Moments & Positive Polynomials in Optimization and more

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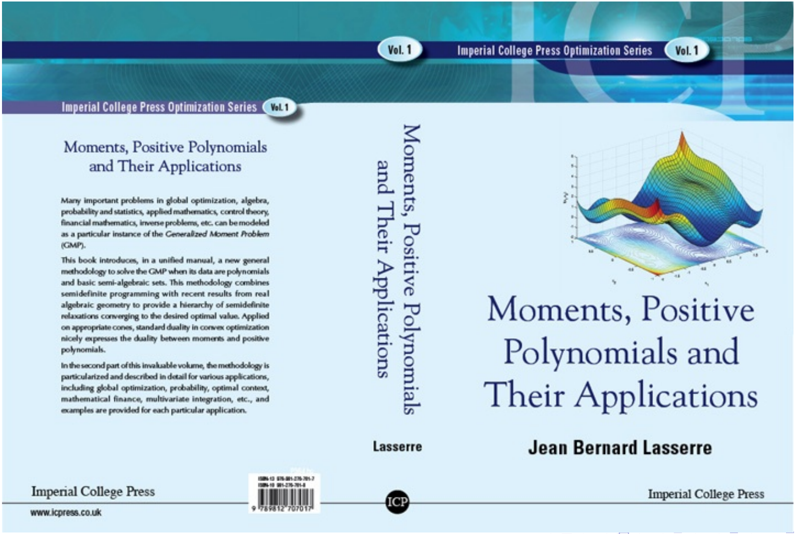
Vol. 1
Imperial College Press Optimization Series
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Moments, Positive Polynomials and Their Applications
Many important problems in global optimization, algebra, probability and statistics, applied mathematics, control theory, financial mathematics, inverse problems, etc. can be modeled as a particular instance of the Generalized Moment Problem (GMP).
This book introduces, in a unified manual, a new general methodology to solve the GMP when its data are polynomials and basic semi-algebraic sets. This methodology combines semidefinite programming with recent results from real algebraic geometry to provide a hierarchy of semidefinite relaxations converging to the desired optimal value. Applied on appropriate cones, standard duality in convex optimization nicely expresses the duality between moments and positive polynomials.
In the second part of this invaluable volume, the methodology is particularized and described in detail for various applications, including global optimization, probability, optimal control, mathematical finance, multivariate integration, etc., and examples are provided for each particular application.
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Moments, Positive Polynomials and Their Applications
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Moments, Positive Polynomials and Their Applications
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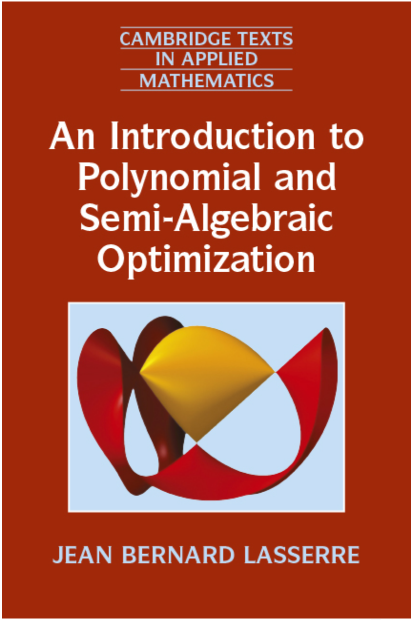
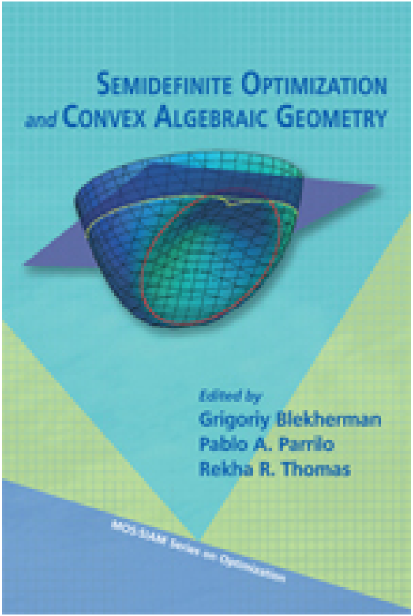

CAMBRIDGE TEXTS IN APPLIED MATHEMATICS
An Introduction to Polynomial and Semi-Algebraic Optimization
JEAN BERNARD LASSERRE


Semidefinite Optimization
and Convex Algebraic Geometry
Edited by
Grigoriy Blekherman
Pablo A. Parrilo
Rekha R. Thomas
MOS-SIAM Series on Optimization

- Why POLYNOMIAL optimization?
- LP- and SDP- CERTIFICATES of POSITIVITY
- The moment-LP and moment-SOS approaches
- Some applications

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Consider the polynomial optimization problem:

$$\mathbf{P}: \quad f^* = \min\{ f(\mathbf{x}) : \quad g_j(\mathbf{x}) \geq 0,\ j = 1, \ldots, m \}$$

for some polynomials f, $g_j \in \mathbb{R}[\mathbf{x}]$.

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After all ... **P** is just a particular case of Non Linear Programming (NLP)!

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True!

... if one is interested with a LOCAL optimum only!!

When searching for a local minimum ...

Optimality conditions and descent algorithms use basic tools from REAL and CONVEX analysis and linear algebra

☞ The focus is on how to improve f by looking at a NEIGHBORHOOD of a nominal point $\mathbf{x} \in \mathbf{K}$, i.e., LOCALLY AROUND $\mathbf{x} \in \mathbf{K}$, and in general, no GLOBAL property of $\mathbf{x} \in \mathbf{K}$ can be inferred.

The fact that f and g_j are POLYNOMIALS does not help much!

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BUT for GLOBAL Optimization

... the picture is different!

Remember that for the GLOBAL minimum f^*:

$$f^* = \sup \{ \lambda : f(\mathbf{x}) - \lambda \geq 0 \quad \forall \mathbf{x} \in \mathbf{K} \}.$$

(Not true for a LOCAL minimum!)

and so to compute f^* ...

☞ one needs to handle EFFICIENTLY the difficult constraint

$$f(\mathbf{x}) - \lambda \geq 0 \quad \forall \mathbf{x} \in \mathbf{K},$$

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REAL ALGEBRAIC GEOMETRY helps!!!!

Indeed, POWERFUL CERTIFICATES OF POSITIVITY EXIST!

Moreover .... and importantly,

Such certificates are amenable to PRACTICAL COMPUTATION!

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SOS-based certificate

$$\text{Let } \mathbf{K} := \{\, \mathbf{x} : \; g_j(\mathbf{x}) \,\geq\, 0, \quad j = 1, \ldots, m \,\}$$

be compact (with $g_1(\mathbf{x}) = M - \|\mathbf{x}\|^2$, so that $\mathbf{K} \subset \mathbf{B}(0, M)$).

Theorem (Putinar's Positivstellensatz)

If $f \in \mathbb{R}[\mathbf{x}]$ is strictly positive ($f > 0$) on $\mathbf{K}$ then:

$$\dagger \quad f(\mathbf{x}) \,=\, \sigma_0(\mathbf{x}) + \sum_{j=1}^{m} \sigma_j(\mathbf{x})\, g_j(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathbb{R}^n,$$

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Dual side: The K-moment problem

Given a real sequence $\mathbf{y} = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, does there exist a Borel measure μ on $\mathbf{K}$ such that

$$\dagger \quad y_\alpha = \int_{\mathbf{K}} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \, d\mu, \qquad \forall \alpha \in \mathbb{N}^n \quad ?$$

If yes then y is said to have
a representing measure supported on $\mathbf{K}$.

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be compact (with $g_1(\mathbf{x}) = M - \|\mathbf{x}\|^2$, so that $\mathbf{K} \subset \mathbf{B}(0, M)$).

Theorem (Dual side of Putinar's Theorem)

A sequence $\mathbf{y} = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, has a representing measure supported on $\mathbf{K}$ IF AND ONLY IF for every $d = 0, 1, \ldots$

$$(\star) \qquad \mathbf{M}_d(y) \succeq 0 \quad \textit{and} \quad \mathbf{M}_d(g_j\, \mathbf{y}) \succeq 0, \quad j = 1, \ldots, m.$$

☞ The real symmetric matrix $\mathbf{M}_2(y)$ is called the MOMENT MATRIX associated with the sequence y

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Remarkably,

the Necessary & Sufficient conditions $(\star)$ for existence of a representing measure are stated only in terms of countably many LMI CONDITIONS on the sequence y ! (No mention of the unknown representing measure in the conditions.)

For instance with $n = 2$, $d = 1$, the moment matrix $\mathbf{M}_2(y)$ reads

$$\mathbf{M}_2(y) = \left[\begin{array}{c|cc|ccc} \overbrace{y_{00}}^{1} & \overbrace{y_{10}}^{x_1} & \overbrace{y_{01}}^{x_2} & \overbrace{y_{20}}^{x_1^2} & \overbrace{y_{11}}^{x_1 x_2} & \overbrace{y_{02}}^{x_2^2} \\ \hline y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ \hline y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{12} & y_{21} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{array}\right]$$

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FUNCTIONAL ANALYSIS

THE K-MOMENT PROBLEM

$$y = (y_\alpha), \quad \alpha \in \mathbb{N}^n$$

$$y_\alpha \overset{?}{=} \int_K x^\alpha \, d\mu \qquad \forall \alpha$$

for some μ

CHARACTERIZE THOSE y

DUALITY $\langle f, y \rangle = \sum_\alpha f_\alpha y_\alpha$

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But unfortunately less powerful ... and with some drawbacks!

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• In addition, polynomials NONNEGATIVE ON A SET $\mathbf{K} \subset \mathbb{R}^n$ are ubiquitous. They also appear in many important applications (outside optimization),

. . . modeled as

particular instances of the so called
Generalized Moment Problem, among which:
Probability, Optimal and Robust Control, Game theory, Signal processing, multivariate integration, etc.

GMP: The primal view

The GMP is the infinite-dimensional LP:

$$GMP: \quad \inf_{\mu_i \in M(\mathbf{K}_i)} \left\{ \sum_{i=1}^{s} \int_{\mathbf{K}_i} f_i \, d\mu_i : \sum_{i=1}^{s} \int_{\mathbf{K}_i} h_{ij} \, d\mu_i \overset{\geq}{\leqq} b_j, \quad j \in J \right\}$$

with $M(\mathbf{K}_i)$ space of Borel measures on $\mathbf{K}_i \subset \mathbb{R}^{n_i}$, $i = 1, \ldots, s$.

GMP: The dual view

The DUAL GMP^* is the infinite-dimensional LP:

$$GMP^*: \quad \sup_{\lambda_j} \left\{ \sum_{j \in J}^{s} \lambda_j\, b_j : f_i - \sum_{j \in J} \lambda_j\, h_{ij} \geq 0 \text{ on } \mathbf{K}_i, \quad i = 1, \ldots, s \right\}$$

And one can see that ...

the constraints of GMP^* state that the functions

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Several examples will follow .... and

Global OPTIM $\rightarrow f^* = \inf_{\mathbf{x}} \{ f(\mathbf{x}) : \mathbf{x} \in \mathbf{K} \}$

is the SIMPLEST example of the GMP

because ...

$$f^* = \inf_{\mu \in M(\mathbf{K})} \{ \int_{\mathbf{K}} f \, d\mu : \int_{\mathbf{K}} 1 \, d\mu = 1 \}$$

• Indeed if $f(\mathbf{x}) \geq f^*$ for all $\mathbf{x} \in \mathbf{K}$ and μ is a probability measure on $\mathbf{K}$, then $\int_{\mathbf{K}} f \, d\mu \geq \int f^* \, d\mu = f^*$.

• On the other hand, for every $\mathbf{x} \in \mathbf{K}$ the probability measure $\mu := \delta_{\mathbf{x}}$ is such that $\int f \, d\mu = f(\mathbf{x})$.

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The moment-LP and moment-SOS approaches

consist of using a certain type of positivity certificate (Krivine-Vasilescu-Handelman's or Putinar's certificate) in potentially any application where such a characterization is needed. (Global optimization is only one example.)

In many situations this amounts to

solving a HIERARCHY of :

- LINEAR PROGRAMS, or
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... of increasing size!.

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LP- and SDP-hierarchies for optimization

Replace $f^* = \sup_{\lambda} \{ \lambda : \; f(\mathbf{x}) - \lambda \geq 0 \quad \forall \mathbf{x} \in \mathbf{K} \}$ with:

The SDP-hierarchy indexed by $d \in \mathbb{N}$:

$$f_d^* = \sup_{\lambda, \sigma_j} \{ \lambda : \; f - \lambda = \underbrace{\sigma_0}_{SOS} + \sum_{j=1}^{m} \underbrace{\sigma_j}_{SOS} \, g_j; \quad \deg(\sigma_j \, g_j) \leq 2d \}$$

or, the LP-hierarchy indexed by $d \in \mathbb{N}$:

$$\theta_d = \sup_{\lambda, c_{\alpha\beta}} \{ \lambda : f - \lambda = \sum_{\alpha,\beta} \underbrace{c_{\alpha\beta}}_{\geq 0} \prod_{j=1}^{m} g_j^{\alpha_j} (1 - g_j)^{\beta_j}; \quad |\alpha + \beta| \leq 2d \}$$

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$$\theta_d = \sup_{\lambda, c_{\alpha\beta}} \{ \lambda : f - \lambda = \sum_{\alpha,\beta} \underbrace{c_{\alpha\beta}}_{\geq 0} \prod_{j=1}^{m} g_j^{\alpha_j} (1 - g_j)^{\beta_j} ; \quad |\alpha + \beta| \leq 2d \}$$

Theorem

Both sequence (f_d^*), and (θ_d), $d \in \mathbb{N}$, are MONOTONE NON DECREASING and when $\mathbf{K}$ is compact (and satisfies a technical Archimedean assumption) then:

$$f^* = \lim_{d\to\infty} f_d^* = \lim_{d\to\infty} \theta_d.$$

Moreover, and importantly,

- GENERICALLY, ... the Moment-SOS hierarchy has finite convergence, that is, $f^* = f_d^*$ for some d.
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• What makes this approach exciting is that it is at the crossroads of several disciplines/applications:

- Commutative, Non-commutative, and Non-linear ALGEBRA
- Real algebraic geometry, and Functional Analysis
- Optimization, Convex Analysis
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- Has already been proved useful and successful in applications with modest problem size, notably in optimization, control, robust control, optimal control, estimation, computer vision, etc. (If sparsity then problems of larger size can be addressed)

- HAS initiated and stimulated new research issues:
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The moment-SOS approach can be applied to problems defined with semi-algebraic functions via the introduction of additional variables (LIFTING)

Examples

$$\begin{aligned} \mathbf{x} \in \mathbf{K};\ |f(\mathbf{x})| \quad &\Leftrightarrow \quad \mathbf{x} \in \mathbf{K};\ f(\mathbf{x})^2 - z^2 = 0; \quad z \geq 0. \\ f(\mathbf{x}) \geq 0 \text{ on } \mathbf{K};\ \sqrt{f(\mathbf{x})} \quad &\Leftrightarrow \quad \mathbf{x} \in \mathbf{K};\ f(\mathbf{x}) - z^2 = 0; \quad z \geq 0. \end{aligned}$$

Similarly to model the function $\mathbf{x} \mapsto g(\mathbf{x}) := \max[f_1(\mathbf{x}), f_2(\mathbf{x})]$,

$$\underbrace{(f_1(\mathbf{x}) - f_2(\mathbf{x}))^2 - z^2 = 0;\ z \geq 0}_{z=|f_1(\mathbf{x})-f_2(\mathbf{x})|} \quad \Leftrightarrow g(\mathbf{x}) = \frac{z}{2} + \frac{f_1(\mathbf{x}) + f_2(\mathbf{x})}{2}$$

etc.

Recall that both LP- and SDP- hierarchies are

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NOT TAILORED to solving specific hard problems!!

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A remarkable property of the SOS hierarchy: I

When solving the optimization problem

$$\mathbf{P}: \quad f^* = \min \{ f(\mathbf{x}) : \; g_j(\mathbf{x}) \geq 0, \; j = 1, \ldots, m \}$$

one does NOT distinguish between CONVEX, CONTINUOUS NON CONVEX, and 0/1 (and DISCRETE) problems! A boolean variable x_i is modelled via the equality constraint "$x_i^2 - x_i = 0$".

In Non Linear Programming (NLP),

modeling a 0/1 variable with the polynomial equality constraint

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Even though the moment-SOS approach DOES NOT SPECIALIZE to each class of problems:

- It recognizes the class of (easy) SOS-convex problems as FINITE CONVERGENCE occurs at the FIRST relaxation in the hierarchy.
- FINITE CONVERGENCE also occurs for general convex problems and GENERICALLY for non convex problems
- $\rightarrow$ (NOT true for the LP-hierarchy.)
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A remarkable property: II

FINITE CONVERGENCE of the SOS-hierarchy is GENERIC!

... and provides a GLOBAL OPTIMALITY CERTIFICATE,

the analogue for the NON CONVEX CASE of the

KKT-OPTIMALITY conditions in the CONVEX CASE!

The no-free lunch rule ...

The size of SDP-relaxations grows rapidly with the original problem size ... In particular:

- $O(n^{2d})$ variables for the d^{th} SDP-relaxation in the hierarchy
- $O(n^d)$ matrix size for the LMIs

$\rightarrow$ In view of the present status of SDP-solvers ... only small to medium size problems can be solved by "standard" SDP-relaxations ...

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• develop more efficient general purpose SDP-solvers ... (limited impact) ... or perhaps dedicated solvers ....?

• exploit symmetries when present ... Recent promising works by De Klerk, Gaterman, Gvozdenovic, Laurent, Pasechnick, Parrilo, Schrijver .. in particular for combinatorial optimization problems. Algebraic techniques permit to define an equivalent SDP of much smaller size.

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• exploit sparsity in the data. In general, each constraint involves a small number of variables κ, and the cost criterion is a sum of polynomials involving also a small number of variables. Recent works by Kim, Kojima, Lasserre, Maramatsu and Waki

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There has been also recent attempts to use other types of algebraic certificates of positivity that try to avoid the size explosion due to the semidefinite matrices associated with the SOS weights in Putinar's positivity certificate

Recent work by :

- Ahmadi et al. ☞ Hierarchy of LP or SOCP programs.
- Lasserre, Toh and Zhang ☞ Hierarchy of SDP with semidefinite constraint of fixed size

EXAMPLES

I. Optimal Control

Consider the OPTIMAL CONTROL (OCP) problem:

$$\rho = \inf_{u} \quad \int_0^T h(\mathbf{x}(t), u(t))\, dt$$

$$\begin{array}{rrcl} \text{s.t.} & \dot{\mathbf{x}}(t) & = & f(\mathbf{x}(t), u(t)), \quad t \in [0, T] \\ & \mathbf{x}(0) & = & x_0 \\ & \multicolumn{3}{l}{\mathbf{x}(t) \in X \subset \mathbb{R}^n;\ u(t) \in U \subset \mathbb{R}^m,} \end{array}$$

that is, the goal is now to compute a function $u : [0, T] \to \mathbb{R}^m$ (in a suitable space).

In general OCP problems are hard to solve, and particularly when STATE CONSTRAINTS $\mathbf{x}(t) \in X$ are present !

By introducing the concept of OCCUPATION MEASURE, there exists a so-called WEAK FORMULATION of the OCP which is an infinite-dimensional LINEAR PROGRAM (LP) on a suitable space of measures, and in fact an instance of the Generalized Problem of Moments.

☞ Under some conditions the optimal values of OCP and LP are the same.

☞ When the vector field f is a polynomial and the sets X and U are compact basic semi-algebraic then the MOMENT-SOS approach can be applied to approximate ρ as closely as desired.

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☞ It yields a HIERARCHY OF SEMIDEFINITE PROGRAMS of increasing size whose associated monotone sequence of optimal values CONVERGES to the optimal value ρ of the OCP.

☞ Lass. J.B., Henrion D., Prieur C., Trelat E. (2008), Nonlinear optimal control via occupation measures and LMI-relaxations, SIAM J. Contr. Optim. 47, pp. 1649–1666.

Extensions & Related works

☞ Compute polynomial Lyapunov Functions

☞ Approximate Regions Of Attraction (ROA) by sets of the form $\{\mathbf{x} : g(\mathbf{x}) \geq 0\}$ for some polynomial g.

☞ Convex Optimization of Non-Linear Feedback Controllers

By several authors ... Ahmadi, Henrion, Korda, Lass., Majumdar, Parrilo, Tedrake, Tobenkin, etc.

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☞ for Estimation problems (seen as Min-max optimization)

☞ for Robust Stability analysis and probabilistic $\mathcal{D}$-Stability Analysis

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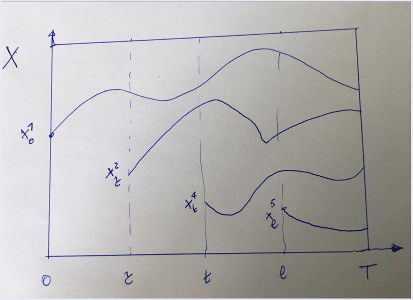
II. Inverse Optimal Control

Given:

- ☞ a dynamical system $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), u(t)),\ t \in [0, T]$
- ☞ State and/or Control constraints $\mathbf{x}(t) \in X$, $u(t) \in U$,
- ☞ a database of recorded feasible trajectories $\{x(t; x_\tau), u(t; x_\tau)\}$ for several initial states $x_\tau \in X$,



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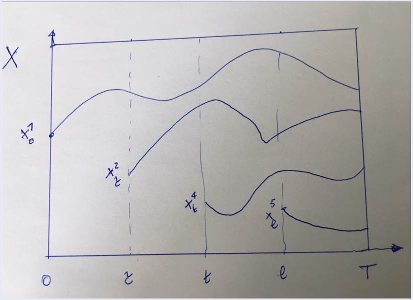

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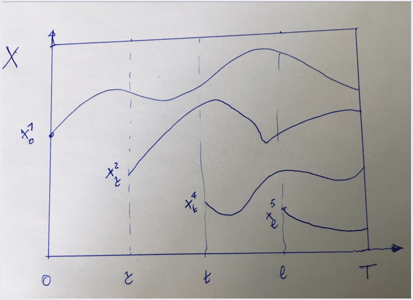

compute a Lagrangian

$h : X \times U \to \mathbb{R}$ for which those trajectories are optimal.

☞ Key idea: I: Hamilton-Jacobi-Bellman (HJB) is the perfect tool to certify GLOBAL OPTIMALITY of the given trajectories in the database.

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Indeed suppose that two functions $\phi : [0, T] \times X \to \mathbb{R}$ and $h : X \times U \to \mathbb{R}$ satisfy:

$$(*) \quad \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} f(x, u) + h(x, u) \geq 0, \quad \forall (x, u, t) \in X \times U \times [0, T]$$

$$(**) \quad \phi(T, x) \;\leq\; 0 \quad \forall x \in X_T.$$

and †

$$\left(\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} f + h \right) (x(t; x_\tau), u(t; x_\tau), \tau) \leq 0; \quad \phi(T, x(T; x_\tau)) \geq 0,$$

for all $(x(t; x_\tau), u(t; x_\tau), \tau)$ in the database

Then

$$\phi(t,z) = \inf_{u} \int_t^T h(\mathbf{x}(s), u(s))\, ds$$

$$\text{s.t.} \quad \begin{aligned} \dot{\mathbf{x}}(s) &= f(\mathbf{x}(s), u(s)), \quad s \in [t, T] \\ \mathbf{x}(s) &\in X \subset \mathbb{R}^n;\ u(s) \in U \subset \mathbb{R}^m \\ \mathbf{x}(t) &= z \end{aligned}$$

☞ and all the trajectories $\{x(t; x_\tau), u(t; x_\tau)\}$ of the database are optimal solutions.

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☞ Key idea II: Look for POLYNOMIALS

$\phi \in \mathbb{R}[x, t]$ and $h \in \mathbb{R}[x, u]$

- that satisfy the relaxed HJB conditions (*) and (**)
- and also satisfy

$$(\dagger) \quad \left(\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} f + h\right)(x(t; x_\tau), u(t; x_\tau), \tau) \leq \epsilon$$

$$(\dagger\dagger) \quad \phi(T, x(T; x_\tau)) \geq -\epsilon,$$

for all $(x(t; x_\tau), u(t; x_\tau), \tau)$ in the database

☞ ... and SOLVE:

$$\rho_d = \min_{\phi,h} \{ \epsilon + \|h\|_1 : \text{s.t. } (*), (**), (\dagger), (\dagger\dagger);\ \deg(\phi), \deg(h) \leq 2d \}$$

where one replaces the nonnegativity conditions (*), (**), (†) and (††) by appropriate positivity certificates.

☞ a HIERARCHY of SEMIDEFINITE PROGRAMS (whose size increases with the degree d).

Pauwels E., Henrion D., Lasserre J.B. (2016) Linear Conic Optimization for Inverse Optimal Control, SIAM J. Control & Optim. 54, pp. 1798–1825.

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III. Approximation of sets with quantifiers

Let $f \in \mathbb{R}[x, y]$ and let $\mathbf{K} \subset \mathbb{R}^n \times \mathbb{R}^p$ be the semi-algebraic set:

$$\mathbf{K} := \{(x, y) : \ x \in \mathbf{B}; \quad g_j(x, y) \geq 0, \quad j = 1, \ldots, m\},$$

where $\mathbf{B} \subset \mathbb{R}^n$ is a box $[-a, a]^n$.

Approximate the set:

$$R_f := \{x \in \mathbf{B} : \ f(x, y) \leq 0 \text{ for all } y \text{ such that } (x, y) \in \mathbf{K}\}$$

as closely as desired by a sequence of sets of the form:

$$\Theta_k := \{\mathbf{x} \in \mathbf{B} : \ J_k(x) \leq 0\}$$

for some polynomials J_k.

☞ Use Putinar Positivity Certificate to build up a hierarchy of semidefinite programs $(\mathbf{Q}_k)_{k\in N}$ of increasing size:

- An optimal solution of $\mathbf{Q}_k$ provides the coefficients of the polynomial J_k of degree $2k$.
- For every k:

$$\Theta_k \;:=\; \{\mathbf{x}\in\mathbf{B}:\; J_k(x)\;\leq\;0\,\}\subset R_f \qquad \text{(inner approximations)}$$

- $\mathrm{vol}(R_f\setminus\Theta_k)\rightarrow 0$ as $k\rightarrow\infty$.

Lass. J.B. (2015) Tractable approximations of sets defined with quantifiers, Math. Program. 151, pp. 507–527.
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- For every k:

$$\Theta_k := \{\mathbf{x} \in \mathbf{B} : J_k(x) \leq 0\} \subset R_f \qquad \text{(inner approximations)}$$

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IV. Convex Underestimators of Polynomials

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• Typically, f is a sum $\sum_k f_k$ where each f_k “sees" only very few variables (say $3, 4$). The same observation is true for each g_j in the constraints:

Hence a very appealing idea is to pre-compute CONVEX UNDER-ESTIMATORS $\widehat{f_k} \leq f_k$ and $\widehat{g_j} \leq g_j$ for each non convex f_k and each non convex g_j, independently and separately!

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Compute a "tight" convex polynomial underestimator $p \leq f$ of a non convex polynomial f on a box $\mathbf{B} \subset \mathbb{R}^n$.

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1. $p(\mathbf{x}) \leq f(\mathbf{x})$ for every $\mathbf{x} \in \mathbf{B}$.
2. p convex on $\mathbf{B} \to \nabla^2 p(\mathbf{x}) \succeq 0$ for all $\mathbf{x} \in \mathbf{B}$,

$$\iff \quad \mathbf{u}^T \nabla^2 p(\mathbf{x})\, \mathbf{u} \geq 0, \; \forall (\mathbf{x}, \mathbf{u}) \in \mathbf{B} \times \mathbf{U},$$

where $\mathbf{U} := \{\, \mathbf{u} : \|\mathbf{u}\|^2 \leq 1\}$.

Hence we have the two "Positivity constraints"

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One possibility is to evaluate the L_1-norm $\displaystyle\int_{\mathbf{B}} |f(\mathbf{x}) - p(\mathbf{x})|\, d\mathbf{x}$

$$\rightarrow \int_{\mathbf{B}} (f(\mathbf{x}) - p(\mathbf{x})\, d\mathbf{x} = \underbrace{\int_{\mathbf{B}} f(\mathbf{x}) d\mathbf{x}}_{constant} - \underbrace{\int_{\mathbf{B}} p(\mathbf{x})\, d\mathbf{x}}_{\text{linear in } p!}$$

Indeed, writing $p(\mathbf{x}) = \sum_{\alpha\in\mathbb{N}^n} p_\alpha\, \mathbf{x}^\alpha$,

$$\int_{\mathbf{B}} p(\mathbf{x})\, d\mathbf{x} = \sum_{\alpha\in\mathbb{N}^n} p_\alpha \underbrace{\int_{\mathbf{B}} \mathbf{x}^\alpha\, d\mathbf{x}}_{\gamma_\alpha},$$

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Hence computing the best degree-d convex polynomial under-estimator of f reduces to solve the CONVEX optimization problem:

$$\mathbf{P}: \quad \rho = \inf_{p \in \mathbb{R}[\mathbf{x}]_d} \sum_{\alpha \in \mathbb{N}^n_d} p_\alpha \, \gamma_\alpha$$

$$\text{s.t.} \quad f(\mathbf{x}) - p(\mathbf{x}) \geq 0, \ \forall \, \mathbf{x} \in \mathbf{B}$$

$$\mathbf{u}^T \nabla^2 p(\mathbf{x}) \, \mathbf{u} \geq 0, \ \forall (\mathbf{x}, \mathbf{u}) \in \mathbf{B} \times \mathbf{U}.$$

☞ which has an optimal solution $p^* \in \mathbb{R}[\mathbf{x}]_d$

Replacing the positivity constraints with Putinar's positivity certificate

☞ yields a HIERARCHY of SEMIDEFINITE PROGRAMS, each with an optimal solution $p^*_\ell \in \mathbb{R}[\mathbf{x}]_d$, and:

Theorem (Lass & T. Phan Thanh (JOGO 2013))

$p^*_\ell \to p^* \in \mathbb{R}[\mathbf{x}]_d$, *as* $\ell \to \infty$

$\to$ Provides the best results in the comparison:

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V. Super-Resolution

Suppose that an unknown SIGNED measure ϕ^* (signal) is supported on finitely many (few) atoms $(\mathbf{x}(k))_{k=1}^p \subset \mathbf{K}$, i.e.,

$$\phi^* = \sum_{k=1}^{p} \gamma_k \, \delta_{\mathbf{x}(k)}, \quad \text{for some real numbers } (\gamma_k).$$

The goal is to find

the SUPPORT $(\mathbf{x}(k))_{k=1}^p \subset \mathbf{K}$ and WEIGHTS $(\gamma_k)_{k=1}^p$ from only FINITELY MANY MEASUREMENTS (moments)

$$q_\alpha = \int_{\mathbf{K}} \mathbf{x}^\alpha \, d\phi^*(\mathbf{x}), \quad \alpha \in \Gamma.$$

Solve the infinite-dimensional LP

$$\mathbf{P}: \quad \inf_{\phi} \{ \|\phi\|_{TV} : \int_{\mathbf{K}} \mathbf{x}^{\alpha} \, d\phi(\mathbf{x}) = q_{\alpha}, \quad \alpha \in \Gamma.$$

Univariate case on a bounded interval $I \subset \mathbb{R}$ (or equivalently on the torus $\mathbb{T} \subset \mathbb{C}$):

If the distance between any two atoms is sufficiently large and sufficiently many (few) moments are available then :

- ϕ^* is the unique solution of **P**, and
- exact recovery is obtained by solving a single SDP.

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Writing the signed measure ϕ on I as $\phi^+ - \phi^-$,

P reads

$$\inf_{\phi^+,\phi^-} \int_I d(\phi^+ + \phi^-) : \int_I \mathbf{x}^k\, d\phi^+(\mathbf{x}) - \int_I \mathbf{x}^k\, d\phi^+(\mathbf{x}) = q_\alpha, \quad \alpha \in \Gamma \}$$

... again an instance of the GMP!

The dual $\mathbf{P}^*$ reads: $\sup_{p\in\mathbb{R}[\mathbf{x}]} \{ \langle p, q\rangle : \sup_{\mathbf{x}\in I} |p(\mathbf{x})| \leq 1 \}$.

Extension to compact semi-algebraic domains $\mathbf{K} \subset \mathbb{R}^n$ via the moment-SOS approach: FINITE RECOVERY is also possible.

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VI. LP on spaces of measures: a rich framework

Consider the infinite dimensional LP:

$$\min_{\phi} \left\{ \int_{\mathbf{K}} f \, d\phi : \quad \phi \leq \mu; \quad \int_{\mathbf{K}} g \, d\phi = b, \, \forall g \in G \right\}$$

where :

- $\mathbf{K} \subset \mathbb{R}^n$ is a basic semi-algebraic set,
- The unknown ϕ is a Borel measure supported on $\mathbf{K}$
- The functions f, and $g \in G$ are polynomials
- All moments of the measure μ are available.

For instance this framework can be used :

- To compute Sharp Upper Bounds on $\mu(\mathbf{K})$ GIVEN some moments of μ.
- To approximate as closely as desired, from below and above, the Lebesgue volume of $\mathbf{K}$, or the Gaussian measure of $\mathbf{K}$ (for possibly non-compact $\mathbf{K}$)
- CHANCE-CONSTRAINTS: Given $\epsilon > 0$ and a prob. distribution μ, approximate AS CLOSELY AS DESIRED

$$\Omega_\epsilon \;:=\; \{\,\mathbf{x}:\; \mathrm{Prob}_\omega(f(\mathbf{x},\omega)\leq 0)\;\geq\; 1-\epsilon\,\}$$

by sets of form : $\Omega^d_\epsilon := \{\,\mathbf{x}:\; h_d(\mathbf{x})\leq 0\,\}$ for some polynomial h_d of degree d.

and more ! ☞ Henrion et al. (SIREV 2009), Lass. (Adv. Appl. Math. (2017)), Lass. (Adv. Comput. Math. (2016)), Lass. (2017) (IEEE Control Systems Letters), ...

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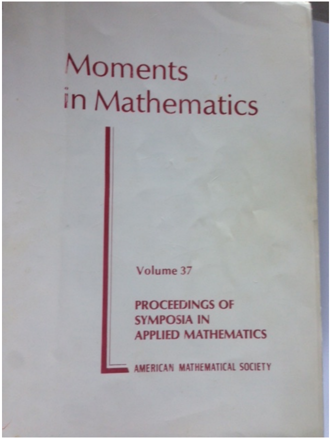
In fact .... the list of potential applications of the GMP is almost ENDLESS!

☞



THANK YOU!