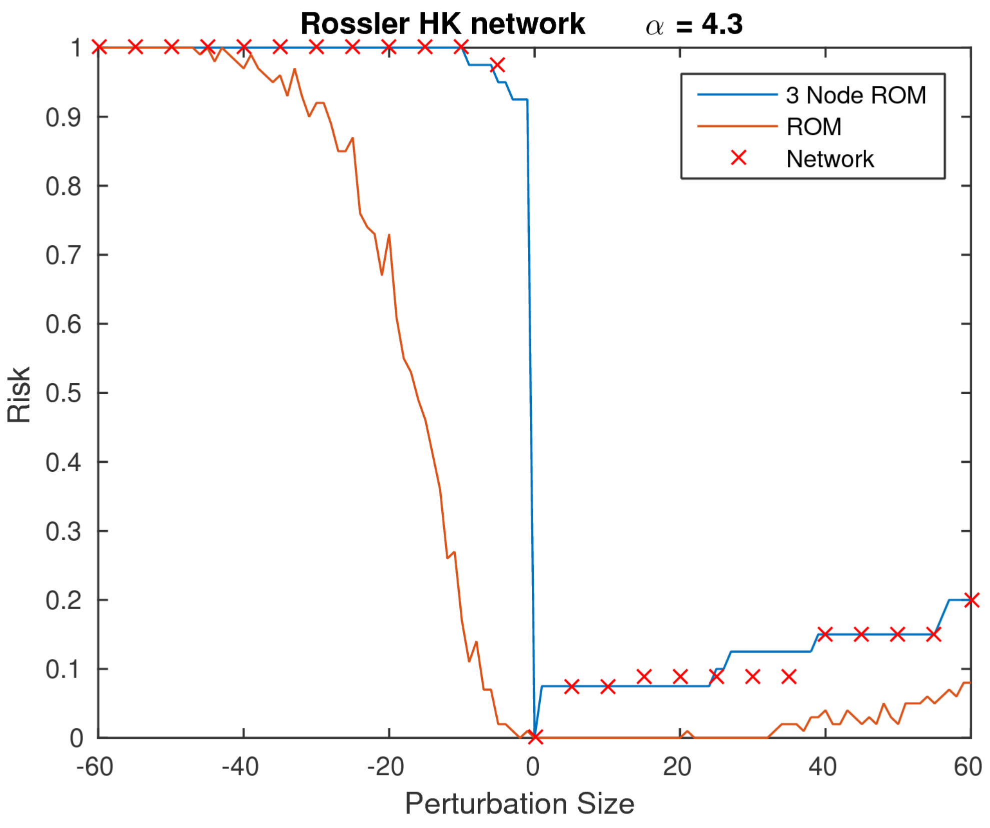
$$\dot{y} = -(-x + y - (ay - by^3 + cy^5)) - y$$





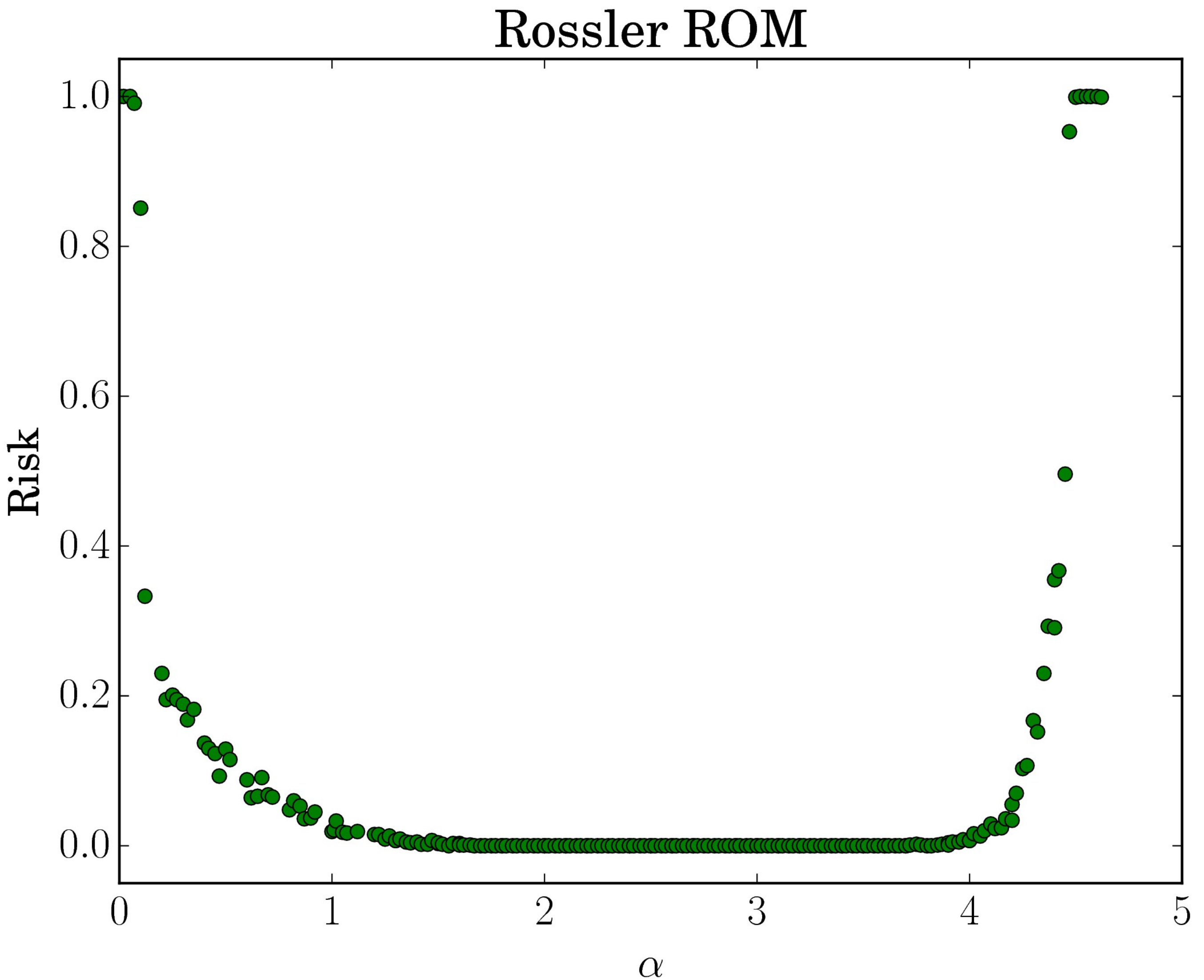
Confirming the ROM Rossler system

The higher order ROM become necessary in the chaotic system



Utilizing the ROM

We can now utilize the ROM to determine how much risk is associated with a particular node being perturbed

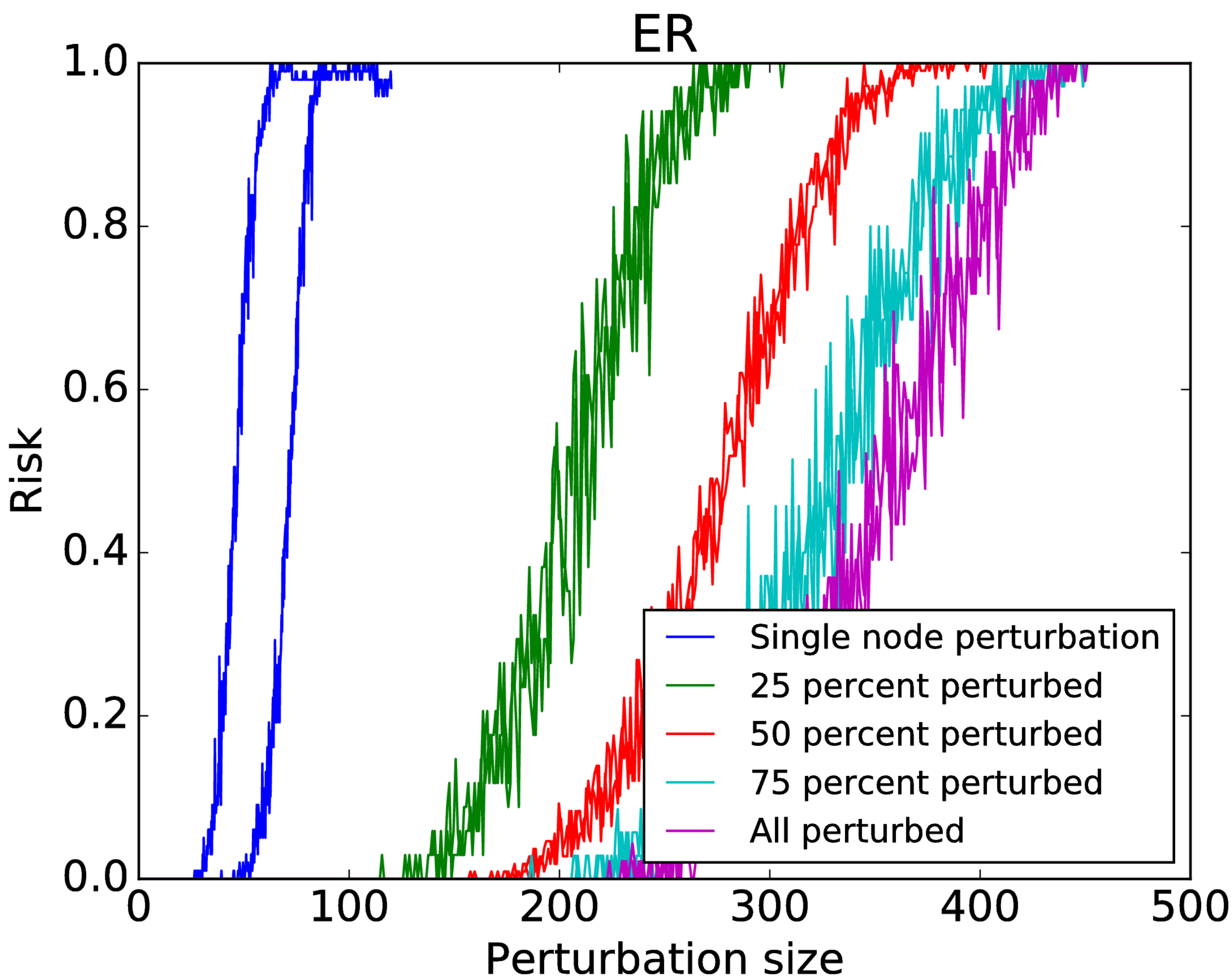


Risk of Desynchronization

- Given a perturbation size, how likely is a desynchronizing perturbation to occur?
- How does increasing the number of nodes perturb relate to the risk of desynchronization?
- How does the degree of the node relate to the risk of desynchronization?

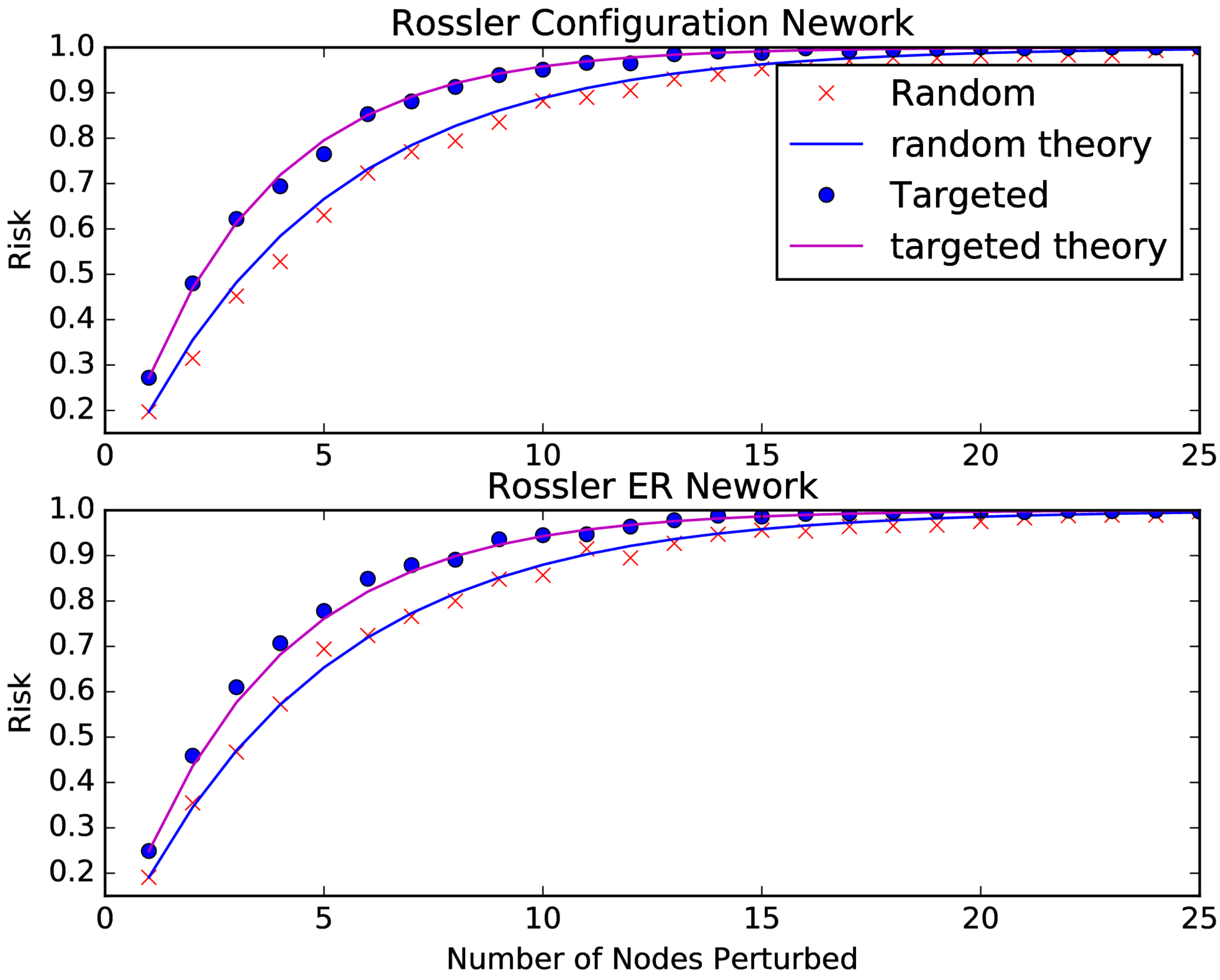
Risk and the size of the perturbation

The magnitude of the perturbation is an important factor in if the system desynchronizes. A large magnitude perturbation almost certainly desynchronizes the system, while a small magnitude perturbation almost certainly allows resynchronization

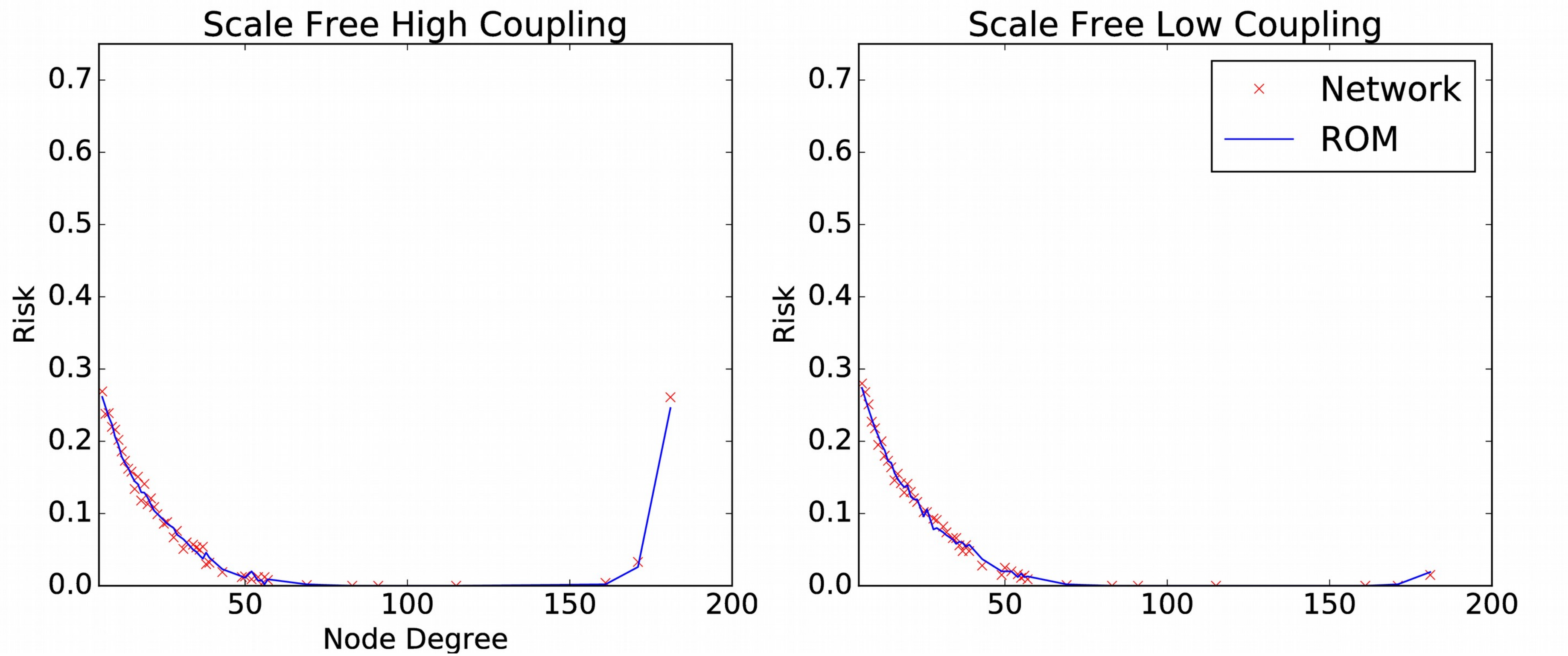


Risk and the number of nodes perturbed

Our assumption about the basin size being the intersection of individual basins works quite well



Risk and Degree



High degree nodes are not necessarily at the least risk of desynchronization!

Potential Applications

Deep brain stimulation (Treatment for Parkinson's disease)

In the power grid, synchronization is desirable and thus decisions on where power lines are added (or perhaps even removed) could be aided

Future Work

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- Expanding proof of the ROM to limit cycle and chaotic dynamics
- Applying the ROM to more real-world systems, such as the Hodgkin-Huxley model
- Implement the ROM to systems with non-identical oscillators, such as the Kuromoto model.

Questions

I wish to thank my advisor Professor Jie Sun
Thank you for you time!