The Square Root Velocity Framework for Curves in a Homogeneous Space

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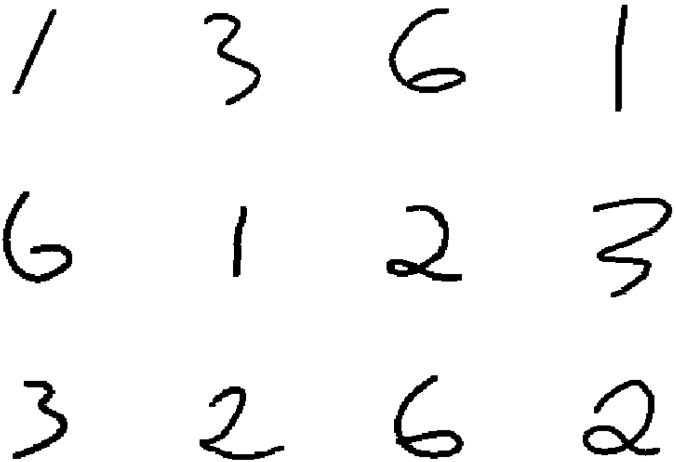
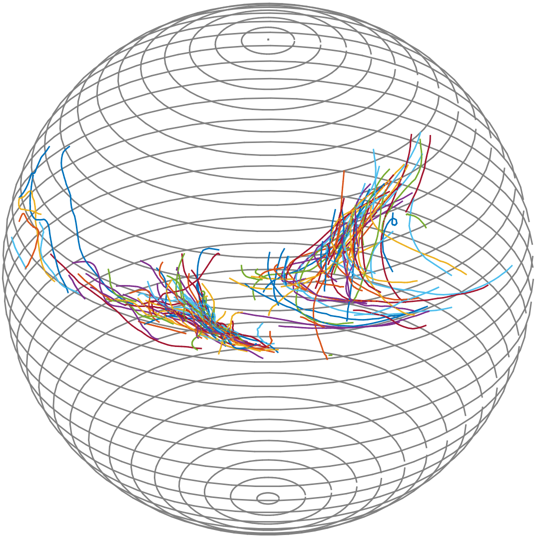
Curves in a Flat Plane



Curves in a Sphere (Hurricane Tracks)



What do we want to do?

Aims:

- Put a metric on the space of parametrized curves in a (flat or curved) manifold. For certain applications, this metric should be invariant under reparametrization, and under rigid motions.
- Define a statistical framework for shape analysis of these curves – e.g., calculate the *mean* of a set of curves, and perform PCA (principal component analysis).

Difficulty:

- The space of unparametrized curves is
  - non-linear
  - infinite dimensional

How?

Our approach:

- Equip the space of parametrized curves with a Riemannan metric that is invariant under reparametrizations and rigid motions.
- This induces a metric on the quotient under the action of the group of reparametrizations and/or the group of rigid motions.
- The lengths of geodesics can be used to measure how similar two curves are.

Previous Work - Curves in $\mathbb{R}^n$ and in Manifolds

- W. Mio et al.[1] introduced a 1-parameter family of first order Sobolev metrics on smooth planar curves in 2007, which they called *elastic metrics*.
- L. Younes et al.[2] analyzed one member of the above family, and gave an elegant way of computing geodesics using this metric in 2008. Their technique applies especially well to closed curves.
    - Their paper applies only to smooth curves whose derivatives never vanish.
    - For many pairs of curves, geodesics between them do not exist.

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[1]Mio, Srivastava, Joshi. On shape of plane elastic curves, *International Journal of Computer Vision*, 2007

[2]Younes, Michor, Shah, Mumford. A metric on shape space with explicit geodesics, *Rendiconti Lincei Matematica e Applicazioni*, 2008

Previous Work - Curves in $\mathbb{R}^n$ and in Manifolds

- A. Srivastava et al.[1] introduced the SRVF (square root velocity framework) in 2011 for absolutely continuous curves in $\mathbb{R}^n$.

  Method: They define a bijection

$$Q : AC(I, \mathbb{R}^n) \to L^2(I, \mathbb{R}^n)$$

$$Q(\alpha)(t) = \begin{cases} \dfrac{\alpha'(t)}{\sqrt{|\alpha'(t)|}} & \alpha'(t) \neq 0 \\ 0 & \alpha'(t) = 0 \end{cases}$$

  where $AC(I, \mathbb{R}^n)$ is the set of all absolutely continuous curves in $\mathbb{R}^n$. This bijection induces a smooth structure and a Riemannian metric on $AC(I, \mathbb{R}^n)$. Restricted to the smooth curves, this metric is identical to one of the elastic metrics defined by Mio et al.

[1] Srivastava, Klassen, Joshi, Jermyn. Shape analysis of elastic curves in Euclidean spaces, *IEEE PAMI*, 2011

Previous Work - Curves in $\mathbb{R}^n$ and in Manifolds

- Characteristics of the SRVF Method:
  - Requires less regularity of curves.
  - $AC(I, \mathbb{R}^n)$ becomes a complete Riemannian Hilbert manifold.
  - Modding out by reparametrizations produces a complete metric space, but not a manifold.
  - Geodesics between open parametrized curves always exist and are easily computed, since geodesics in $L^2(I, \mathbb{R}^n)$ are straight lines.

Previous Work - Curves in $\mathbb{R}^n$ and in Manifolds

- J. Su et al.[1] introduced the TSRVF (transported square root velocity function) for Riemannian manifolds in 2014.
  - The method is computationally efficient;
  - It requires the choice of a refence point in the manifold;
  - The method introduces distortions for curves that are far away from the reference point;
  - The metric depends on the chosen reference point. (Hence, the metric is not invariant under rigid motions of the manifold.)

[1] Su, Kurtek, Klassen, Srivastava. Statistical analysis of trajectories on Riemannian manifolds: bird migration, hurricane tracking and video surveillance, *Annals of Applied Statistics*, 2014.

Previous Work - Curves in $\mathbb{R}^n$ and in Manifolds

- Z. Zhang et al.[1] introduced a different adaptation of the SRVF to manifold valued curves in 2015.
- A. Le Brigant et al.[2] introduced a more intrinsic metric on manifold-valued curves using SRVF in 2015.
  - These methods avoid the arbitrariness and distortion resulting from the choice of a reference point;
  - These methods have great computational costs.

[1] Zhang, Su, Klassen, Le, Srivastava. Video-based action recognition using rate invariant analysis of covariance, *arXiv*, 2015

[2] Le Brigant, Arnaudon, Barbaresco. Reparameterization Invariant Metric on the Space of Curves, *International Conference on Networked Geometric Science of Information*, 2015

Previous Work - Curves in $\mathbb{R}^n$ and in Manifolds

- E. Celledoni et al. applied the SRVF to Lie groups[1] and Homogeneous Spaces[2] in 2015-17.
    - Their method avoids the arbitrariness and distortion resulting from the choice of a reference point;
    - Their method is computationally efficient;
    - For homogeneous spaces, their method has only been implemented for sets of paths that all start at the same point.

[1] Celledoni, Eslitzbichler, Schmeding. Shape analysis on Lie groups with applications in computer animation, *J. Geom. Mech.*, 2015

[2] Celledoni, Eidnes, Schmeding, Shape Analysis on Homogeneous Spaces, *arXiv*, 2017.

Homogeneous Spaces

Our work focuses on curves in **homogeneous spaces** and the curves are allowed to start at arbitrary points. It is similar to Celledoni et al., but uses a twisting construction to capture the topological non-triviality of the tangent bundle of the homogeneous space.

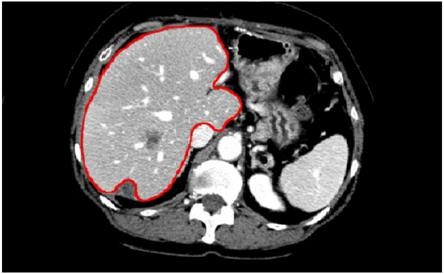
Why are Homogeneous Spaces Important

Many Riemannian manifolds appearing in applications are **homogeneous spaces**. For example:

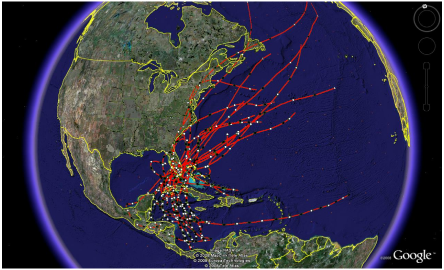
- Euclidean spaces $\mathbb{R}^n$
- spheres S^n
- hyperbolic spaces $\mathbb{H}^n$
- Grassmannian manifolds
- the space of $n \times n$ positive definite symmetric matrices with determinant 1 (PDSM)

Why are Homogeneous Spaces Important

In applications: Many manifolds involved are homogeneous.



Curves in $\mathbb{R}^2$



Curves on S^2

Definition of Homogeneous Space

In this presentation, we define a *homogeneous space* to be a quotient

$$M = G/K,$$

where G is a finite dimensional Lie group and K is a compact Lie subgroup. G acts transitively on M from the left by $g * hK = ghK$.

Let

$$\mathfrak{g} = \text{ the Lie algebra of } G$$

$$\mathfrak{k} = \text{ the Lie algebra of } K$$

and let $\pi : G \to M$ denote the quotient map.

In this situation, we can always endow G with a Riemannian metric that is left-invariant under multiplication by G and bi-invariant under multiplication by K. This metric induces a metric on the quotient $M = G/K$ that is invariant under the left action of G.

Absolutely continuous curves in M

Denote by $AC(I, M)$ the set of absolutely continuous curves $\beta : I \to M$, where $I = [0, 1]$.

Our goal is to put a smooth structure and a Riemannian metric on $AC(I, M)$ that is invariant under the left action of G and also under the right action of the reparametrization group $\Gamma = Diff_+(I, I)$.

Our method is to contruct a bijection between $AC(I, M)$ and a space that can easily be endowed with these structures. We now state this bijection.

Main Bijection

Let $\mathfrak{k}^\perp$ denote the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$.

Theorem

There is a bijection

$$\Phi : (G \times L^2(I, \mathfrak{k}^\perp))/K \to AC(I, M),$$

which we will define in the following slides. We will also define a Hilbert manifold structure and a Riemannian metric on $(G \times L^2(I, \mathfrak{k}^\perp))/K$, which yield the desired structures on $AC(I, M)$. These structures are preserved by the action of the reparametrization group Γ and by the left action of G.

Building the Bijection: First Step

Following the SRVF, we define a bijection

$$Q : AC(I, G) \to G \times L^2(I, \mathfrak{g})$$
$$Q(\alpha) = (\alpha(0), q),$$

where

$$q(t) = \begin{cases} L_{\alpha(t)^{-1}} \dfrac{\alpha'(t)}{\sqrt{||\alpha'(t)||}} & \alpha'(t) \neq 0 \\ 0 & \alpha'(t) = 0 \end{cases}$$

where $L_{\alpha(t)^{-1}}$ denotes left translation by $\alpha(t)^{-1}$ in the Lie group G.

Building the Bijection: Second Step

Let $AC^{\perp}(I, G)$ denote the set of absolutely continuous curves in G that are perpendicular to each coset gK that they meet. Then it is immediate that Q induces a bijection

$$AC^{\perp}(I, G) \to G \times L^2(I, \mathfrak{k}^{\perp}).$$

Building the Bijection: Third Step

Theorem

Given $\beta \in AC(I, M)$ *and* $\alpha_0 \in \pi^{-1}(\beta(0))$*, there is a unique horizontal lift* $\alpha \in AC^{\perp}(I, G)$ *satisfying* $\beta = \pi \circ \alpha$ *and* $\alpha(0) = \alpha_0$*.*

Because the right action of K on G preserves $AC^{\perp}(I, G)$ and acts freely and transitively on $\pi^{-1}(\beta(0))$, we obtain the following:

Theorem

π *induces a bijection*

$$AC^{\perp}(I, G)/K \to AC(I, M).$$

Main Bijection proved!

The bijections on the last two slides imply the main bijection:

$$\Phi : (G \times L^2(I, \mathfrak{k}^\perp))/K \to AC(I, M).$$

The formula for the K-action appearing on the left side of this bijection is

$$(g, q) * y = (gy, y^{-1} q y),$$

where $g \in G$, $q \in L^2(I, \mathfrak{k}^\perp)$, and $y \in K$.

Smooth structure and Riemannian metric

$L^2(I, \mathfrak{k}^\perp)$ is a Hilbert space; hence it is a Hilbert manifold with the obvious Riemannian metric.

G has already been given the structure of a finite dimensional Riemannian manifold.

Hence, $G \times L^2(I, \mathfrak{k}^\perp)$ is a Riemannian Hilbert manifold.

Since K is a compact Lie group acting freely by isometries, it follows that the quotient map induces a Riemannian metric on

$$(G \times L^2(I, \mathfrak{k}^\perp))/K,$$

making it into a Riemannian Hilbert manifold.

Geodesics and distance

A geodesic in $G \times L^2(I, \mathfrak{k}^\perp)$ is the product of a geodesic in G with a straight line in $L^2(I, \mathfrak{k}^\perp)$, making geodesics in this product space easy to compute, in general. The length of such a geodesic is computed from the lengths of its two factors by the Pythagorean Theorem.

A geodesic in $(G \times L^2(I, \mathfrak{k}^\perp))/K$ is the image of a geodesic in $G \times L^2(I, \mathfrak{k}^\perp)$ that is perpendicular to the K-orbits that it meets.

Method for obtaining geodesics in $AC(I, M)$

Given elements β_0 and β_1 in $AC(I, M)$, we identify them via Φ with the corresponding K-orbits in $G \times L^2(I, \mathfrak{k}^\perp)$; denote these orbits by $[\alpha_0, q_0]$ and $[\alpha_1, q_1]$.

1. Determine $y \in K$ that minimizes
$$d((\alpha_0, q_0), (\alpha_1, q_1) * y).$$
Because K is a compact Lie group, this optimization is generally straightforward using a gradient search.
2. Calculate the geodesic in $G \times L^2(I, \mathfrak{k}^\perp)$ from (α_0, q_0) to $(\alpha_1, q_1) * y$.
3. Under the above correspondences, this geodesic will give a shortest geodesic between β_0 and β_1 in $AC(I, M)$.

Unparametrized Curves, or "Shapes," in M

Define the *shape space*, or the space of unparametrized curves, by

$$\mathcal{S}(I, M) = AC(I, M)/ \sim$$

where

$$\beta_1 \sim \beta_2 \Leftrightarrow \mathrm{Cl}(\beta_1 \Gamma) = \mathrm{Cl}(\beta_2 \Gamma)$$

Recall that $\Gamma = \mathit{Diff}_+(I)$; closure is with respect to the metric we have put on $AC(I, M)$. The distance function on $\mathcal{S}(I, M)$ is given by

$$d([\beta_1], [\beta_2]) = \inf_{\gamma_1, \gamma_2 \in \bar{\Gamma}} d(\beta_1 \circ \gamma_1, \beta_2 \circ \gamma_2)$$

where $\bar{\Gamma} =$

$$\{\gamma \in AC(I, I) : \gamma'(t) \geq 0, \gamma(0) = 0, \gamma(1) = 1\}.$$

Existence of Optimal Reparametrizations

For $M = \mathbb{R}^n$, the existence of optimal reparametrizations $\gamma_1, \gamma_2 \in \bar{\Gamma}$ has been proved in the following two cases:

- If at least one of the curves β_1 or β_2 is PL: proved by Lahiri, Robinson and Klassen in 2015.
- If both β_1 and β_2 are C^1: proved by M. Bruveris in 2016.

Both of these theorems have easy generalizations to the case of an arbitrary homogeneous space M.

Example: the Sphere S^n

S^n is a homogeneous space since

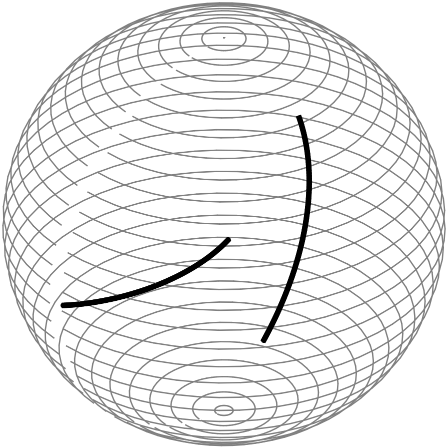
$$S^n \cong SO(n+1)/SO(n)$$

We use the usual metric on $SO(n+1)$:

$$\langle u, v \rangle_g = \mathrm{tr}(u^t v),$$

which has the required invariance properties.

Geodesic Between Two Unparametrized Curves on S^2



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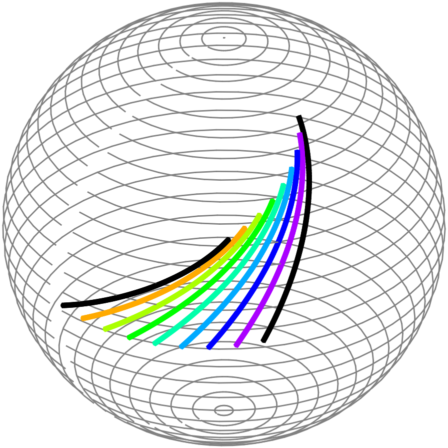
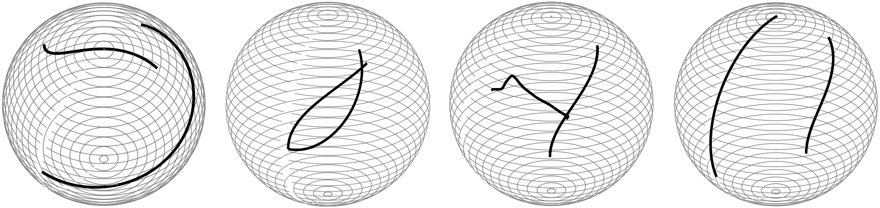
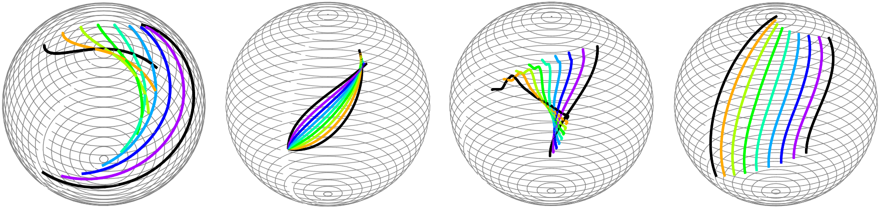


More Examples on S^2



More Examples on S^2



Applications to Hurricane Tracks

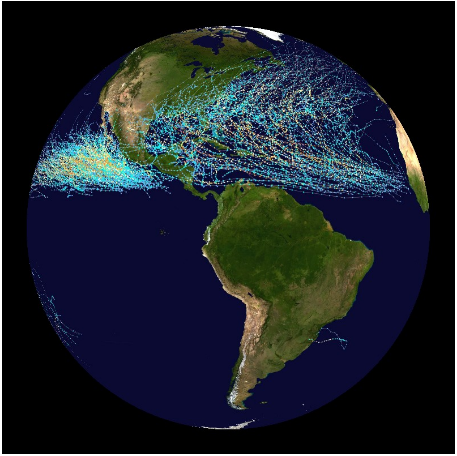
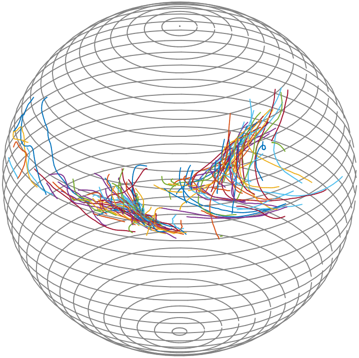


Figure: Hurricane tracks cumulative from 1950 to 2005 obtained from the National Hurricane Center website.

Multi-dimensional Scaling



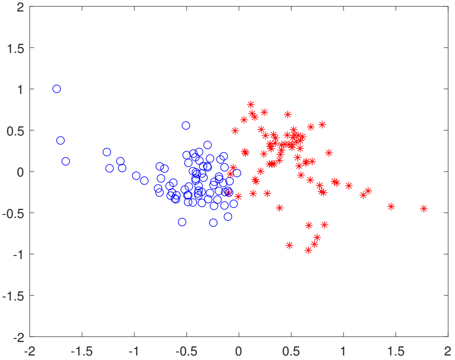


Figure: Left: 75 hurricane tracks from the Atlantic hurricane database and 75 hurricane tracks from the Northeast and North Central Pacific hurricane database. Right: multi-dimensional scaling in two dimensions. Atlantic $\star$; Pacific $\circ$.

Karcher Means

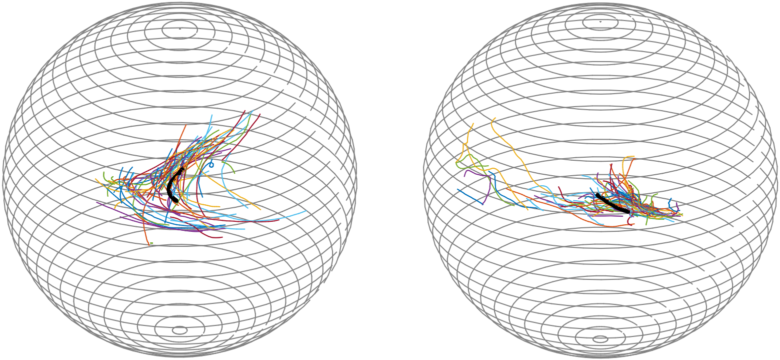
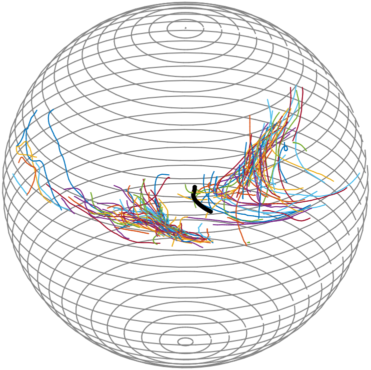
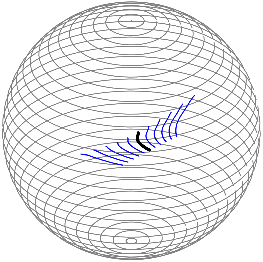
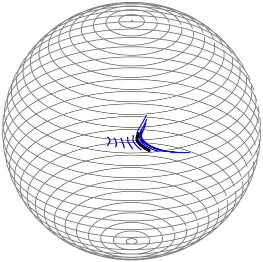


Figure: The Karcher means (up to reparametrization) of 75 hurricane paths in the Atlantic (left) and Pacific (right).

The First Two Principal Directions

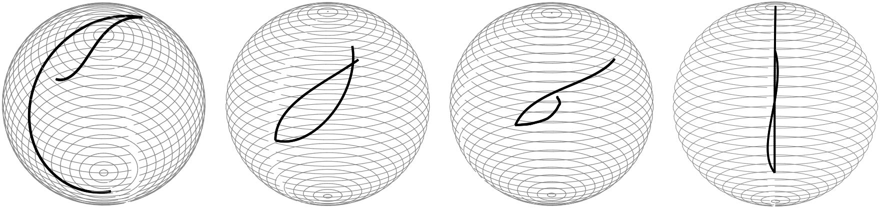






Geodesics in the Space of Unparametrized Curves Modulo Rigid Motions

$SO(3)\,(= G)$ acts on $S^2\,(= M)$ as its group of rigid motions. We can compute geodesics in the corresponding quotient space:



Geodesics in the Space of Unparametrized Curves Modulo Rigid Motions

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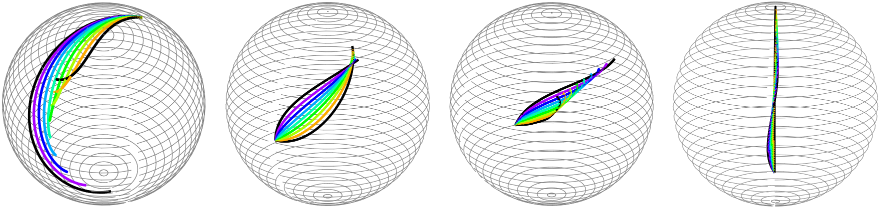

Summary:

- We have adapted the SRVF to compare curves in a homogeneous space.
- Our method is computationally efficient.
- Our method overcomes some of the drawbacks of previous methods.

End

Thank You