The Optimal Design of Wall-Bounded Heat Transport

Ian Tobasco (Michigan)

Joint work with Charlie Doering (Michigan)

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I.T. & C. Doering, *Optimal wall-to-wall transport by incompressible flows*, PRL '17
I.T. & C. Doering, *On the optimal design of wall-to-wall heat transport*, submitted
Andre Souza, I.T., & C. Doering, *Optimal 2D wall-to-wall transport — numerics*, in prep

Heat transport in a fluid with velocity $\mathbf{u}(\mathsf{x}, t)$ occurs by two mechanisms:

advection at rate $||\mathbf{u}||/L$ **diffusion** at rate κ/L^2

Together, they determine $T(\mathsf{x}, t) =$ temperature through

$$\partial_t T + \mathrm{div}(\mathbf{u}T - \kappa\nabla T) = 0$$

We recognize the *heat flux*

$$\mathbf{J} = \mathbf{u}T - \kappa\nabla T$$

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advection

diffusion

Together, they determine $T(\mathbf{x}, t) =$ temperature through

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The Péclet number

$$Pe = \frac{\text{rate of advection}}{\text{rate of diffusion}} = \frac{||\mathbf{u}||/L}{\kappa/L^2} \gg 1$$

Heat transport in a fluid layer

$$T = 0, \quad \mathbf{u} = \mathbf{0}$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \Delta T$$
$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \Delta \mathbf{u} + \mathbf{f}$$
$$\operatorname{div} \mathbf{u} = 0$$

$$T = 1, \quad \mathbf{u} = \mathbf{0}$$

$$\mathbf{u} = \hat{\mathbf{i}} u + \hat{\mathbf{j}} v + \hat{\mathbf{k}} w$$
$$\mathbf{J} = \mathbf{u} T - \nabla T$$

Question: Which forces $\mathbf{f}$ produce the largest transport of heat,

$$\max_{\mathbf{f}} \langle \mathbf{J} \cdot \hat{\mathbf{k}} \rangle ?$$

Notation for averaging:

$$\langle \cdot \rangle = \limsup_{\tau \to \infty} \frac{1}{\tau |\text{fluid layer}|} \int_0^\tau \int_{\text{fluid layer}} \cdot \, dx dt$$

To flow $\geq$ not to flow

$$T = 0, \quad \mathbf{u} = \mathbf{0}$$

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The *Nusselt number* is defined as enhancement of heat transport

$$Nu(\mathbf{u}) = \frac{\text{total vertical heat flux}}{\text{conductive vertical heat flux}} = \frac{\langle \mathbf{J} \cdot \hat{\mathbf{k}} \rangle}{\langle -\nabla T \cdot \hat{\mathbf{k}} \rangle} \geq 1$$

We seek to **maximize** it... the answer is $+\infty$ w/o constraints

Enstrophy budget

$$T = 0, \quad \mathbf{u} = \mathbf{0}$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \Delta T$$
$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \Delta \mathbf{u} + \mathbf{f}$$
$$\text{div } \mathbf{u} = 0$$

$$\mathbf{u} = \hat{\mathbf{i}} u + \hat{\mathbf{j}} v + \hat{\mathbf{k}} w$$

$$T = 1, \quad \mathbf{u} = \mathbf{0}$$

A natural constraint is on the **power** expended to sustain fluid flow

From the momentum eqn.,

$$\langle \mathbf{f} \cdot \mathbf{u} \rangle = \left\langle |\nabla \mathbf{u}|^2 \right\rangle$$

average power expended = average "enstrophy"

The wall-to-wall optimal transport problem

$$T = 0, \quad \mathbf{u} = \mathbf{0}$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \Delta T$$
$$\text{div } \mathbf{u} = 0$$

$$T = 1, \quad \mathbf{u} = \mathbf{0}$$

$$\mathbf{u} = \hat{\mathbf{i}} u + \hat{\mathbf{j}} v + \hat{\mathbf{k}} w$$
$$Nu = \langle \mathbf{J} \cdot \hat{\mathbf{k}} \rangle$$

Problem: Maximize the wall-to-wall heat transport Nu amongst all incompressible flows sat. a given enstrophy budget,

$$\max_{\substack{\mathbf{u}(\mathbf{x},t) \\ \langle |\nabla \mathbf{u}|^2 \rangle^{1/2} = Pe \\ \text{b.c.s}}} \quad Nu(\mathbf{u})$$

What do optimizers look like?

$$T = 0, \quad \mathbf{u} = \mathbf{0}$$

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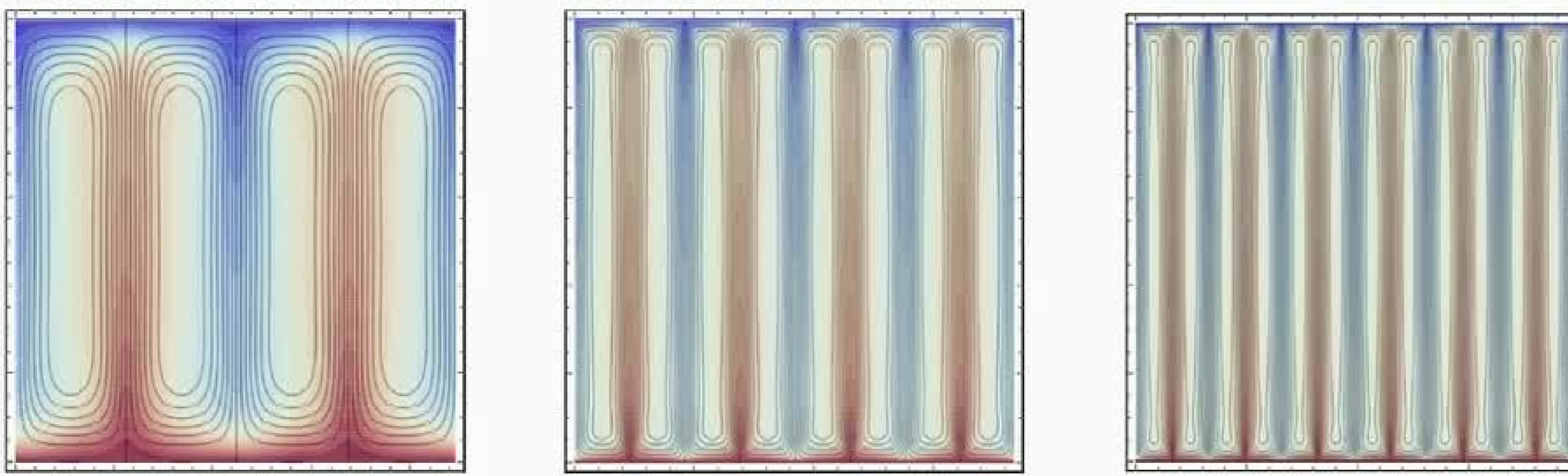

[1]

[1] A. Souza, PhD thesis 2016

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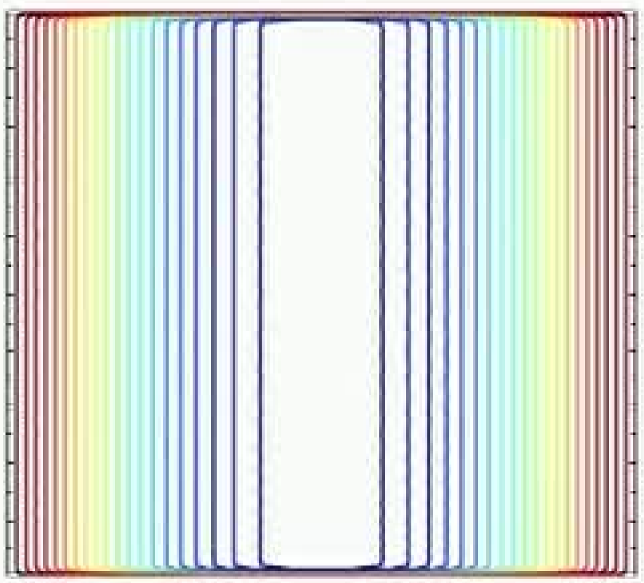
stress-free b.c.
$\partial_z u = w = 0$



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$\max Nu \sim Pe$

$l_{bulk} \sim Pe^{-1/2}$

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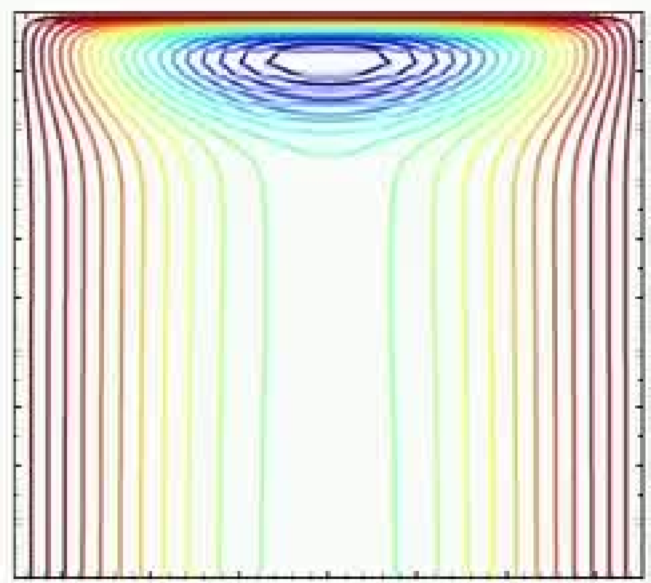

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$Nu \sim Pe^{0.58}$

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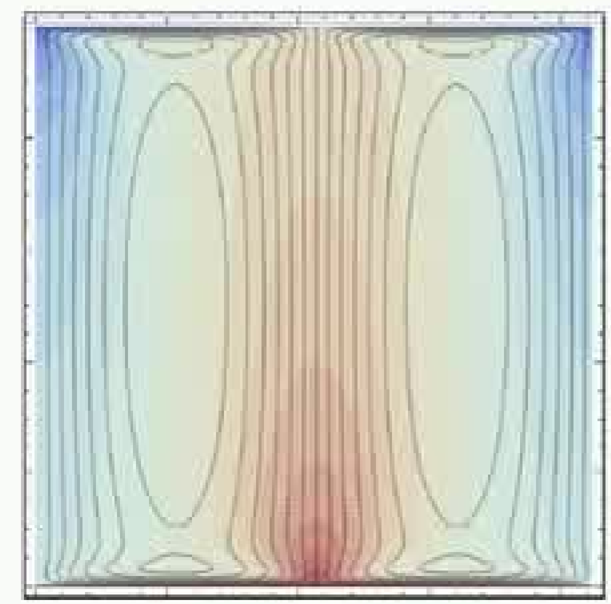

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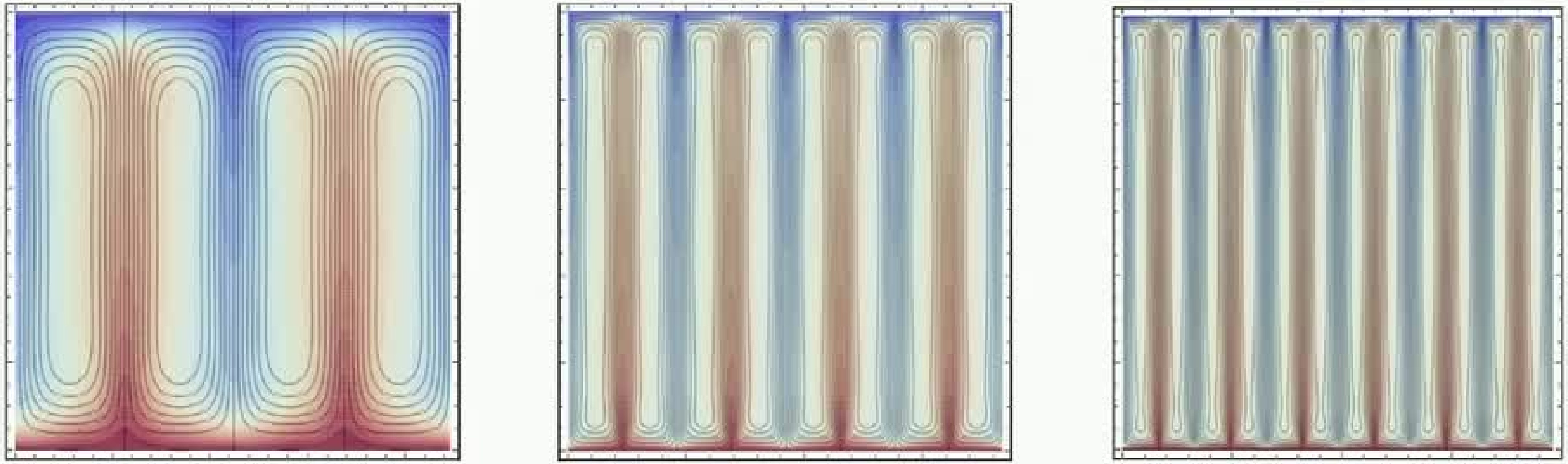

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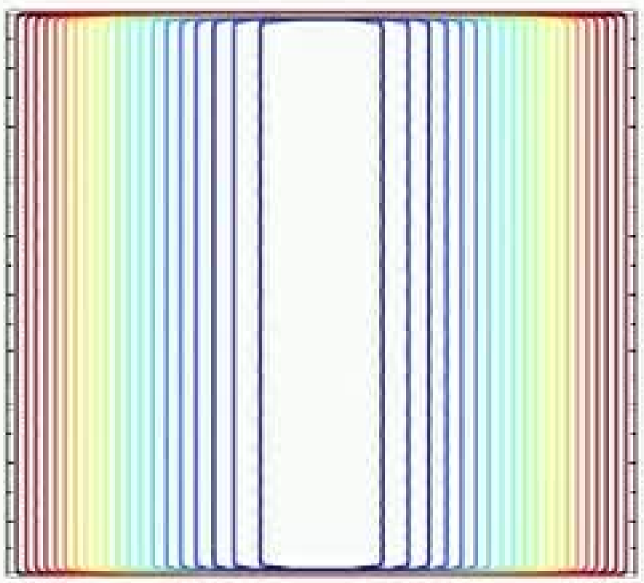

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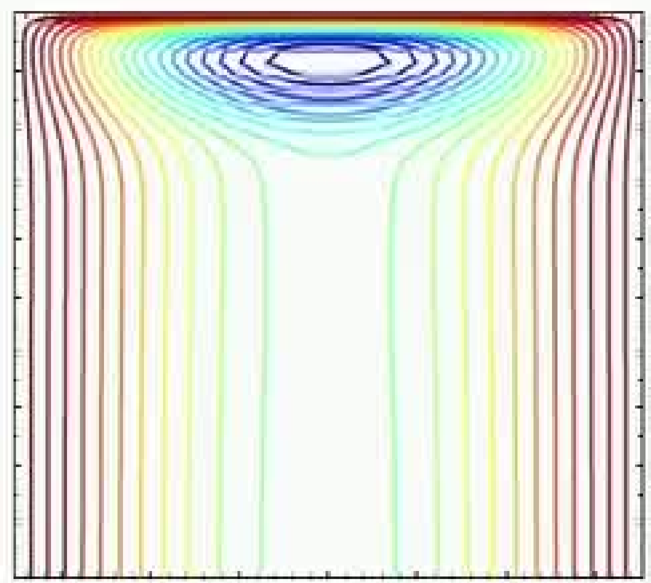

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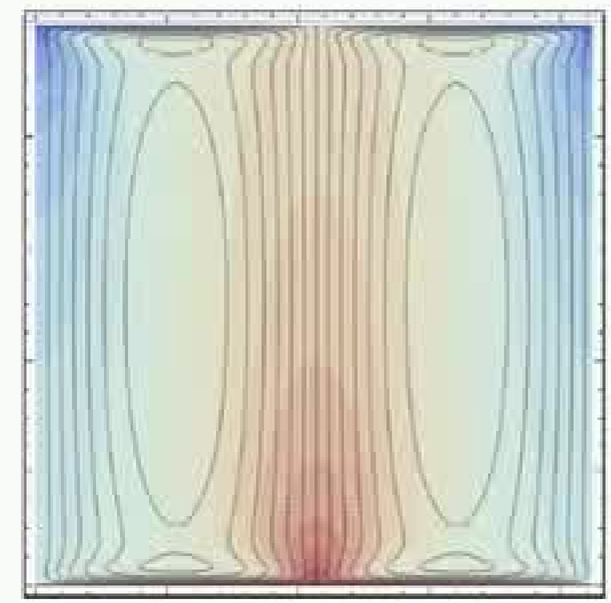

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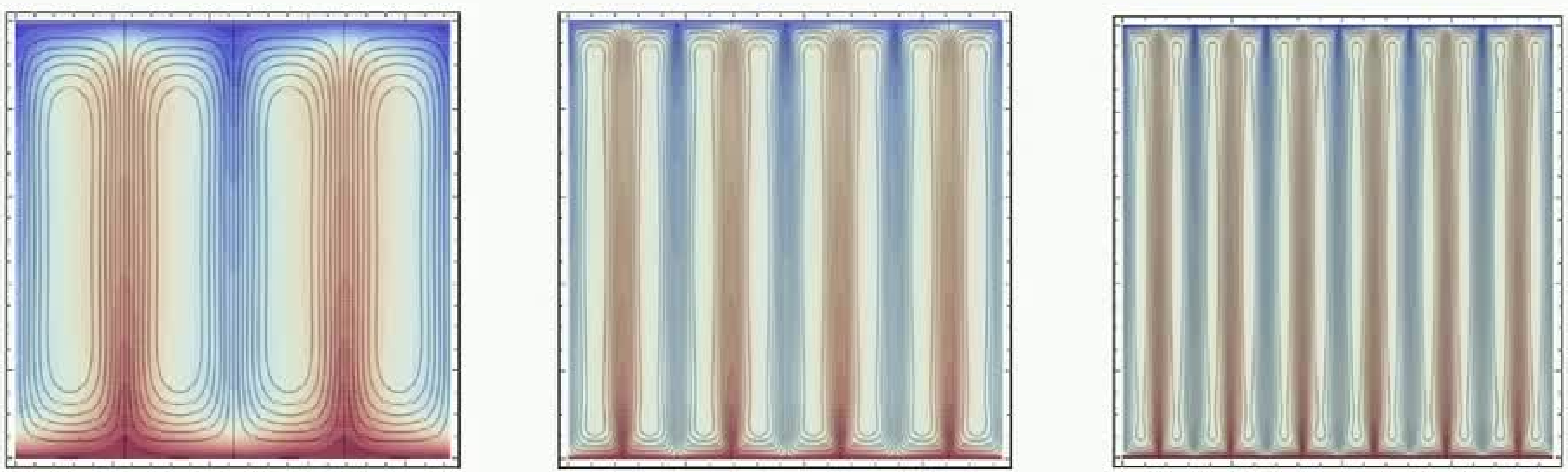

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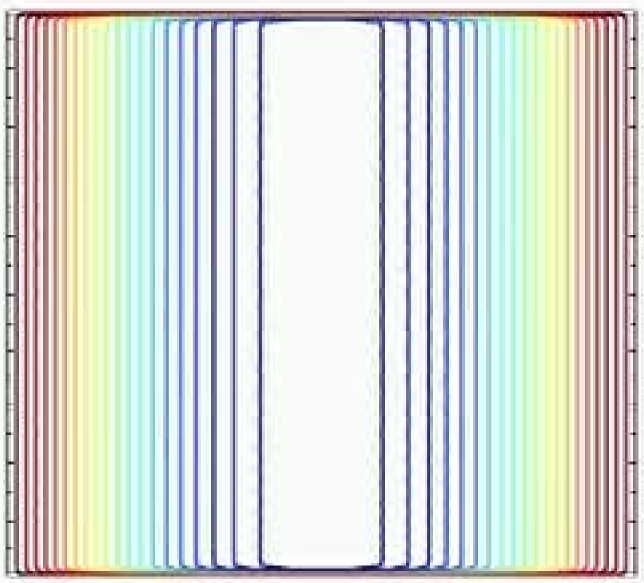

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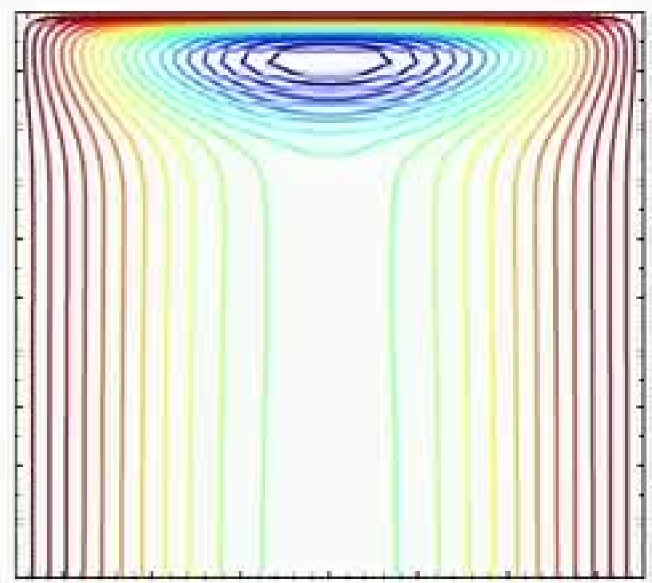

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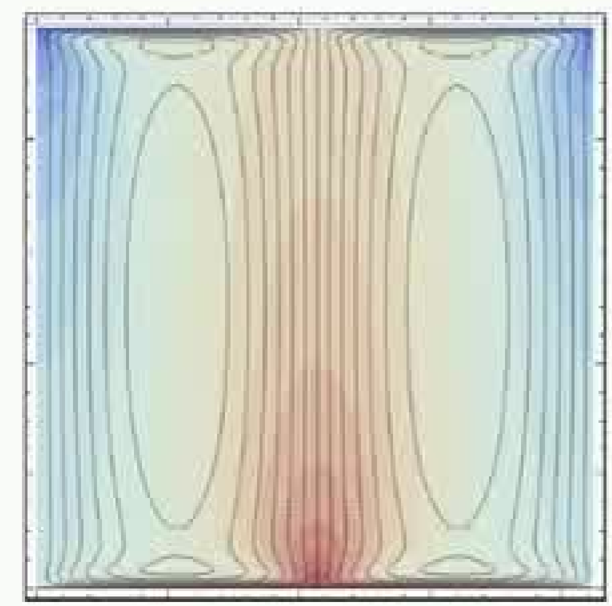

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What must optimizers obey?

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Theorem (Souza & Doering, '16)

$$\max_{\substack{\mathbf{u}(\mathbf{x},t) \\ \langle |\nabla \mathbf{u}|^2 \rangle^{1/2} = Pe \\ b.c.s}} \quad Nu(\mathbf{u}) \leq C\, Pe^{2/3}$$

$\exists$ multiple proofs:

- a modification of the "background method" (C. Doering & P. Constantin, Phys Rev E '96)
- an elementary "conservation law" argument (C. Seis, JFM '15)

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K.E. budget $\langle|\mathbf{u}|^2\rangle^{1/2} = Pe$ | enstrophy budget $\langle|\nabla\mathbf{u}|^2\rangle^{1/2} = Pe$ | enstrophy budget $\langle|\nabla\mathbf{u}|^2\rangle^{1/2} = Pe$

stress-free b.c. $\partial_z u = w = 0$ | stress-free b.c. $\partial_z u = w = 0$ | no-slip b.c. $u = w = 0$

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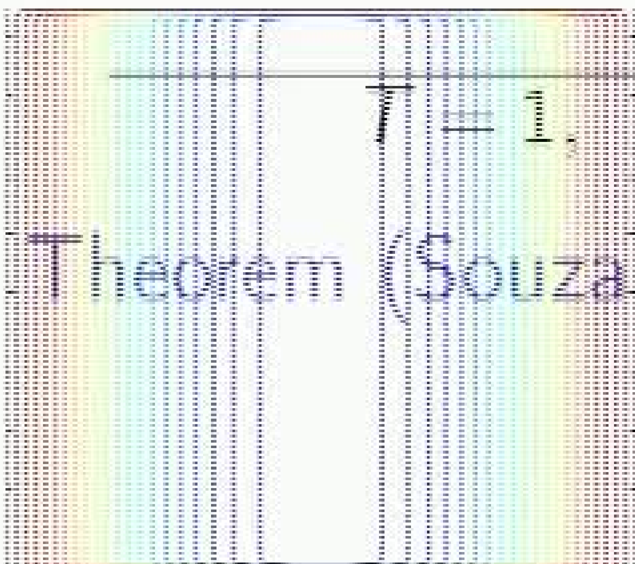

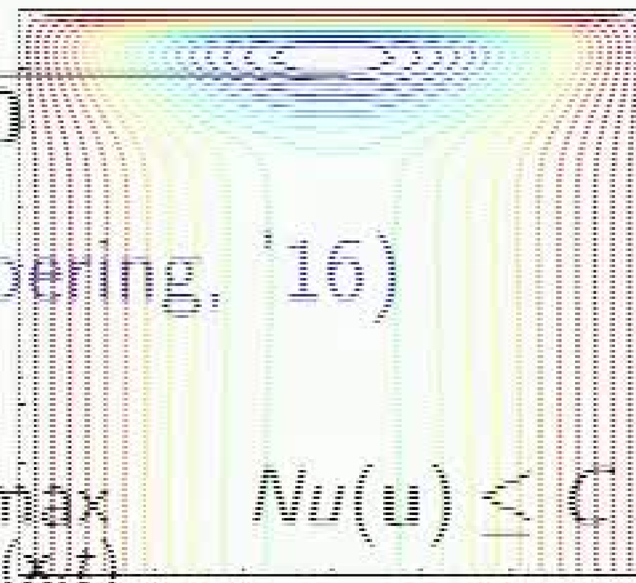

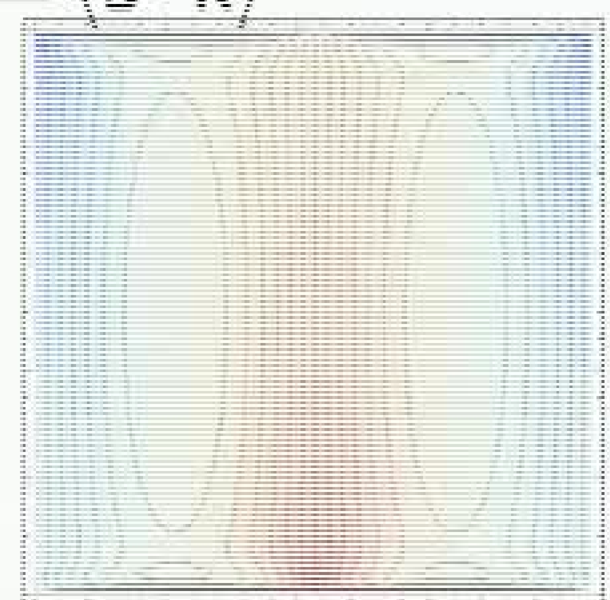

Theorem (Souza & Doering, '16)

$$\max_{\substack{\mathbf{u}(\mathbf{x},t)\\ \langle|\nabla\mathbf{u}|^2\rangle^{1/2}=Pe\\ b.c.s}} Nu(\mathbf{u}) \le C\, Pe^{2/3}$$

$\max Nu \sim Pe$ | $Nu \sim Pe^{0.58}$ | $Nu \sim Pe^{0.54}$

$l_{bulk} \sim Pe^{-1/2}$ | $l_{bulk} \sim Pe^{-0.36}$ | $l_{bulk} \sim Pe^{-0.37}$

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$\exists$ multiple proofs:

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So what's the optimal rate?

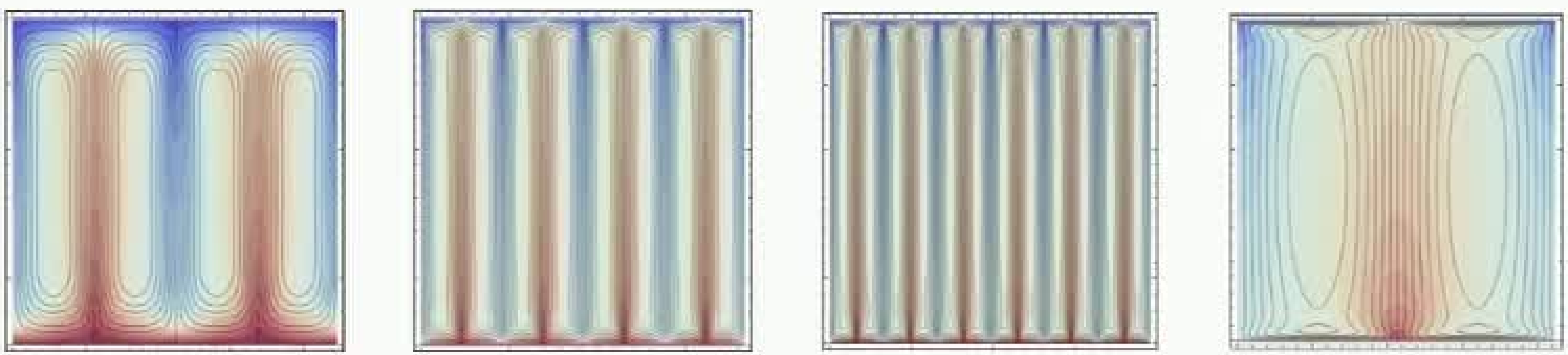
$$T = 0, \quad \mathbf{u} = \mathbf{0}$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \Delta T$$

$$\operatorname{div} \mathbf{u} = 0$$

$$T = 1, \quad \mathbf{u} = \mathbf{0}$$

$$\max_{\substack{\mathbf{u}(\mathbf{x},t) \\ \langle |\nabla \mathbf{u}|^2 \rangle^{1/2} = Pe \\ \text{b.c.s}}} \quad Nu(\mathbf{u})$$



$\cdots$??

$Nu \sim Pe^{1/2}$ $\qquad$ $Nu \sim Pe^{0.54}$

$Pe = 4 \times 10^2$ $\qquad$ 5×10^3 $\qquad$ 1.3×10^4 $\qquad$ 4×10^4

Main result

Theorem (T. & Doering, '17)

Up to logarithmic corrections, the optimal rate of heat transport satisfies

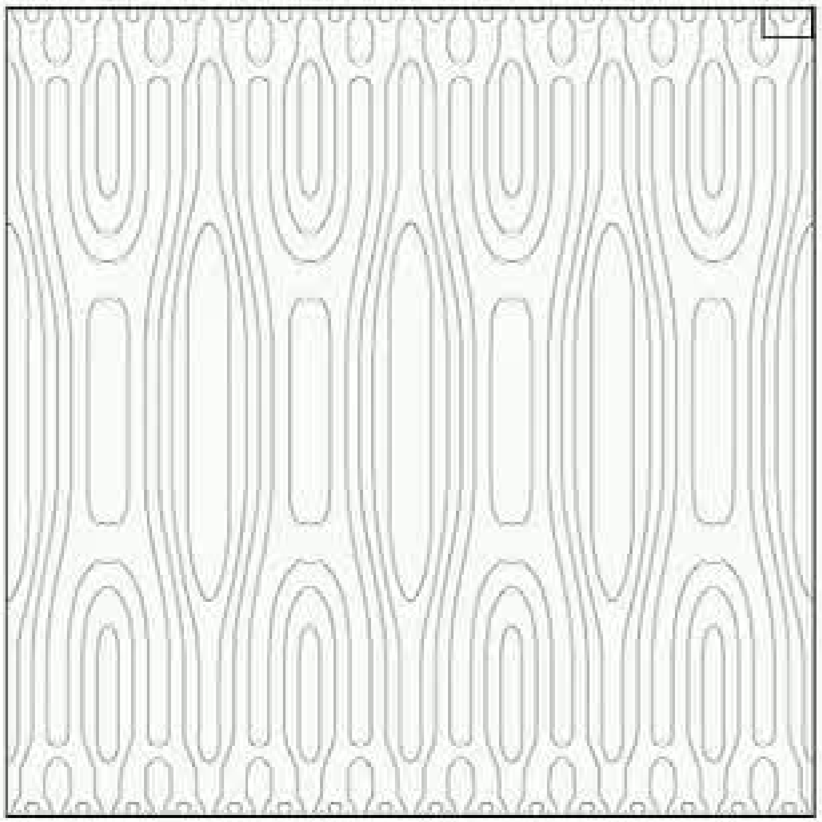
$$\max_{\substack{\mathbf{u}(\mathbf{x},t) \\ \langle|\nabla \mathbf{u}|^2\rangle^{1/2}=Pe \\ \mathbf{u}|_{\partial\Omega}=\mathbf{0}}} Nu(\mathbf{u}) \sim Pe^{2/3} \quad \text{as } Pe \to \infty.$$

More precisely, there exist constants C, C' depending only on the domain such that

$$C\frac{Pe^{2/3}}{\log^{4/3} Pe} \leq \max_{\substack{\mathbf{u}(\mathbf{x},t) \\ \langle|\nabla \mathbf{u}|^2\rangle^{1/2}=Pe \\ \mathbf{u}|_{\partial\Omega}=\mathbf{0}}} Nu(\mathbf{u}) \leq C' Pe^{2/3}$$

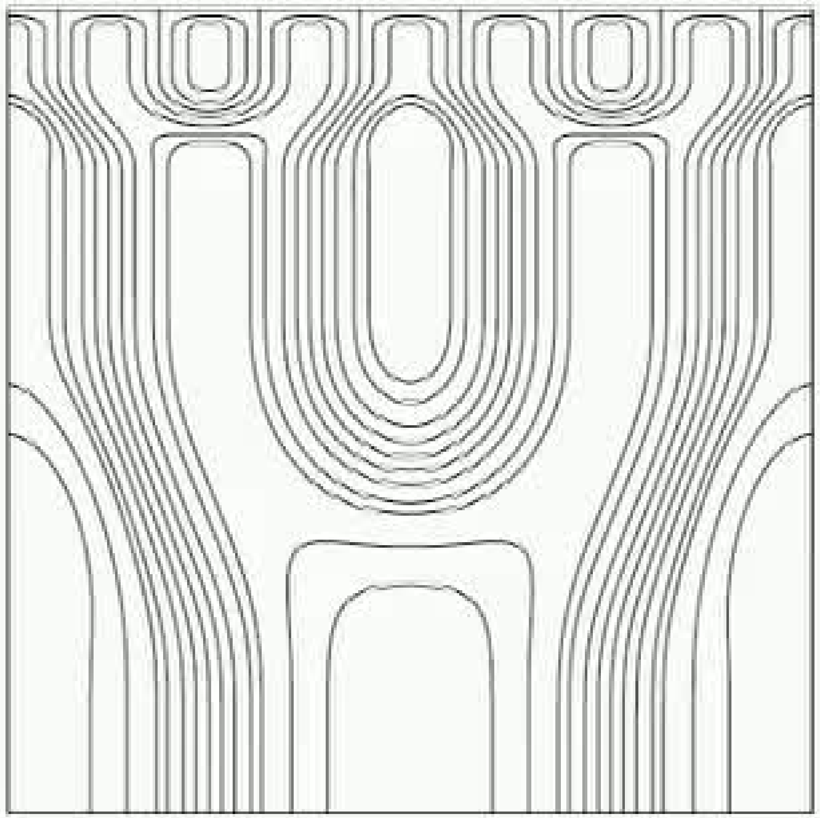
for $Pe \gg 1$.

What do our flows look like?



$$\mathbf{u} = \nabla^{\perp}\psi$$

$$l_k \lesssim \delta_k$$
$$l_{bl} \sim \delta_{bl}$$



Streamlines refine self-similarly from bulk to boundary layer

In the kth stage of refinement,

$$\psi(x,z) = f\left(\frac{z-z_k}{\delta_k}\right) \cdot l_k \Psi\left(\frac{x}{l_k}\right) + g\left(\frac{z-z_k}{\delta_k}\right) \cdot l_{k+1} \Psi\left(\frac{x}{l_{k+1}}\right)$$

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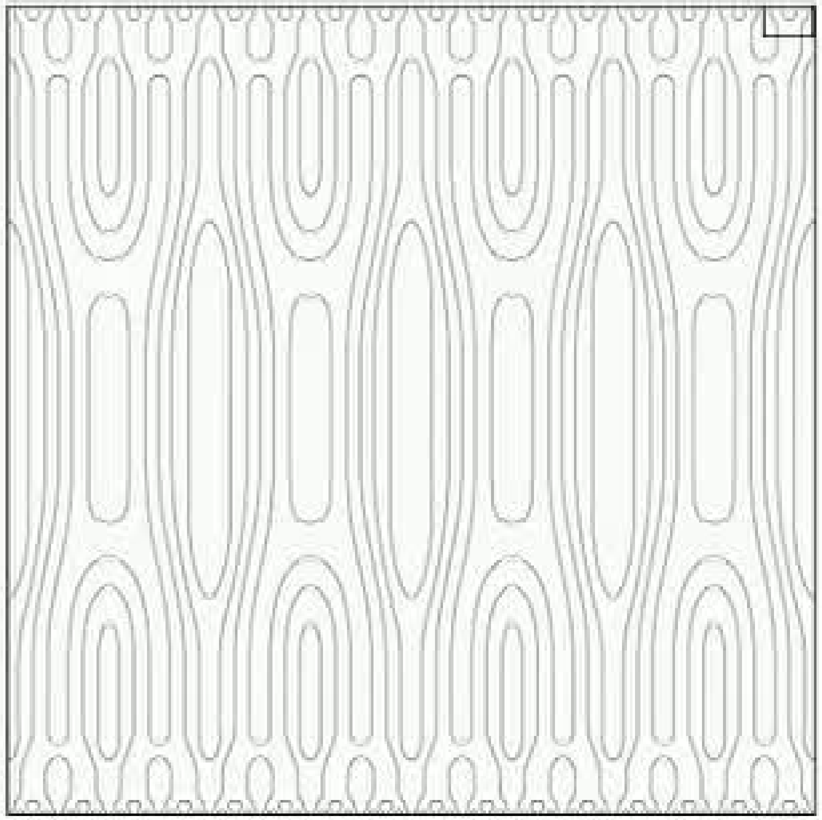
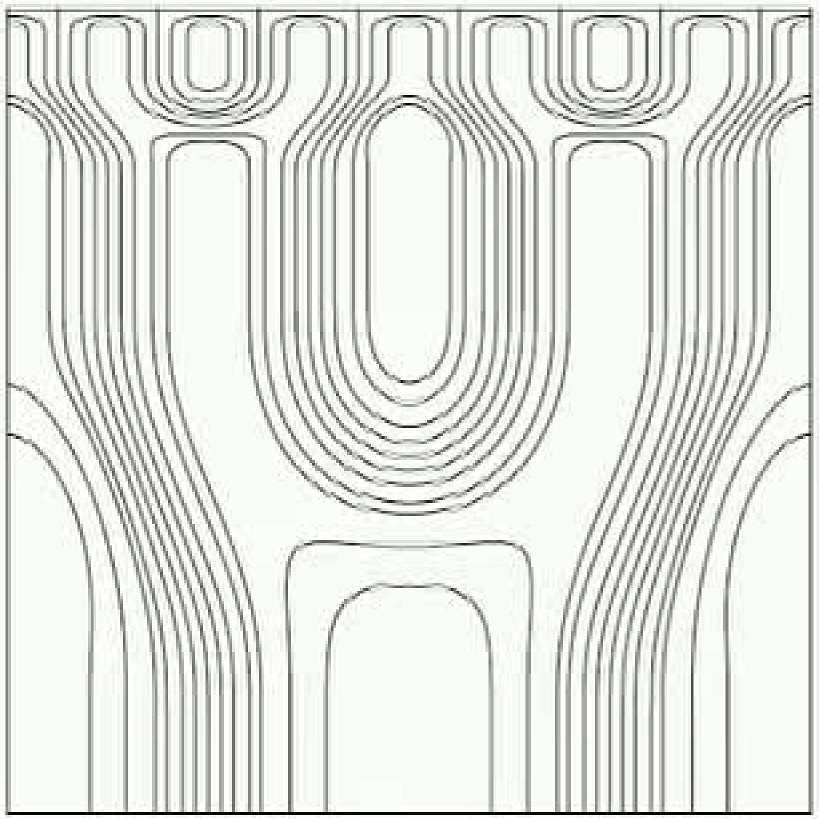


$$\mathbf{u} = \nabla^{\perp}\psi$$

$$\ell(z)$$



Horizontal lengthscales satisfy

$$l_{bl} \sim \frac{\log^{1/3} Pe}{Pe^{2/3}} \qquad l_{bulk} \sim \frac{\log^{1/6} Pe}{Pe^{1/3}}$$

and

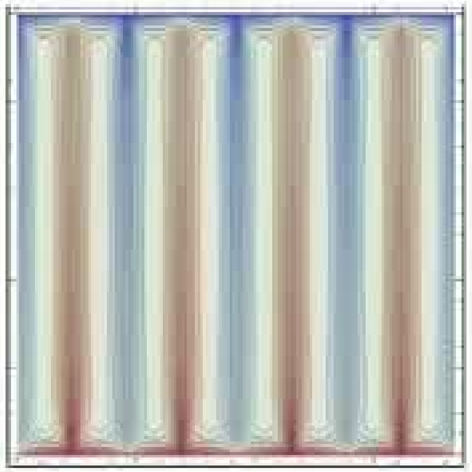
$$\ell(z) \sim \frac{\log^{1/6} Pe}{Pe^{1/3}}(1-z)^{1/2}$$

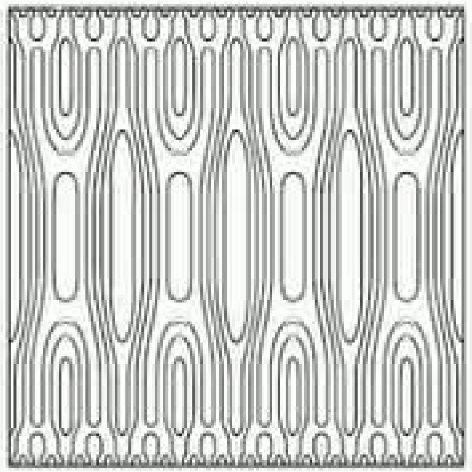
Brief sketch of the proof

Main challenges

As $Pe \to \infty$, our designs feature

- increasingly fine lengthscales
- an increasing number of distinct lengthscales





Simplify by taking $\mathbf{u}(x)$ indpt. of time (and why should time-dependence help?)

Main goals: Motivate our "branched" flow designs, and estimate their heat transport Nu in the advection-dominated limit $Pe \to \infty$.

Punchline: The analysis of optimal heat transport is analogous to pattern formation in micromagnetics, elasticity theory, etc.

Step 1: Obtain a general variational principle for heat transport

A non-local Dirichlet principle for heat transport

$$T = 0, \quad \mathbf{u} = \mathbf{0}$$

$$\mathbf{u} \cdot \nabla T = \Delta T$$
$$\operatorname{div} \mathbf{u} = 0$$

$$T = 1, \quad \mathbf{u} = \mathbf{0}$$

$$\mathbf{u} = \hat{\mathbf{i}} u + \hat{\mathbf{j}} v + \hat{\mathbf{k}} w$$
$$Nu = \langle \mathbf{J} \cdot \hat{\mathbf{k}} \rangle$$

Lemma

There exist dual variational principles for heat transport by a steady divergence-free flow.

$$Nu(\mathbf{u}) - 1 = \min_{\eta : \eta|_{\partial\Omega} = 0} \fint |\nabla \eta|^2 + |\nabla \Delta^{-1}(-w + \mathbf{u} \cdot \nabla \eta)|^2$$
$$= \max_{\xi : \xi|_{\partial\Omega} = 0} \fint 2w\xi - |\nabla \Delta^{-1} \mathbf{u} \cdot \nabla \xi|^2 - |\nabla \xi|^2$$

Step 2: Recognize optimal heat transport as "energy-driven pattern formation"

...what plays the role of "free energy"?

A useful change of variables

Consider the general class of steady wall-to-wall problems,

$$\max_{\substack{\mathbf{u}(\mathbf{x}) \\ ||\mathbf{u}||=Pe \\ \text{b.c.s}}} \; Nu(\mathbf{u})$$

where, e.g.,

$$||\mathbf{u}||^2 = \fint_\Omega |\mathbf{u}|^2 \quad \text{in the energy-constrained case}$$

$$||\mathbf{u}||^2 = \fint_\Omega |\nabla \mathbf{u}|^2 \quad \text{in the enstrophy-constrained case}$$

Now we know the variational principle

$$\max_{\substack{\mathbf{u}(\mathbf{x}) \\ ||\mathbf{u}||=Pe \\ \text{b.c.s}}} \; Nu(\mathbf{u}) = 1 + \max_{\substack{\mathbf{u}(\mathbf{x}),\xi(\mathbf{x}) \\ ||\mathbf{u}||=Pe \\ \text{b.c.s}}} \left\{ \fint 2w\xi - |\nabla\Delta^{-1}\mathrm{div}\,\mathbf{u}\xi|^2 - |\nabla\xi|^2 \right\}$$

A useful change of variables

Consider the general steady wall-to-wall problem

Application: Enstrophy-constrained wall-to-wall problem

$$\max_{\substack{\mathbf{u}(\mathbf{x}) \\ \fint_\Omega |\nabla \mathbf{u}|^2 = Pe^2 \\ \mathbf{u}|_{\partial\Omega}=0}} Nu(\mathbf{u})$$

is equivalent to solving

$$\min_{\substack{\mathbf{u}(\mathbf{x}),\xi(\mathbf{x}) \\ \fint_\Omega w\xi = 1 \\ \mathbf{u}|_{\partial\Omega}=0,\ \xi|_{\partial\Omega}=0}} \fint_\Omega |\nabla \Delta^{-1} \operatorname{div} \mathbf{u}\xi|^2 + \frac{1}{Pe^2} \fint_\Omega |\nabla \mathbf{u}|^2 \cdot \fint_\Omega |\nabla \xi|^2$$

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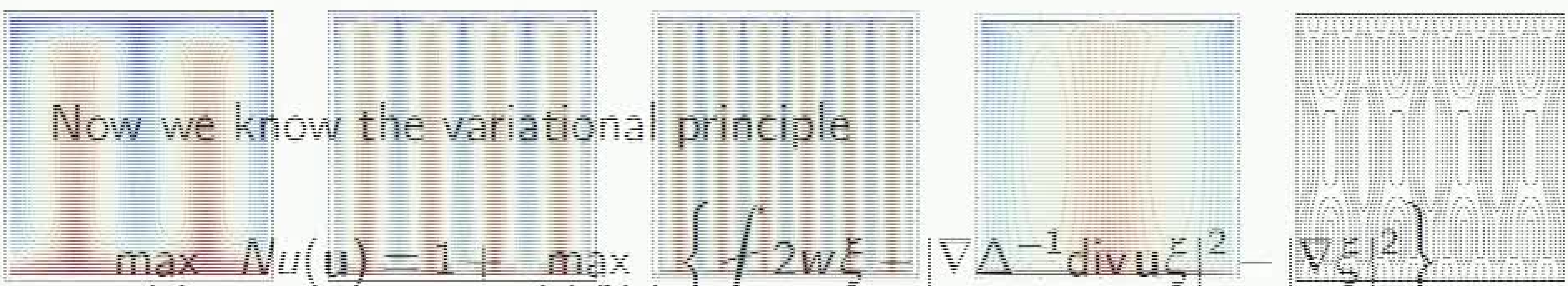

Now we know the variational principle

$$\underbrace{\max_{\substack{\mathbf{u}(\mathbf{x}) \\ ||\mathbf{u}||=Pe \\ \text{b.c.s}}} Nu(\mathbf{u}) = 1 + \max_{\substack{\mathbf{u}(\mathbf{x}),\xi(\mathbf{x}) \\ ||\mathbf{u}||=Pe \\ \text{b.c.s}}}}_{Nu \sim Pe^{1/2}} \left\{ \fint 2w\xi - |\nabla \Delta^{-1} \operatorname{div} \mathbf{u}\xi|^2 - |\nabla \xi|^2 \right\}$$

$Nu \sim Pe^{0.54}$ $\quad$ $Nu \sim Pe^{2/3}$

A useful change of variables

Consider the general class of steady wall-to-wall problems,

$$\max_{\substack{\mathbf{u}(\mathbf{x}) \\ ||\mathbf{u}||=Pe \\ \text{b.c.s}}} Nu(\mathbf{u})$$

where, e.g.,

$$||\mathbf{u}||^2 = \fint_\Omega |\mathbf{u}|^2 \quad \text{in the energy-constrained case}$$

$$||\mathbf{u}||^2 = \fint_\Omega |\nabla \mathbf{u}|^2 \quad \text{in the enstrophy-constrained case}$$

Now we know the variational principle

$$\max_{\substack{\mathbf{u}(\mathbf{x}) \\ ||\mathbf{u}||=Pe \\ \text{b.c.s}}} Nu(\mathbf{u}) = 1 + \max_{\substack{\mathbf{u}(\mathbf{x}),\xi(\mathbf{x}) \\ ||\mathbf{u}||=Pe \\ \text{b.c.s}}} \left\{ \fint 2w\xi - |\nabla \Delta^{-1} \mathrm{div}\, \mathbf{u}\xi|^2 - |\nabla \xi|^2 \right\}$$

A useful change of variables

Application: The enstrophy-constrained wall-to-wall problem

$$\max_{\substack{\mathbf{u}(\mathbf{x}) \\ \fint_\Omega |\nabla \mathbf{u}|^2 = Pe^2 \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}}} \quad Nu(\mathbf{u})$$

is **equivalent** to solving

$$\min_{\substack{\mathbf{u}(\mathbf{x}),\xi(\mathbf{x}) \\ \fint_\Omega w\xi = 1 \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}, \xi|_{\partial\Omega} = 0}} \quad \fint_\Omega |\nabla \Delta^{-1} \operatorname{div} \mathbf{u}\xi|^2 + \frac{1}{Pe^2} \fint_\Omega |\nabla \mathbf{u}|^2 \cdot \fint_\Omega |\nabla \xi|^2$$

$Nu \sim Pe^{1/2}$ $\qquad$ $Nu \sim Pe^{0.54}$ $\qquad$ $Nu \sim Pe^{2/3}$

A useful change of variables

Consider the general class of steady wall-to-wall problems,

$$\max_{\substack{\mathbf{u}(\mathbf{x}) \\ ||\mathbf{u}||=Pe \\ \text{b.c.s}}} \; Nu(\mathbf{u})$$

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Now we know the variational principle

$$\max_{\substack{\mathbf{u}(\mathbf{x}) \\ ||\mathbf{u}||=Pe \\ \text{b.c.s}}} \; Nu(\mathbf{u}) = 1 + \max_{\substack{\mathbf{u}(\mathbf{x}),\xi(\mathbf{x}) \\ ||\mathbf{u}||=Pe \\ \text{b.c.s}}} \left\{ \fint 2w\xi - |\nabla \Delta^{-1} \mathrm{div}\, \mathbf{u}\xi|^2 - |\nabla \xi|^2 \right\}$$

A useful change of variables

<u>Application:</u> The enstrophy-constrained wall-to-wall problem

$$\max_{\substack{\mathbf{u}(\mathbf{x}) \\ \fint_\Omega |\nabla \mathbf{u}|^2 = Pe^2 \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}}} \quad Nu(\mathbf{u})$$

is **equivalent** to solving

$$\min_{\substack{\mathbf{u}(\mathbf{x}),\xi(\mathbf{x}) \\ \fint_\Omega w\xi = 1 \\ \mathbf{u}|_{\partial\Omega}=\mathbf{0},\xi|_{\partial\Omega}=0}} \quad \fint_\Omega |\nabla \Delta^{-1} \mathrm{div}\, \mathbf{u}\xi|^2 + \frac{1}{Pe^2} \fint_\Omega |\nabla \mathbf{u}|^2 \cdot \fint_\Omega |\nabla \xi|^2$$

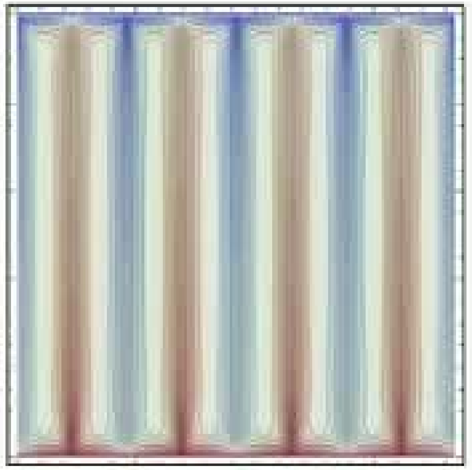
$Nu \sim Pe^{1/2}$ $\quad$ $Nu \sim Pe^{0.54}$ $\quad$ $Nu \sim Pe^{2/3}$

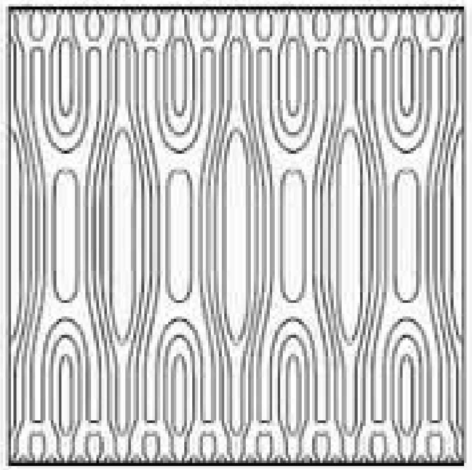
Step 3: The heat transport of branched flow designs

The branching construction

Recall: Our main result states that

$$\max_{\substack{\mathbf{u}(\mathbf{x},t)\\ \fint_\Omega |\nabla \mathbf{u}|^2 = Pe^2 \\ \mathbf{u}|_{\partial\Omega}=\mathbf{0}}} Nu(\mathbf{u}) \sim Pe^{2/3} \quad \text{up to logs}$$

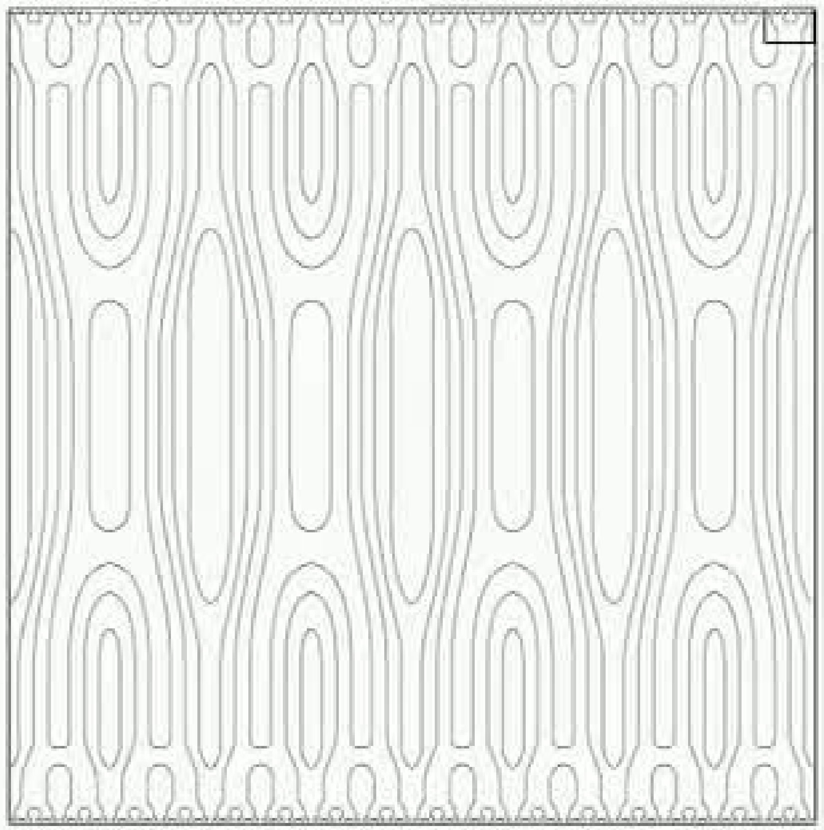




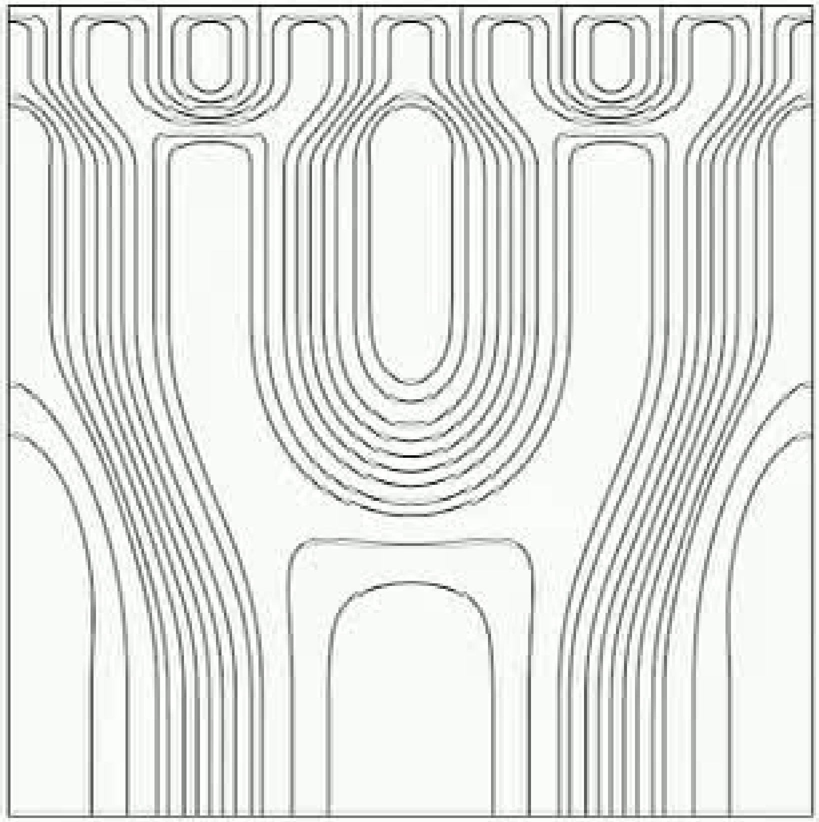
We just showed: It is equivalent to prove

$$\min_{\substack{\mathbf{u}(\mathbf{x}),\xi(\mathbf{x})\\ \fint_\Omega w\xi = 1\\ \mathbf{u}|_{\partial\Omega}=\mathbf{0},\xi|_{\partial\Omega}=0}} \fint_\Omega |\nabla \Delta^{-1} \operatorname{div} \mathbf{u}\xi|^2 + \frac{1}{Pe^2}\fint_\Omega |\nabla \mathbf{u}|^2 \cdot \fint_\Omega |\nabla \xi|^2 \sim \frac{1}{Pe^{2/3}}$$

The branching construction



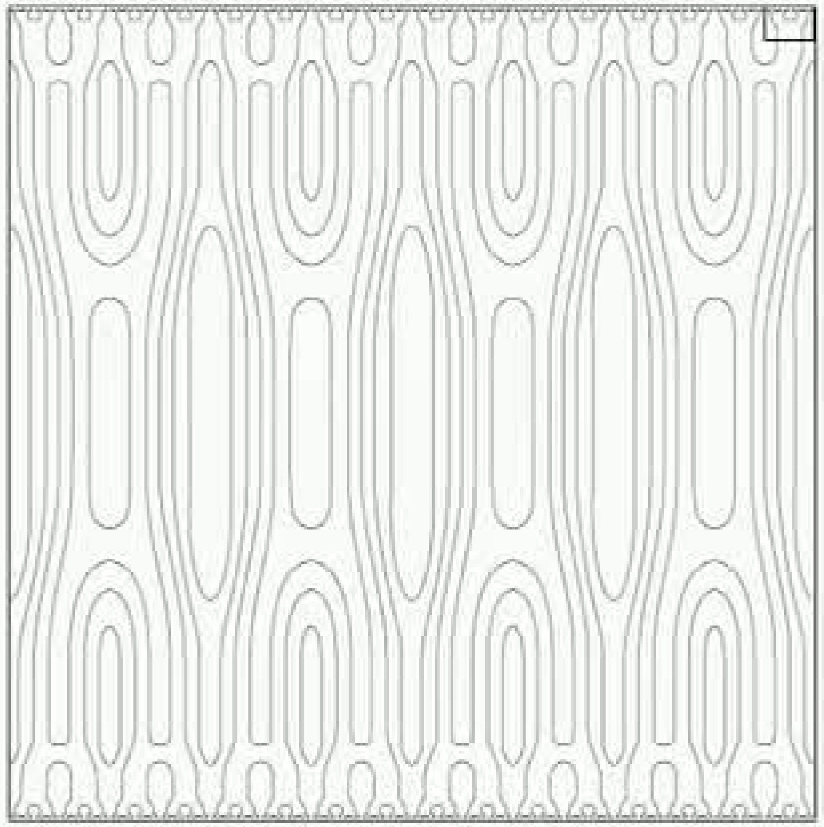
$$\mathbf{u} = \nabla^\perp \psi$$
$$\xi = w$$
$$\ell(z)$$



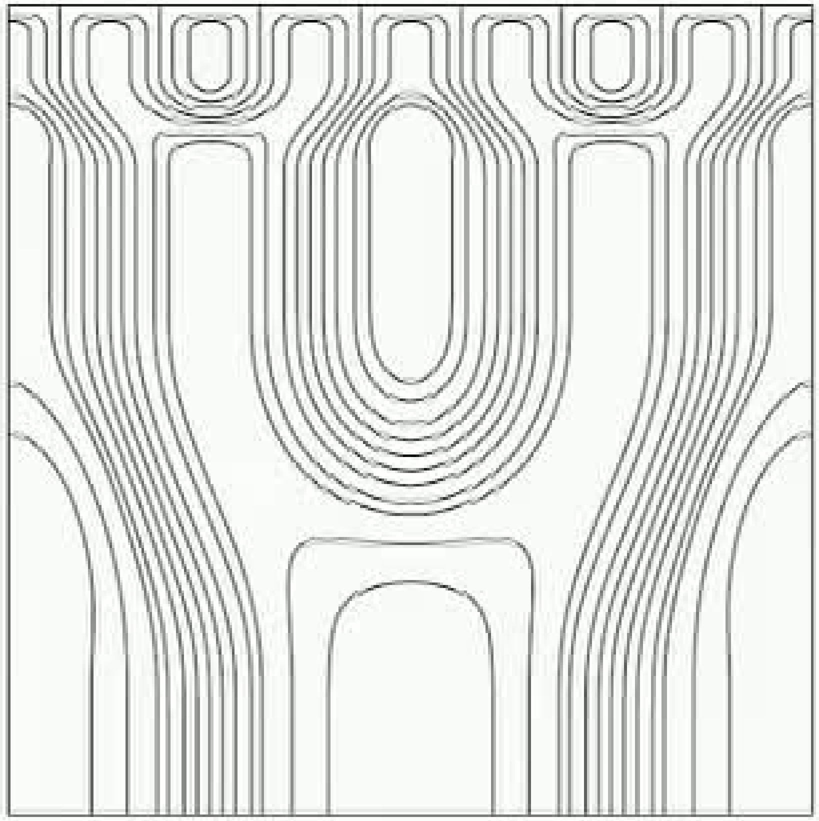
$$\fint_\Omega |\nabla \Delta^{-1} \mathrm{div}\, \mathbf{u}\xi|^2 + \frac{1}{Pe^2} \fint_\Omega |\nabla \mathbf{u}|^2 \cdot \fint_\Omega |\nabla \xi|^2$$
$$\lesssim l_{bl} + \int_{z_{bulk}}^{z_{bl}} (\ell')^2\, dz + \frac{1}{Pe^2} \left(\frac{1}{l_{bulk}^2} + \int_{z_{bulk}}^{z_{bl}} \frac{1}{\ell^2}\, dz + \frac{1}{l_{bl}} \right)^2$$

where $\ell = \ell(z) =$ horizontal lengthscale

The branching construction



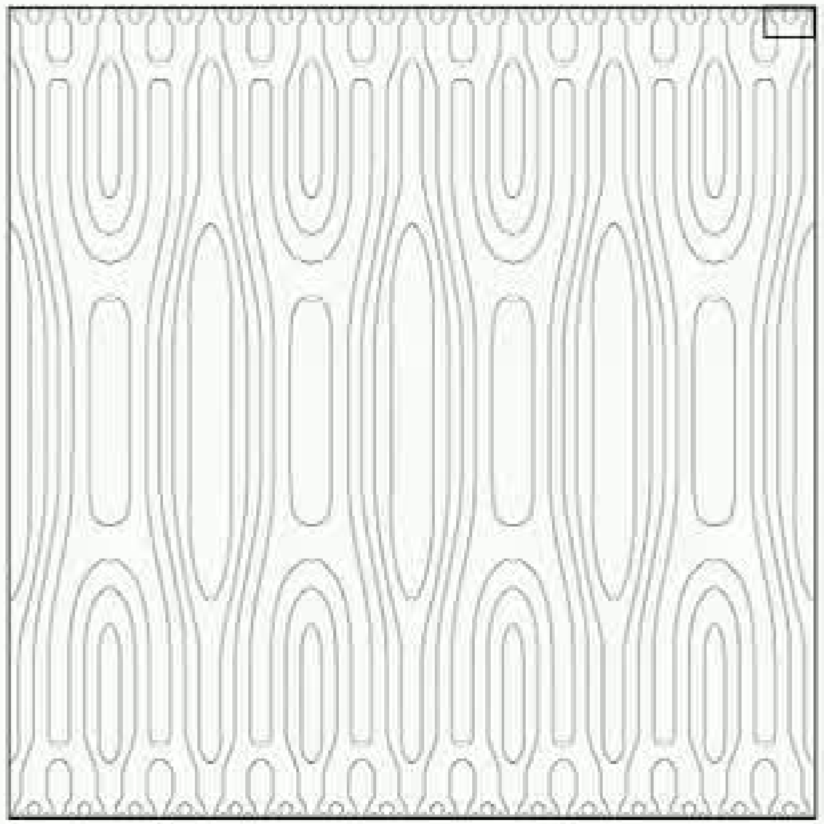
$$\mathbf{u} = \nabla^\perp \psi$$
$$\xi = w$$
$$\ell(z)$$



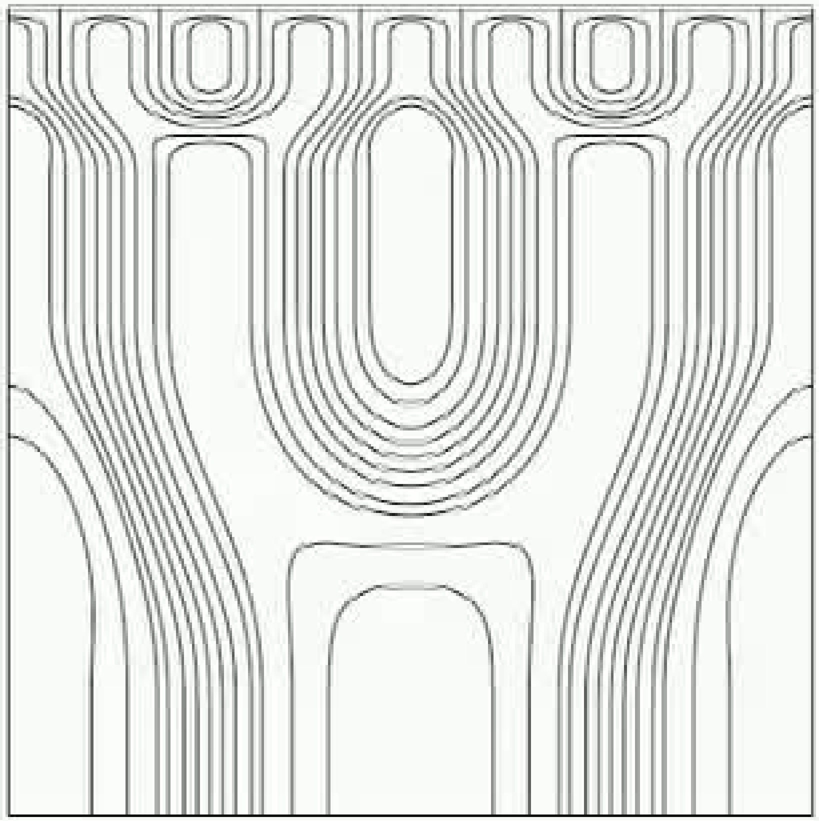
$$\fint_\Omega |\nabla \Delta^{-1} \mathrm{div}\, \mathbf{u} \xi|^2 + \frac{1}{Pe^2} \fint_\Omega |\nabla \mathbf{u}|^2 \cdot \fint_\Omega |\nabla \xi|^2$$
$$\lesssim l_{bl} + \int_{z_{bulk}}^{z_{bl}} (\ell')^2\, dz + \frac{1}{Pe^2} \left(\frac{1}{l_{bulk}^2} + \int_{z_{bulk}}^{z_{bl}} \frac{1}{\ell^2}\, dz + \frac{1}{l_{bl}} \right)^2$$

where $\ell = \ell(z) =$ horizontal lengthscale

The branching construction



$$\mathbf{u} = \nabla^\perp \psi$$
$$\xi = w$$
$$\ell(z)$$



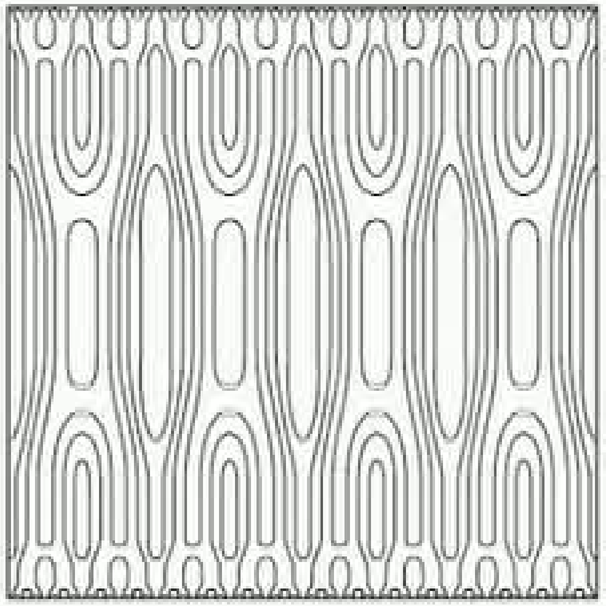
$$\min_{\substack{\ell(z) \\ \ell(z_{bulk}) = l_{bulk} \\ \ell(z_{bl}) = l_{bl}}} \left\{ l_{bl} + \int_{z_{bulk}}^{z_{bl}} (\ell')^2 \, dz + \frac{1}{Pe^2} \left(\frac{1}{l_{bulk}^2} + \int_{z_{bulk}}^{z_{bl}} \frac{1}{\ell^2} \, dz + \frac{1}{l_{bl}} \right)^2 \right\} \sim \frac{\log^{4/3} Pe}{Pe^{2/3}}$$

$$\ell(z) \sim \frac{\log^{1/6} Pe}{Pe^{1/3}} (1-z)^{1/2}$$

$$l_{bulk} \sim \frac{\log^{1/6} Pe}{Pe^{1/3}} \qquad l_{bl} \sim \frac{\log^{1/3} Pe}{Pe^{2/3}}$$

Concluding remarks

- For enstrophy-constrained transport max $Nu \sim Pe^{2/3}$ up to logs
- Extensive 2D numerics finds $Nu \sim Pe^{0.54} \approx Pe^{6/11}$
- Proof combines
  1. The old *a priori* upper bound max $Nu \lesssim Pe^{2/3}$
  2. A new functional analytic framework for optimal heat transport
  3. A new branching construction achieving $Nu \gtrsim Pe^{2/3-}$
- We were inspired by the analysis of branching in materials science, e.g., micromagnetics



[1]

[1] A. Hubert & R. R. Schäfer, *Magnetic Domains* 1998

Other examples of branching flows in fluid dynamics?

An old scientific question...

Does nature achieve optimal transport?

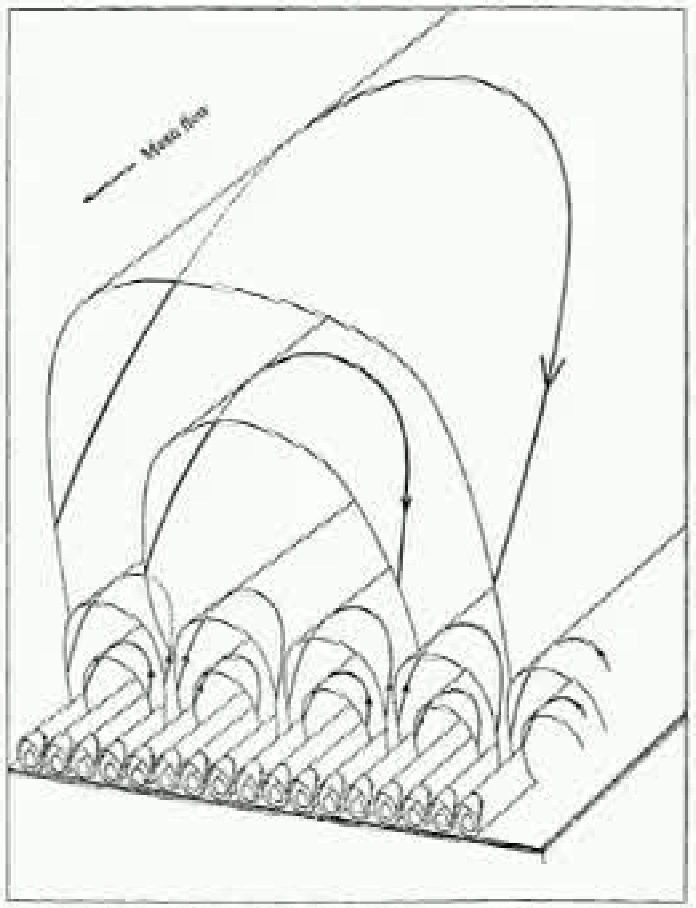
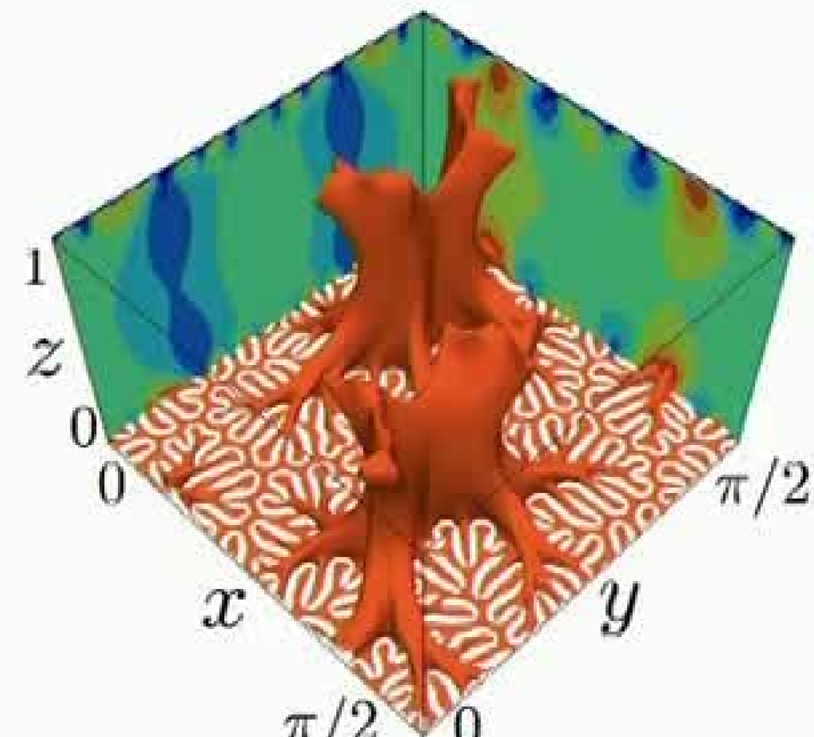


FIGURE 3. Qualitative sketch of the boundary-layer region of the vortex field yielding maximum transport of momentum.

F. H. Busse, *Bounds for turbulent shear flow*, JFM '70



S. Motoki, G. Kawahara, & M. Shimizu, *Maximal heat transfer between two parallel plates*, arxiv 1801.04588